On Gauss-Bonnet black hole entropy

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Abstract

We investigate the entropy of black holes in Gauss-Bonnet and Lovelock gravity using the Noether charge approach, in which the entropy is given as the integral of a suitable \((n - 2)\) form charge over the event horizon. We compare the results to those obtained in other approaches. We also comment on the appearance of negative entropies in some cases, and show that there is an additive ambiguity in the definition of the entropy which can be appropriately chosen to avoid this problem.

1 Introduction

There has recently been considerable interest in the study of higher-curvature corrections to the Einstein-Hilbert action, particularly in the context of brane worlds [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. These terms are important because their presence can lead to qualitative changes in the physics as seen from the point of view of observers on the brane (see, e.g., [14, 15]). In addition, such terms appear in the low-energy effective theories obtained from string theory [16, 17]; it is thus very natural to include them in the context of models motivated by string theory.

Typically, only certain special corrections are considered. At quadratic order, the combination which is considered is the Gauss-Bonnet term,

\[
\mathcal{L} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2.
\]  

(1)

In four dimensions this is a topological invariant; in higher dimensions, it is the most general quadratic correction which preserves the property that the equations of motion involve only second derivatives of the metric. This combination is thus particularly tractable, and explicit solutions to the resulting equations of motion have been found. It has also been shown that this gives a useful Lagrangian description of the leading correction to the low-energy effective action in heterotic string theory [16, 17]. The

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special nature of the Gauss-Bonnet term is more apparent when it is written in a differential forms notation; it is simply

\[ L = \varepsilon_{a_1 \ldots a_n} R^{a_1 a_2} R^{a_3 a_4} e^{a_5} \ldots e^{a_n}, \]  

(2)

where \( n \) is the spacetime dimension, \( e^a \) is a vielbein one-form, \( e^a = e^a_\alpha dx^\alpha \), and \( R^{a_1 a_2} \) is the curvature two-form, which is related to the Riemann tensor by \( R^{\alpha \beta}_{\gamma \delta} = R^{a_1 a_2} e^a_\alpha e^b_\beta \).

At higher orders, the natural generalisation is to consider the dimensionally continued Euler densities

\[ L^{(p)} = \varepsilon_{a_1 \ldots a_n} R^{a_1 a_2} \ldots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \ldots e^{a_n}. \]  

(3)

The first \( L^{(1)} \) is just the Einstein-Hilbert action. The Lagrangian density constructed from Einstein gravity plus these higher-order corrections defines the Lovelock theory [18]:

\[ L = \sum_{p=0}^{[n/2]} \alpha_p L^{(p)}. \]  

(4)

This is the most general Lagrangian invariant under local Lorentz transformations, constructed from the vielbein, the spin connection, and their exterior derivatives, without using the Hodge dual; such that the field equations for the metric are second order [19, 20, 21]. We will only consider in detail the choice of \( \alpha_p \) discussed in [22, 23], which gives a unique cosmological constant (i.e., a unique constant-curvature vacuum solution).

Since the equations of motion are of second order, it is possible to find explicit solutions, and black hole solutions of these theories have been found by several authors [24, 25, 26, 27, 22, 28, 23, 29, 30]. In the context of brane-world models, these black hole solutions play an important role, as non-trivial cosmological evolutions are obtained by considering the motion of a brane in a bulk black hole solution. The higher-order corrections in the action lead to modifications in the formula for the entropy of these black hole solutions; the entropy is no longer simply given by the area of the black hole’s event horizon. The entropy of the black hole solutions was calculated in [26, 23, 30]. In [26], the Euclidean approach was used, while in [23, 30] the entropy was calculated from the other thermodynamic quantities using the first law of thermodynamics. The thermodynamics of these solutions was further studied in [31, 32, 33], with the same results for the entropy.

