Non-extensive statistics and the stellar polytrope index

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We use physical constraints imposed from the H-Theorem and from the negative nature of the heat capacity of self-gravitating thermodynamically isolated systems to investigate some possible limits on the stellar polytrope index \( n \) within the domain of a classical non-extensive kinetic theory.

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Polytropic distribution functions provide the simplest models for self-gravitating stellar systems \(^1\). These models are characterized by an equation of state of the form \( p = K \rho^n \), where \( K \) and \( n \) are constants. The polytropic index \( n \) is defined by the following expression:

\[
\gamma = 1 + \frac{1}{n}
\]

which in the limit \( n \to \infty \) or, equivalently, \( \gamma = 1 \) results in an equation of state identical to the one of an isothermal body of gases, i.e., \( p = K \rho \). The distribution function associated to these systems is given by

\[
f(q) = \frac{\rho_1}{(2\pi \sigma^2)^{\frac{n}{2}}} e^{\frac{\rho_1}{\sigma^2}}
\]  

(1)

where \( \rho_1 \) and \( \sigma = (K_0 T/m)^{1/2} \) are, respectively, the density and the velocity dispersion. The quantity \( \epsilon \) is equal to a constant minus the total energy per unit mass of an individual star moving in the global galactic gravitational potential. In fact, \( \epsilon \) is the relative energy of a star defined by \( \epsilon = \Psi - \frac{v^2}{2} \), where \( \Psi \) is the relative potential, given by \( \Psi = \phi_0 - \phi \), \( \phi \) is the star potential energy and \( \phi_0 \) is an arbitrary constant. In particular, the integrability condition \( \int f d^3 \nu < \infty \) results in the constraint \( n > 1/2 \) (for a complete study on the gaseous polytropes and isothermal gas spheres, see \(^2\)).

On the other hand, a non-extensive statistical formalism has been proposed as a possible extension of the classical one, which intends to study more properly systems that possess long-range interactions. Such a formalism is based on the entropy formula \(^3\)

\[
S_q = -k_B \sum_i p_i^q \ln_q p_i,
\]  

(2)

where \( p_i \) is the probability of the \( i \)th microstate and \( q \) is the non-extensive parameter. The Boltzmann-Gibbs extensive formula, \( S_1 = -k_B \sum_i p_i \ln p_i \), is readily recovered in the extensive limit \( q = 1 \) and the \( q \)-logarithmic function above is defined as

\[
\ln_q f = \frac{f^{1-q} - 1}{1-q}, \quad f > 0.
\]  

(3)

Some attempts to build a kinetic counterpart for this non-extensive statistical formalism has been recently proposed in Refs. \(^4\) \(^5\). The main result of these works consists in showing that this extended formulation leads to a new velocity distribution for free particles given by

\[
f_0(v) = B_q \left[ 1 - (1-q) \frac{mv^2}{2k_B T} \right]^{1/q},
\]  

(4)

where the quantity \( B_q \) is a \( q \)-dependent normalization constant whose expressions for \( q \leq 1 \) and \( q \geq 1 \) are respectively given by

\[
B_{q \leq 1} = \mathcal{G} \left[ \frac{1/q + 1/2}{(1/q - 2)} \right] \left( \frac{m}{2 \pi k_B T} \right)^{3/2}
\]  

(5a)

and

\[
B_{q \geq 1} = \mathcal{H} \left[ \frac{1/q + 1/2}{(1/q - 2)} \right] \left( \frac{m}{2 \pi k_B T} \right)^{3/2}
\]  

(5b)

where the coefficients \( \mathcal{G} \) and \( \mathcal{H} \) are given by \( \mathcal{G} = n_p (1 - q)^{1/2} \left[ (2mq^2) \frac{(2\pi k_B T)^q}{(2\pi k_B T)^q} \right] \) and \( \mathcal{H} = n_p (q - 1)^{3/2} \), \( T \) is the temperature and \( n_p \) is the particle number density. This \( q \)-distribution can be derived at least from two different methods, namely, a generalization of the Maxwell ansatz, \( f(v) \neq f(v_x)f(v_y)f(v_z) \), which follows from the introduction of statistical correlations between the components of the velocities \(^6\) \(^7\) and a new formulation for the Boltzmann H-theorem, which requires \( q > 0 \) \(^8\).

