Dirty black holes: Spacetime geometry and near-horizon symmetries

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Abstract

We consider the spacetime geometry of a static but otherwise generic black hole (that is, the horizon geometry and topology are not necessarily spherically symmetric). It is demonstrated, by purely geometrical techniques, that the curvature tensors, and the Einstein tensor in particular, exhibit a very high degree of symmetry as the horizon is approached. Consequently, the stress-energy tensor will be highly constrained near any static Killing horizon. More specifically, it is shown that — at the horizon — the stress-energy tensor block-diagonalizes into “transverse” and “parallel” blocks, the transverse components of this tensor are proportional to the transverse metric, and these properties remain invariant under static conformal deformations. Moreover, we speculate that this geometric symmetry underlies Carlip’s notion of an asymptotic near-horizon conformal symmetry controlling the entropy of a black hole.

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1 Introduction

Roughly 30 years has elapsed since the initial proposal that black holes behave as thermodynamic systems; nonetheless, Bekenstein’s black hole entropy $[^1][^2]$, 

$$S_{BH} = \frac{1}{4} \text{[Horizon area in Planck units]},$$

is still one of the most puzzling (yet intriguing!) concepts in theoretical gravity $[^3]$. It will be a challenge to any prospective fundamental theory to provide an unambiguous and universal explanation for this entropy at the level of state counting $[^4]$. In spite of the relative success in this regard of, for instance, string theory $[^5]$ and loop quantum gravity $[^6]$, it is safe to say that the microscopic origin of $S_{BH}$ remains a decidedly open question.

It seems likely that the microstates underlying black hole entropy will only be fully understood at the level of quantum gravity. On the other hand, there is a growing suspicion that these states (whatever they may be) are actually controlled by a classically inherited symmetry $[^7]$. This notion can, in large part, be attributed to Strominger’s realization $[^8]$ that the black hole entropy in three spacetime dimensions $[^9]$ can be calculated by way of Cardy’s formula $[^10]$; notably, a formula which counts the density of states for a two-dimensional conformal field theory. Significant to this calculation was an earlier observation made by Brown and Henneaux $[^11]$: the diffeomorphism invariance of three-dimensional (anti-de Sitter) gravity can be manifested as a two-dimensional conformal field theory that, in some sense, “lives” at the boundary of the spacetime.

An obvious limitation of Strominger’s work is that it directly applies to only a three-dimensional theory of gravity. Nonetheless, progress in four (as well as an arbitrary number of) dimensions of spacetime has since been made by Carlip $[^12]$ $[^13]$ $[^14]$ and Solodukhin $[^15]$. In these studies, the conformal field theory is now regarded as living at the black hole horizon. Although the horizon is not a true physical boundary (for instance, a free-falling observer is not even aware of its existence), it is indeed a place where boundary conditions can and should be set $[^7]$. Moreover, when one considers issues of locality, the horizon is naturally preferred over asymptotic infinity as the boundary that harbors the relevant degrees of freedom. Unfortunately, the precise choice of conditions — which, at this point, is somewhat ambiguous — will greatly influence any such determination of the entropy. Hence, substantial progress in this program will likely require a better understand-
ing of precisely what classical symmetry is at play. Most recently, Carlip has suggested that the Einstein–Hilbert action acquires a “new” asymptotic conformal symmetry in the neighborhood of the horizon [14]. Although this seems intuitively correct and is supported by the elegance of his calculation, this symmetry still lacks the type of fundamental (classical) explanation that was alluded to above.

Perhaps, the key to understanding Carlip’s notion of a conformal symmetry can be found in the Einstein field equations rather than the action per se. To motivate this perspective, let us take note of the following observation: any static (black hole) Killing horizon with the proper property of spherical symmetry is known to possess the following symmetry as the horizon is approached [16, 17]:

\[ G^{tt} - G^{rr} \to 0 , \]

where \( G_{\mu\nu} \) represents the Einstein (curvature) tensor, while \( t \) and \( r \) are the usual Schwarzschild (or, in the case of “dirty” black holes, \(^1\) Schwarzschild-like) temporal and radial coordinates. After imposing Einstein’s field equations, one then immediately obtains, at the horizon,

\[ T^{tt} - T^{rr} = 0 , \]

where \( T_{\mu\nu} \) is the stress-energy tensor. Hence, restricting attention to the \( r-t \)-plane (where all of the “interesting” near-horizon physics should presumably take place [7]), we have \( T_\perp \propto g_\perp \).