Our aim in the present paper is to investigate the entropy of these black hole solutions using the general approach to black hole entropy for arbitrary Lagrangian theories introduced by Iyer and Wald [34, 35]. This is referred to as the Noether charge approach, because the entropy is always expressed in terms of the integral of an \((n-2)\)-form charge \( Q \) over the event horizon. This approach thus extends the relation between entropy and geometry from general relativity to arbitrary metric theories of gravity. The expressions for the entropy we obtain from the Noether charge approach agree with those obtained in [26, 23, 30]. Since the Noether charge entropy satisfies the first law of thermodynamics by construction, this is unsurprising.
In fact, one can think of this Noether charge approach as a more formal version of the integration of the first law used in [23, 30]. However, by using this approach, we will gain some insight into the geometric significance of these expressions for the black hole entropy. Similar results were obtained by the conical singularity method in [36].

We will also address the appearance of negative entropies in these calculations. In [37, 38, 39], it was observed that the entropy assigned to some solutions could be negative. In [39], it was suggested that some of these solutions which are assigned negative entropy should be discarded as unphysical. We will argue instead that this occurrence of negative entropies reflects a deficiency in the methods used to calculate the entropy. As noted also in [38], there is an additive ambiguity in the definition of entropy in the approach used in [23, 30, 39]; one can add an arbitrary constant to the entropy of all the solutions in some family of solutions without affecting the first law. By changing the choice of this constant, one can clearly arrange to have all solutions have positive entropy. That is, we have to decide which classical solution we assign zero entropy; if we make this choice appropriately, all solutions have positive entropy.

We had initially hoped that using the Noether charge approach would resolve this ambiguity. However, since the Noether charge calculation precisely reproduced the previous calculation, this cannot be true. In fact, we will see that there is a precisely corresponding ambiguity in this approach: we can redefine $Q$ by adding to it a closed but not exact $(n-2)$-form. The existence of such a form is a consequence of the non-trivial topology of the black hole solutions. Thus, the negative entropies are again removed by an appropriate choice of zero entropy.

In the next section, we will review the black hole solutions we will consider, and describe the thermodynamic results obtained in [23, 30]. In section 3, we will briefly review the Noether charge approach to black hole entropy, and apply it to these solutions. In section 4, we point out that some solutions have been assigned negative entropy, and explain how we can correct this problem by changing our choice of zero entropy solution in both approaches.

## 2 Black hole solutions

Black hole solutions of a theory with higher-curvature corrections to the Lagrangian were first written down in [24], where the first three terms in the Lagrangian (4) were kept: that is, the usual action for Einstein gravity with a cosmological constant was modified by the addition of the Gauss-Bonnet term,

$$I = \frac{1}{16\pi G} \int d^n x \sqrt{-g} \left( R - 2\Lambda + \alpha (R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta} R^{\alpha\beta} + R^2) \right).$$

We will sometimes write the cosmological constant as $\Lambda = -(n - 1)(n - 2)/2l^2$, since the interest from the brane world point of view is mainly in the case where the cosmological constant is negative.

Attention was restricted to static solutions, so the ansatz for the metric is

$$ds^2 = -F^2(r)dt^2 + \frac{1}{G^2(r)}dr^2 + r^2 h_{\mu\nu}dx^\mu dx^\nu.$$
where $h_{\mu\nu}$ is a positive definite metric independent of $r, t$, and $\mu, \nu$ run over $n - 2$ values. In [24], spherical symmetry was considered, so $h_{\mu\nu}$ was taken to be the metric on an $S^{n-2}$. In [30], this was extended to consider $h_{\mu\nu}$ a metric of constant curvature $(n - 2)(n - 3)k$ where $k = 0, \pm 1$. We denote the volume of this space by $\Sigma_k$. The physically relevant solution of the equations of motion has

$$F^2(r) = G^2(r) = k + \frac{r^2}{2\tilde{\alpha}} \left( 1 - \sqrt{1 + \frac{64\pi G \tilde{\alpha} M}{(n - 2)\Sigma_k r^{n-1}} + \frac{8\tilde{\alpha} \Lambda}{(n - 1)(n - 2)} } \right), \quad (7)$$