In the astrophysical domain, both the non-extensive statistical formalism and its kinetic counterpart have been applied in a wide range of problems. The very first application of this non-extensive statistics was done in connection with stellar polytropes \(^9\), with several authors suggesting different expressions for the polytropic index \( n \) as a function of the non-extensive parameter \( q \) \(^9\) \(^8\) \(^10\) \(^11\). Recently, a numerical study of stellar dynamical evolution for self-gravitating systems was performed by Taruya and Sakagami \(^12\) while Chavanis \(^7\) considered the Tsallis’ entropy as a particular \( H \)-function corresponding to isothermal stellar systems and stellar polytropes. In this latter approach, the maximization of the \( H \)-function at fixed mass and energy reveals a

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thermodynamical analogy with the study of the dynamical stability in collisionless stellar systems. The non-extensive kinetic approach has been used to study the Jeans’ gravitational instability.

Another recent application of this non-extensive kinetic theory was to investigate the negative nature of the heat capacity for isolated self-gravitating system. This study was performed by considering the following steps: first, consider a simple analogy between self-gravitating system and ideal gas (IG) or, in other words, the equivalence between IG kinetical and internal energies, given by \( \frac{1}{2} m < v^2 > = \frac{3}{2} k_B T \), where \( m \) is the mass of a particle (e.g., a star). If such a system is composed by \( N \) particles, its total kinetic energy is \( K = \frac{3}{2} N k_B T \) which, according to the virial theorem, is equal to minus total energy, i.e., \( E = -K \). Therefore, the heat capacity of the system is given by (see [1] for details)

\[
C_V = \frac{dE}{dT} = \frac{3}{2} N k_B. \tag{6}
\]

Second, in order to investigate the \( q \)-dependence of the heat capacity, we consider a cloud of ideal gas within the non-relativistic gravitational context, which is analog to a self-gravitating collisionless gas. The kinetic energy of this system is simply given by \( K = \frac{1}{2} m < v^2 > \), with the statistical content included in the average square velocity of the particles \( < v^2 > \). Indeed, the non-extensivity may be introduced through a new derivation of expectation value \([21]\)

\[
<v^2> = \frac{\int_{-v_m}^{v_m} f(q) v^2 d^3v}{\int_{-v_m}^{v_m} f(q) d^3v}, \tag{7}
\]

where \( v_m = \left( \frac{2 k_B T}{m(1-q)} \right)^{1/2} \) is a thermal cutoff on the maximum value allowed for the velocity of the particles \( (q < 1) \), whereas for the power law without cutoff \( (q > 1) \) \( v_m \rightarrow \infty \). This \( q \)-expectation value can be easily evaluated for \( q \neq 1 \), resulting in

\[
<v^2> = \frac{6}{5-3q} \frac{k_B T}{m}, \text{ for } q < 5/3. \tag{8}
\]

Now, combining the \( q \)-expectation value for the square velocity of the particles with the definition of the heat capacity \( C_V \) one finds

\[
C_V = -\frac{3}{5-3q} N k_B, \tag{9}
\]

which clearly places an upper limit to the non-extensive parameter, i.e., \( q < 5/3 \). Following this reasoning, it is therefore natural that our kinetic approach uses the free particle distribution, with the long range nature of gravity being introduced via the virial theorem (see [10] for details). We note that, in the domain of ensemble theory, there are some analyses in the literature dealing with the classical IG within this nonextensive scenario \([17, 18, 19, 20]\). In particular, Ref. [18] showed that the IG naturally exhibits negative heat capacity and that its pressure and particle number density obey a polytropetype relation. Here, however, we assume that all IG properties are accounted for by the \( q \)-Maxwellian (see [8, 10] for details).

In this Letter, by considering the different expressions relating the polytropic index \( n \) and the non-extensive parameter \( q \) we discuss some possible physical constraints on \( n \) which arise from the second law of thermodynamics \( (q > 0) \) and the negativeness of the heat capacity \( (q < 5/3) \). We show that the existent expressions between \( n \) and \( q \) (see [8, 9, 10]) can be naturally linked with the heat capacity imposing limits on the polytropic index. It is worth mentioning that the constraint \( q > 0 \), arising from the generalized \( H \)-theorem, does not provide a strong case for limiting the polytropic index \( n \). This corresponds to a weak condition of the generalized \( H \)-theorem since it is possible to show that there are two arrow of time for \( q > 0 \) and \( q < 0 \).