The above observation is certainly interesting but, alas, the condition of spherical symmetry seems quite restrictive. [The condition of staticity is also restrictive. However, for a slowly evolving black hole, as long as the evolution rate is small compared with the surface gravity, it could always be argued that the spacetime is approximately static — or, at the very least, approximately stationary — in the neighborhood of the horizon.] The main purpose of the current paper is to rectify this situation by establishing a suitable analogue to equation (2) [and, hence, equation (3)] for a static but otherwise generic Killing horizon.

\(^1\)In the current context, a dirty black hole is meant to imply a generic static and spherically symmetric spacetime for which a central black hole is surrounded by arbitrary matter fields [16, 18, 19]. In the upcoming analysis, we specifically lift the condition of spherical symmetry. For example, a ring of material might be placed around the equator of what would otherwise be a Schwarzschild black hole, distorting its horizon into an ovaloid shape.
With the above discussion in mind, our aim is to develop a general analysis for studying the geometric structure of a generic static Killing horizon. We begin, in Section 2, by using the natural time coordinate in a static [(3+1)-dimensional] spacetime to slice this geometry into space plus time. Next, we twice employ the Gauss–Codazzi and Gauss–Weingarten equations; allowing us to decompose the spacetime Einstein tensor in terms of the geometrical properties of a spatial 2-surface embedded in a constant-time slice. In Section 3, we then assume the existence of a Killing horizon and consider the near-horizon limit of the generic formalism. By way of purely geometrical arguments, we are able to formulate a clear description of the near-horizon geometry. The Einstein tensor turns out to indeed have a high degree of symmetry (at the horizon), and we proceed to elaborate upon the implications of this outcome. Extremal horizons are discussed in Section 4, while the effect of conformal transformations is discussed in Section 5. Finally, Section 6 contains further discussion and a comment on future directions of study.

We mention, in passing, that our analysis also enables an independent and physically transparent verification of the “zeroth law of black hole thermodynamics” (that is, the constancy of the surface gravity \(^2\)) for static Killing horizons. The main argument is presented in Section 3, with some supporting calculations in an appendix.

\section{Generic static spacetimes}

\subsection{The (3+1)-geometry}

We will begin by considering the geometry of a (3+1)-dimensional spacetime which is constrained to be static but is otherwise completely generic. (Conditions as appropriate for the existence of a black hole Killing horizon will be imposed later on.) Note that the methodology of this section is based largely on the techniques of \cite{21, 22, 23, 24, 25}.

Given any static spacetime, one can always decompose the metric into a block-diagonal form as follows \cite{26, 27, 20}:

\begin{align}
\text{ds}^2 &= g_{\mu\nu} \, dx^\mu dx^\nu \\
&= -N^2 \, dt^2 + g_{ij} \, dx^i dx^j .
\end{align}

\footnote{See, for instance, Wald \cite{20}.}
Notation: Greek indices run from 0–3 and refer to the complete spacetime; whereas Latin indices in the middle of the alphabet (i, j, k, ...) run from 1–3 and refer to the spacelike coordinates. Also, Latin indices at the beginning of the alphabet (a, b, c, ...) will run from 1–2 and refer to directions parallel to a soon-to-be-defined arbitrary spacelike 2-surface. Furthermore (for an arbitrary geometrical object $X$), we will use $X_\alpha$ or $\nabla_\alpha X$ to denote a spacetime covariant derivative, $X_i$ to denote a three-space covariant derivative, and $X_\alpha$ to denote a two-space covariant derivative (which is always taken on the aforementioned 2-surface). Finally, capitalized Latin indices at the beginning of the alphabet ($A$, $B$, $C$, ...) will run from 0–1 and refer to the two directions perpendicular to the arbitrary spacelike 2-surface; essentially, the $t$–$n$-plane defined by the time direction and the normal direction.