where $\tilde{\alpha} = (n - 3)(n - 4)\alpha$. Since horizons occur where $F^2(r) = 0$, we see that black hole solutions with $k = 0$ or $k = -1$ are only possible when the cosmological constant is negative. For negative or zero cosmological constant, the largest root of $F^2(r) = 0$ defines the black hole event horizon $r = r_+$. For positive cosmological constant, we take $k = +1$, and the largest root is the cosmological horizon $r = r_c$, and the next root is the event horizon $r = r_+$. In [30], the mass and temperature of these black hole solutions was obtained in the usual way. The parameter $M$ in the above solution is the mass, which can be expressed in terms of the horizon radius $r_+$ by

$$M = \frac{(n - 2)\Sigma_k r^{n-3}_+}{16\pi G} \left( k + \frac{\tilde{\alpha}}{r^2_+} + \frac{r^2_+}{l^2} \right). \quad (8)$$

We will assume that $M \geq 0$. In [31, 32], it was argued that the energy for the $k = -1$ case differs from this mass by a background subtraction; this does not affect the analysis of the entropy. The temperature is obtained by determining the natural periodicity in the Euclidean time direction, giving

$$T = \frac{(n - 1)r^2_+ + (n - 3)kl^2r^2_+ + (n - 5)\tilde{\alpha}k^2l^2}{4\pi^2r^2_+(r^2_+ + 2\tilde{\alpha}k)}. \quad (9)$$

These expressions were then used to obtain an expression for the entropy by integrating up the first law. That is, we assume that the family of black hole solutions parametrised by $M$, or alternatively $r_+$, satisfies the first law of thermodynamics $dM = TdS$, and we obtain an expression for the entropy by integrating this relation over the family of solutions,

$$S = \int T^{-1}dM = \int_0^{r_+} T^{-1} \frac{\partial M}{\partial r_+} dr_+. \quad (10)$$

In the second step, the choice of lower limit of integration represents the additive ambiguity in the definition of the entropy referred to in the introduction. This specification of lower limit represents the reasonable assumption that, as in the usual formula for the entropy in Einstein gravity, we should take the entropy to go to zero when the area of the horizon vanishes. This gives a formula

$$S = \frac{r^{n-2}_+\Sigma_k}{4G} \left( 1 + \frac{2\tilde{\alpha}k(n - 2)}{(n - 4)r^2_+} \right). \quad (11)$$
for the entropy of these black hole solutions.

We wish to point out two features of this black hole entropy formula. First, even though the entropy was obtained purely by thermodynamic arguments, it has evident relations to the geometry of the event horizon: in the case $k = 0$, where the horizon is flat, the Gauss-Bonnet term has no effect on the expression for the entropy, which is simply the area of the event horizon. Furthermore, the correction term is controlled in general by the curvature of the event horizon. In the present approach, it is difficult to see why this should be the case. In the next section, we will use the Noether charge approach to obtain a more geometrical understanding of this observation. Secondly, we note that the entropy is not necessarily positive. As observed in [37, 38, 39], if $\alpha k < 0$, the second term in (11) is negative, and this term can make the whole expression negative for sufficiently small black holes. We will explore this observation in more detail in section 4.