In order to investigate such constraints we first present the different expressions for the polytropic index \( n \) as a function of the non-extensive parameter \( q \) discussed in FIG. 1: The quantity \( C_V/Nk_B \) for a self-gravitating collisionless gas obeying the non-extensive Tsallis’ \( q \)-statistic is shown as a function of the polytropic index \( n \). The polytropic index and non-extensive parameter \( q \) are related by Eq. (10). As indicated, the shadowed regions stand for the constraints from the second law of thermodynamics and the negative nature of \( C_V \).
the polytropic index and non-extensive parameter \( q \) are related by Eq. (13).

The case (i) arises from an extension of Padmanabhan’s classical analysis of the Antonov instability for the case of polytropic distribution function \( \text{[22]} \). By using kinetic arguments of the Tsallis’ non-extensive formalism, the case (ii) is obtained from an extension of the polytropic Lane-Emden spheres as well as from a study of maximum entropy solutions of the Vlasov-Poisson equations describing self-gravitating systems. We emphasize that the polytropic family of solutions to the collisionless Boltzmann equation (parameterized by the index \( n \)) is always the same, which means that expressions (10) and (11) constitute only different ways of parameterizing (in terms of non-extensive parameter \( q \)) this polytropic family. In both cases, Eqs. (10) and (11), the same choice of the statistical average, the so-called \( q \)-expectation value, has been used. As expected, for all the above expressions the Maxwellian isothermal spheres are obtained in the limit \( q = 1 \) or, equivalently, \( n = \infty \).

By combining Eqs. (10) and (11) with Eq. (9), one obtains the following expressions relating the heat capacity and the polytropic index:

\[
\text{(i) } C_V = \frac{(3 - 6n)}{4n + 4} Nk_B, \tag{12}
\]

and

\[
\text{(ii) } C_V = \frac{(9 - 6n)}{4n} Nk_B. \tag{13}
\]

The case (i) results in the bound on \( q \) that by imposing the constraints on the non-extensive parameter \( q \) from the second law of thermodynamics \( q > 0 \) and the negative nature of \( C_V \) \( (q < 5/3) \) it is possible to limit regions in the \( C_V - n \) plane. To better visualize such constraints, in Figs. 1 and 2 we show, respectively for the cases (i) and (ii), the dimensionless quantity \( C_V/Nk_B \) as a function of the polytropic index \( n \).

In Figures 1 [case (i)] and 2 [case (ii)] we show that the constraint \( q > 0 \) results, respectively, in the forbidden intervals for the polytropic index \( \frac{1}{3} < n < \frac{2}{3} \) and \( \frac{1}{3} < n < \frac{3}{5} \) while the limit resulting from Eq. (9), i.e., \( q < \frac{5}{3} \) implies \( 0 < n < \frac{1}{3} \) and \( 0 < n < \frac{3}{5} \). In all cases the bound on \( q \) from the second law of thermodynamics \( (q > 0) \) restricts only negative regions in the plane \( C_V/Nk_B - n \) while, as expected, the thermodynamical limit \( C_V < 0 \) forbids any possible positive region in the plots. In particular, we also note that the constraints on \( n \) from Eq. (12) (case i) seem to be in agreement with the theorem proved by Antonov and Dremus \( \text{[1]} \) who showed that the equilibrium distribution is dynamically stable for values of the polytropic index \( n > 3/2 \) or, equivalently, \( q > 0 \) (see Fig. 1). The bounds on the polytropic index \( n \) obtained in this Letter are summarized in Table I.

Finally, we emphasize that in a more quantitative case, i.e., in which a self-gravitating ideal gas is contained in a spherical container of radius \( R \) our constraints on the polytropic index \( n \) must be valid only in the negative region of the heat capacity. As widely known, when the potential energy is taking into account or, equivalently, the virial theorem is modified to introduce a pressure term, \( 2K + W \approx p(R) \), the heat capacity can assume both positive and negative values. We also note that a possible connection with stability of spherical systems can be introduced through the Doremus-Feix-Baumann theorem which implies, for the extensive case, that polytropes with \( \frac{1}{2} < n < \frac{3}{2} \) are unrealistic (see, however, \( \text{[23, 24]} \)).
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[4] See also http://tsallis.cat.cbpf.br/biblio.htm for an up to date bibliography.