Let us next consider the three-geometry of space on a constant-time slice. As it so happens, the property of staticity tightly constrains the manner in which this three-geometry can be embedded into the spacetime. For instance, by way of a standard textbook [26, page 518], we have the following results:

\[(3+1) R_{ijkl} = (3) R_{ijkl}, \]
\[(3+1) R_{iijk} = 0, \]
\[(3+1) R_{titj} = \frac{N_{ij}}{N}. \]

Here (and throughout), the “hat” on an index indicates that we are looking at appropriately normalized components; for instance,

\[X_i = X_i \sqrt{-g^{tt}} = \frac{X_i}{\sqrt{-g^{tt}}} = \frac{X_i}{N}. \]

One can interpret this choice of normalization as using the orthonormal basis attached to the fiducial observers (FIDOS).

Now decomposing the spacetime metric in terms of the spatial 3-metric, the set of vectors $e^\mu_i$ tangent to the time slice [the triad or drei-bein], and the vector $V^\mu = \frac{1}{N} (\partial/\partial t)^\mu$ normal to the time slice, we have

\[(3+1) g^{\mu\nu} = e^{\mu}_i e^{\nu}_j g^{ij} - V^\mu V^\nu, \]

which can be used to readily deduce the following contractions:

\[(3+1) R_{ij} = (3) R_{ij} - \frac{N_{ij}}{N}, \]
\[ (3+1) R_{ti} = 0 , \]  
\[ (3+1) R_{\hat{t}i} = \frac{g^{ij} N_{ij}}{N} = \frac{(3) \Delta N}{N} , \]  
\[ (3+1) R = (3) R - 2 \frac{(3) \Delta N}{N} . \]  
These results enable us to calculate the various components of the Einstein tensor (cf. [26, page 552]):

\[ (3+1) G_{ij} = (3) G_{ij} - \frac{N_{ij}}{N} + g_{ij} \left\{ \frac{(3) \Delta N}{N} \right\} , \]  
\[ (3+1) G_{\hat{t}i} = 0 , \]  
\[ (3+1) G_{\hat{t}\hat{t}} = + \frac{1}{2} (3) R . \]  
It should be re-emphasized that this decomposition is generic to any static spacetime.

### 2.2 The 3-geometry

Let us now (arbitrarily) choose a particular 2-surface in the constant-time slice and utilize Gaussian normal coordinates in the surrounding region; that is,

\[ g_{ij} \, dx^i \, dx^j = dn^2 + g_{ab} \, dx^a \, dx^b , \]  
where \( n = \hat{n} \) represents the spatial direction normal to the specified 2-surface. It can then be shown — see for example [26, page 514, equations (21.75-21.76) and page 516, equation (21.82)] — that

\[ (3) R_{abcd} = (2) R_{abcd} - (K_{ac} K_{bd} - K_{ad} K_{bc}) , \]  
\[ (3) R_{\hat{a}\hat{b}c} = -(K_{\hat{a}bc} - K_{\hat{a}cb}) , \]  
\[ (3) R_{\hat{a}\hat{a}\hat{b}b} = \frac{\partial K_{ab}}{\partial n} + (K^2)_{ab} , \]  
where the \textit{extrinsic} curvature, \( K_{ab} \), is given in Gaussian normal coordinates by \(^3\)

\[ K_{ab} = - \frac{1}{2} \frac{\partial g_{ab}}{\partial n} . \]  
\(^3\)Note that we are using Misner–Thorne–Wheeler sign conventions. In particular, see page 552 of [26].
It should be noted that the above curvature expressions are all independent of the spacetime dimensionality. Nevertheless, two transverse dimensions is somewhat special as, in this case, equation (19) reduces to

$$^{(3)} R_{abcd} = \frac{1}{2} R_{\parallel} (g_{ac} g_{bd} - g_{ad} g_{bc}) - (K_{ac} K_{bd} - K_{ad} K_{bc}) .$$

(23)

Here, we are using $R_{\parallel}$ to denote the Ricci scalar of the two-dimensional surfaces of constant $n$ and $t$; that is, the Ricci scalar of the 2-surfaces parallel to the arbitrarily chosen 2-surface. Strictly speaking, these results have validity only on the specified 2-surface and in some limited region surrounding the 2-surface; holding only as long as the Gaussian normal coordinate system does not break down. (Such a breakdown tends to occur because the normal geodesics typically intersect after a certain distance.) This is, however, not a significant restriction on the subsequent analysis.