The extension to consider black hole solutions when further corrections are turned on was explored in [23, 28, 29], where the full Lovelock Lagrangian (1) was considered for particular choices of the coefficients $\alpha_p$,

$$ \alpha_p = \frac{l^{2(p-q)}}{2(n-2p)(n-2)\Omega_{n-2} G_q} \binom{q}{p} $$

(12)

for $p \leq q$ and zero for $p > q$, where $q$ is an integer less than or equal to $(n-1)/2$. The static spherically symmetric black hole solutions of the resulting theories were found in [23]; these were extended to non-trivial horizon topology in [28, 29]. They are given by the same ansatz (6), with $h_{\mu\nu}$ a metric of constant curvature $k = 0, \pm 1$ and

$$ F^2(r) = G^2(r) = k + \frac{r^2}{l^2} - \left( \frac{2G_q M + \delta_{n-2q,1}}{r^{n-2q-1}} \right)^{1/q}. $$

(13)

The entropy of these black hole solutions was found by a similar argument to the one employed in the Gauss-Bonnet example; the result is

$$ S = \frac{2\pi q}{G_q} \int_0^{r_+} r^{n-2q-1}(k + \frac{r^2}{l^2})^{q-1} dr. $$

(14)

Here, we once again see that for $k = 0$, the correction to the usual entropy formula drops out; more generally, the correction is determined by the curvature of the horizon. In the next section, we will also reproduce this result from the Noether charge point of view.

3 Black hole entropy as a Noether charge

In general relativity, the black hole entropy is directly connected to the geometry: it is simply the area of the event horizon in appropriate units. This relation forms part

\footnote{The entropy for the more general Lovelock Lagrangian was studied in [40, 41]. For simplicity, we will restrict to the above choices.}
of a very strong relation between thermodynamics and geometry, particularly in the Euclidean approach to black hole thermodynamics, where the black hole entropy can be seen to arise from the non-trivial topology of the Euclidean black hole solutions. As remarked above, the black hole entropy in higher-curvature theories of gravity takes a more complicated form. It would clearly be very interesting to understand the geometrical origin of the expressions (11,14).

In [34, 35], a geometrical expression for the black hole entropy in an arbitrary diffeomorphism-invariant theory of gravity was obtained, and shown to satisfy the first law in general. This approach is based on a Noether current constructed from a diffeomorphism variation of the Lagrangian.

We consider an arbitrary Lagrangian density $L$ depending on a collection of dynamical fields $\phi = (g_{\mu\nu}, \psi)$, where $\psi$ represents matter degrees of freedom. The variation of the Lagrangian density under a general variation $\delta \phi$ of the dynamical fields was shown to be given by an equation of motion term plus a surface term,

$$\delta L = E \delta \phi + d\Theta(\phi, \delta \phi).$$

(15)

A Noether current was then defined by

$$J = \Theta(\phi, L_\xi \phi) - \xi \cdot L,$$

(16)

for $\xi^\mu$ some smooth vector field. That is, we take the surface term in the variation when the infinitesimal variation arises from a diffeomorphism, $\delta \phi = L_\xi \phi$, and subtract the vector field contracted with $L$. This current was shown to be a closed form on-shell, so we can define an $(n-2)$-form charge $Q$ by $J = dQ$.

This Noether charge $Q$ then plays the central role in defining the entropy. It can always be expressed as

$$Q = W_a(\phi)\xi^a + X_{ab}(\phi)\nabla_{[a}\xi_{b]} + Y(\phi, L_\xi \phi) + dZ(\phi, \xi),$$

(17)

where

$$X_{ab} = -\frac{\delta L}{\delta R_{ab}},$$

(18)

that is, $X_{ab}$ is equal to the variation of the Lagrangian with respect to the curvature, if we were to treat the curvature as an independent field. This expression for the Noether charge is clearly not unique; the form of $Z$ is completely arbitrary, and there are other ambiguities arising from ambiguities in the definitions of $L$ and $\Theta$.

It was then proposed in [34, 35] that we define the entropy of an asymptotically flat black hole solution with a bifurcate Killing horizon by

$$S = 2\pi \int_\Sigma Q[t] = 2\pi \int_\Sigma X^{cd} \epsilon_{cd},$$

(19)

where the integral is over the bifurcation $(n-2)$-surface in the black hole solution, and $t$ is the horizon Killing field, normalised to have unit surface gravity. In the

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\(^2\text{Here } \nabla_{[a}\xi_{b]} = e^a_c e^b_d \nabla_{[a}\xi_{b]}\).
second relation, we have used the fact that $t$ is a Killing vector which vanishes on $\Sigma$, and $\epsilon_{cd}$ is the binormal to the surface $\Sigma$. The main result of [34, 35] is that this entropy automatically satisfies the first law of thermodynamics,

$$\frac{\kappa}{2\pi} \delta S = \delta \mathcal{E} - \Omega_H^{(\mu)} \delta \mathcal{J}^{(\mu)},$$

(20)

where $\kappa$ is the surface gravity and $\mathcal{E}$ and $\mathcal{J}^{(\mu)}$ are energy and angular momenta defined in terms of surface integrals at infinity. This approach thus gives a satisfactory definition of the black hole entropy in terms of an integral over the black hole’s event horizon. An interesting attempt to relate this general entropy formula to a microscopic description appeared in [42, 43].

We would now like to apply this approach to the calculation of the entropy of the black hole solutions discussed in the previous section. For the Einstein+Gauss-Bonnet theory, the appropriate $Q$ was worked out in [35]:

$$Q_{\alpha_1...\alpha_{n-2}} = -\epsilon_{\gamma\delta\alpha_1...\alpha_{n-2}} \left( \frac{1}{16\pi} \nabla^\gamma \xi^\delta + 2\alpha (R \nabla^\gamma \xi^\delta + 4 \nabla^{[\beta} \xi^{\delta]\gamma} R^{\beta} + R^{\gamma\delta\alpha_1...\alpha_{n-2}} \nabla^\delta \xi^\alpha) \right).$$

(21)

Putting this into (19) and evaluating the integral over the bifurcation surface $r = r_+$ in the black hole geometry (6, 7) does reproduce the entropy (11) obtained previously. It is pleasing to see the two approaches agree—although hardly surprising, since (11) was calculated by assuming the entropy satisfies the first law, and (19) does so.

However, our hope was that performing the calculation in terms of the Noether charge would offer some insight into the evident relation between the entropy formula (11) and the geometry of the horizon. Although the Noether charge formulation allows us to express the entropy as a surface integral over the horizon, the origin of the $k$ dependence in (11) remains somewhat obscure. To obtain further insight into this question, we will redo the calculation in the language of differential forms.

To expose the geometric structure, we want to calculate the integral (19) over the bifurcation surface in a general metric of the form (6), without at first fixing the forms of $F(r)$, $G(r)$ or $h_{\mu\nu}$. We will consider the general Lagrangian (4) for the Lovelock theory. It is easy to see that

$$X_{ab} = - \sum_{p=1}^{[n/2]} \rho \alpha_p \epsilon_{aba_3...a_n} R^{a_3 a_4} ... R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} ... e^{a_n}. $$

(22)

For the metric (6), a suitable choice of orthonormal basis is

$$e^1 = F dt, \quad e^2 = \frac{1}{G} dr, \quad e^m = \tilde{e}^m, \quad (23)$$

where $\tilde{e}^m$, $m = 3, \ldots, n$ is a suitable orthonormal basis corresponding to the induced metric $r^2 h_{\mu\nu}$ on the $(n - 2)$ surface. The spin connection computed for this vielbein has

$$\omega^{mn} = \tilde{\omega}^{mn}, \quad \omega^{2m} = -\frac{G}{2r} \tilde{e}^m; \quad (24)$$

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and hence
\[ R^{mn} = \tilde{R}^{mn} - \left(\frac{G}{2r}\right)^2 \tilde{e}^m \tilde{e}^n. \] (25)

We will not need the other components of the curvature two-form. Hence
\[ X_{12} = - \sum_{p=1}^{\lfloor n/2 \rfloor} p \alpha_p \epsilon_{123\ldots a_n} R^{a_3 a_4} \ldots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \ldots e^{a_n} \] (26)
\[ = - \sum_{p=1}^{\lfloor n/2 \rfloor} p \alpha_p R^{m_1 m_2} \ldots R^{m_2 \ldots m_{2p-2}} e^{m_{2p-1}} \ldots e^{m_{n-2}} \] (27)
\[ = - \sum_{p=1}^{\lfloor n/2 \rfloor} p \alpha_p \left( R^{m_1 m_2} - \left(\frac{G}{2r}\right)^2 \tilde{e}^{m_1} \tilde{e}^{m_2} \right) \ldots \left( R^{m_{2p-3} m_{2p-2}} - \left(\frac{G}{2r}\right)^2 \tilde{e}^{m_{2p-3}} \tilde{e}^{m_{2p-2}} \right) \] \times \tilde{e}^{m_{2p-1}} \ldots \tilde{e}^{m_{n-2}} \] (28)
\[ = - \sum_{p=1}^{\lfloor n/2 \rfloor} p \alpha_p \sum_{j=0}^{p-1} (-1)^{p-1-j} \binom{p-1}{j} \left( \frac{G}{2r} \right)^{2(p-1-j)} \tilde{L}^{(j)}. \] (29)

We see that the term in \( X_{cd} \) that is relevant to the evaluation of the entropy is a certain combination of the Euler densities evaluated on the \((n-2)\)-dimensional submanifold. This is the key to understanding the very suggestive form of (11).

In the particular case of the Gauss-Bonnet theory (5), we can use the fact that \( \alpha_p = 0 \) for \( p > 2 \) and that \( G(r) \) vanishes on the event horizon in the solution (7) to write
\[ S = 2\pi \int_\Sigma X^{cd} \epsilon_{cd} = 4\pi \int_\Sigma X^{12} = 4\pi \int_\Sigma (\alpha_1 \tilde{L}^{(0)} + 2\alpha_2 \tilde{L}^{(1)}). \] (30)

That is, the entropy contains a term proportional to the area of the event horizon (since \( \tilde{L}^{(0)} \) is just the horizon volume form) and a term proportional to the Ricci scalar of the induced metric on the horizon (since \( \tilde{L}^{(1)} \) is the Einstein-Hilbert Lagrangian). Explicitly, comparing (5) to the general form (4) gives
\[ \alpha_0 = \frac{(n-1)(n-2)}{16\pi G n! l^2}, \quad \alpha_1 = \frac{1}{16\pi G (n-2)!}, \quad \alpha_2 = \frac{\alpha}{16\pi G (n-4)!}, \] (31)
and so
\[ S = \frac{1}{4G} \int_\Sigma d^{n-2}x \sqrt{\tilde{h}} \left( 1 + 2\alpha \tilde{R} \right), \] (32)
where \( \tilde{R} \) is the Ricci scalar of the metric \( \tilde{h}_{\mu\nu} = r^2 h_{\mu\nu} \) on the bifurcation surface \( \Sigma \).

Taking the metric \( h_{\mu\nu} \) to be a constant-curvature metric, this will reproduce the formula (11) for the entropy. We see that the fact that the correction is proportional to \( k \) arises from the more general statement that the correction to the usual entropy

\[^3\text{This result is also interestingly reminiscent of the study of codimension two braneworlds in [15], where it was found that a Gauss-Bonnet action in the bulk induced Einstein gravity on the brane. We do not yet fully understand the relation between the two results.}\]
The formula is simply proportional to the integral of the scalar curvature of the horizon for black hole solutions of this form. This general connection to horizon curvature was also obtained by the conical singularity method in [36].

We can easily extend this result to the case considered in [23, 28, 29], where we take the $\alpha_p$ to be given by (12). In this case, we again have that $G(r)$ vanishes on the event horizon, so we can simplify the general formula for the entropy to

$$S = 2\pi \int_\Sigma \sum_{p=1}^q p \alpha_p L^{(p-1)}.$$ 

(33)

If we again take the metric $h_{\mu\nu}$ to have constant curvature, the curvature of the horizon will be given by

$$\tilde{R}^{mn}|_\Sigma = \frac{k}{r_+^2} e^m e^n,$$ 

(34)

where $k = 0, \pm 1$, and we obtain the entropy

$$S = 2\pi \sum_{p=1}^q p \alpha_p \left( \frac{k}{r_+^2} \right)^{p-1} \int_\Sigma \varepsilon_h$$ 

(35)

$$= \frac{2\pi q}{G_q} \sum_{p=1}^q \frac{q!(p-1)!\varepsilon_p - 2p}{(n - 2p)!n^{2q}}$$ 

(36)

$$= \frac{2\pi q}{G_q} \sum_{p=0}^{q-1} \left( \frac{q - 1}{p} \right) \frac{k^p r_+^{n-2p-2}}{(n - 2p - 2)!2q - 2p - 2}$$ 

(37)

$$= \frac{2\pi q}{G_q} \int_{r_+}^{r_+ + 0} r^{n-2q-1} \left( k + \frac{r^2}{l^2} \right)^{q-1} dr$$ 

(38)

reproducing the expression (14). Thus, we see that from the Noether charge point of view, the dependence of the entropy on the curvature of the horizon, which appeared somewhat obscure in previous calculations, arises very naturally.

## 4 Negative entropy

We would now like to explore the problem of negative entropy. As observed in [37, 38, 39], if we consider the entropy we have assigned to the Gauss-Bonnet black holes (11), we see that the entropy is negative if

$$\frac{2\tilde{\alpha}k(n - 2)}{(n - 4)r_+^2} < -1.$$ 

(39)

This condition can be satisfied for sufficiently small black holes in either of two ways: if $\tilde{\alpha} < 0$, $k = +1$, or if $\tilde{\alpha} > 0$, $k = -1$, which latter is possible only for negative cosmological constant.

Now in both these cases, there is a lower bound on the size of the black hole—physical black hole solutions do not exist for all values of $r_+$—so it is still a non-trivial question whether there actually exist any solutions with negative entropy.
In the case where $\tilde{\alpha} < 0, k = 1$, it was observed in [30, 39] that there was a minimum horizon radius, $r_+^2 > -2\tilde{\alpha}$. For any value of $\alpha$, this still leaves a finite range of values where the black hole will have negative entropy:

$$-2\tilde{\alpha} < r_+^2 < -2\tilde{\alpha}\frac{(n-2)}{(n-4)}.\quad (40)$$

In [39], it was suggested that these solutions be discarded. However, they have no obvious pathologies as classical solutions. Therefore, we cannot justify discarding them; the fact that they appear to have negative entropy should be interpreted as a problem with our prescription for calculating the entropy.

In the case where $\tilde{\alpha} > 0, k = -1$, which requires $\Lambda < 0$, there is also a minimum size black hole: if we consider the solution with $M = 0$, we can see from (7) that there is still a horizon at

$$r_{+\text{min}}^2 = \frac{l^2}{2}\left(1 + \sqrt{1 - \frac{4\tilde{\alpha}}{l^2}}\right).\quad (41)$$

Now this is a vacuum solution, and this horizon is just an acceleration horizon, corresponding to the fact that the coordinates we have used do not cover the whole of AdS$_n$. Nonetheless, when we consider $M > 0$, the event horizons of our black hole solutions will always have $r_+ > r_{+\text{min}}$. The question of whether there are any physical negative entropy solutions is now a little more non-trivial. The entropy of the solution with horizon radius $r_{+\text{min}}$ vanishes when

$$\tilde{\alpha} = l^2\frac{n(n-4)}{4(n-2)^2}.\quad (42)$$

Thus, for values of $\tilde{\alpha}$ in

$$\frac{n(n-4)}{(n-2)^2}\frac{l^2}{4} < \tilde{\alpha} < \frac{l^2}{4},\quad (43)$$

there are negative-entropy black holes with

$$r_{+\text{min}}^2 < r_+^2 < 2\tilde{\alpha}\frac{(n-2)}{(n-4)}.\quad (44)$$

The upper bound on the range of $\tilde{\alpha}$ is the maximum value for which this theory has a well-defined vacuum solution [30].

Since there is a minimum value of the size of the event horizon in the cases where we are encountering negative entropies, there is an obvious solution to this problem; we can simply change the lower limit of integration in (10), defining the entropy to be

$$S = \int_{r_{+\text{min}}}^{r_+} T^{-1} \left(\frac{\partial M}{\partial r_+}\right) dr_+,\quad (45)$$

where $r_{+\text{min}}$ is the minimum value that the horizon radius takes in a given family of solutions.\(^4\) This will add an overall constant to the entropy assigned to all the black

\(^4\)This additive ambiguity in the entropy was also noted in [38].
hole solutions in a particular family of solutions. This will reduce to the previous expression (10) for the usual cases, and will by construction assign a positive entropy to all the solutions, since the integrand is positive, so $S$ is an increasing function of $r_+$. This assigns zero entropy to the solution at $r_+ = r_{+\text{min}}$, which has a non-zero horizon area, which may seem unnatural. However, we saw in the particular example discussed above that this horizon was just an acceleration horizon in an AdS$_n$ solution, representing the failure of our coordinates to cover the whole solution, so this might be a reasonable assignment. It is certainly more reasonable than ascribing a negative entropy to some of the solutions.

Thus, the problem of negative entropies can easily be resolved within this previous framework. However, the main point of our paper was to re-calculate these entropies from the Noether charge point of view; in doing so, we have successfully reproduced the sometimes negative answers of [30, 28, 23]. But in this approach, we have a purely geometrical expression for the entropy of a particular solution, not an integral over the family of solutions. So how can we fix this problem in the Noether charge approach?

In fact, there is a precisely corresponding ambiguity in this approach. The Noether charge $Q$ was defined by $J = dQ$, so it is only defined up to the addition of a closed form. Previously, the ambiguity has been viewed as one of adding an exact form $dZ$, which clearly cannot affect the calculation of the entropy. However, for our black hole metric (6), there is a natural closed but not exact $(n-2)$-form: the volume form $\epsilon_h$ associated with the metric $h_{\mu\nu}$. Note that we must take the $r$-independent metric $h_{\mu\nu}$ rather than the induced metric $r^2 h_{\mu\nu}$ for this to be a closed form. The integral of this form over the bifurcation two-surface $\Sigma$ is clearly non-zero. Hence, if we redefine $Q$ by

$$Q' = Q - \frac{S_{\text{min}}}{\Sigma_k} \epsilon_h,$$

it will change the entropy:

$$S' = S - S_{\text{min}}.$$  (47)

The coefficient $S_{\text{min}}$ must be a constant, independent of the parameters in a particular solution (but possibly depending on the parameters $\alpha_p$ in the Lagrangian), so this represents exactly the same ambiguity—the freedom to shift the entropy of all the black hole solutions in a particular family of solutions by an overall constant. The entropy $S'$ will still satisfy the first law, as $S$ does and $\delta S_{\text{min}} = 0$.

Why does this ambiguity not appear in discussions of the entropy for more familiar cases? Here we should note that $\epsilon_h$ is a well-defined form on the black hole solutions only because the surface $r = 0$, where it might become ill-defined, is not part of the spacetime manifold. Thus, if we consider a family of solutions which includes a true vacuum solution, where $r = 0$ is a regular point, we lose the freedom to add such a term. So from this point of view, we could argue that the Noether charge approach does have one additional piece of information: it knows we should take $S_{\text{min}} = 0$ when our family of solutions has $r_{+\text{min}} = 0$. But in cases where $r_{+\text{min}} \neq 0$, there is a real ambiguity, and we have to choose $S_{\text{min}}$.

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References