Let us next decompose the spatial 3-metric in terms of the specified 2-metric, the set of vectors $e^i_a$ tangent to the 2-surface [the zwei-bein], and the vector $n^i$ normal to the 2-surface. That is,

$$^{(2+1)} g^{ij} = e^i_a e^j_b g_{ab} + n^i n^j .$$

(24)

Also taking note of the useful identity

$$\text{tr} \left( \frac{\partial K}{\partial n} \right) = \frac{\partial \text{tr}(K)}{\partial n} - 2\text{tr}(K^2) ,$$

(25)

where the trace $\text{tr}$ is performed using the 2-metric $g_{ab}$ and its inverse $g^{ab}$, we are able to effect the following contractions:

$$^{(3)} R_{ab} = \frac{1}{2} R_{\parallel} g_{ab} + \frac{\partial K_{ab}}{\partial n} + 2(K^2)_{ab} - (\text{tr}K) K_{ab} ,$$

(26)

$$^{(3)} R_{\hat{a}\hat{b}} = K_{a} - K_{ab}^{\;\hat{b}} ,$$

(27)

$$^{(3)} R_{\hat{a}\hat{n}} = \frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2) ,$$

(28)

$$^{(3)} R = R_{\parallel} + 2 \frac{\partial \text{tr}(K)}{\partial n} - \text{tr}(K^2) - (\text{tr}K)^2 .$$

(29)

We can now evaluate the various components of the three-space Einstein tensor (cf, [26 page 552]):

$$^{(3)} G_{ab} = \frac{\partial K_{ab}}{\partial n} + 2(K^2)_{ab} - (\text{tr}K) K_{ab}$$
\[-g_{ab} \left\{ \frac{\partial \text{tr}(K)}{\partial n} - \frac{1}{2} \text{tr}(K^2) - \frac{1}{2} (\text{tr}K)^2 \right\}, \quad (30)\]

\[^{(3)}G_{\hat{n}a} = K_{\hat{a}b} K_{ab}, \quad (31)\]

\[^{(3)}G_{\hat{n}\hat{n}} = -\frac{1}{2} R_{\parallel} ^{\hat{n}} - \frac{1}{2} \text{tr}(K^2) + \frac{1}{2} (\text{tr}K)^2. \quad (32)\]

By way of this decomposition, we can further elaborate on the form of the spacetime (3+1) Einstein tensor. For example,

\[^{(3+1)}G_{ab} = -\frac{N_{ab}}{N} + g_{ab} \left[ \frac{(3)\Delta N}{N} \right] \]

\[+ \frac{\partial K_{ab}}{\partial n} + 2(K^2)_{ab} - \text{tr}(K) K_{ab} \]

\[+ g_{ab} \left\{ - \frac{\partial \text{tr}(K)}{\partial n} + \frac{1}{2} \text{tr}(K^2) + \frac{1}{2} (\text{tr}K)^2 \right\}. \quad (33)\]

However, utilizing the definition of the extrinsic curvature and the Gauss–Weingarten equations, \(^4\)

\[N_{|ab} = N_{ab} - K_{ab} N_{|n}, \quad (34)\]

\[N_{|na} = \partial_n N_{|a} + K_{a} \ ^{b} \ N_{|b}, \quad (35)\]

we then have

\[^{(3)}\Delta N = g^{kl} N_{|kl} \quad (36)\]

\[= g^{ab} N_{|ab} - (g^{ab} K_{ab}) N_{|n} + N_{|nn} \quad (37)\]

\[= (2)\Delta N - \text{tr}(K) N_{|n} + N_{|nn}. \quad (38)\]

Putting everything together, we can finally write

\[^{(3+1)}G_{ab} = -\frac{N_{ab}}{N} + K_{ab} \frac{N_{|n}}{N} \]

\[+ g_{ab} \left[ \frac{(2)\Delta N}{N} + \frac{N_{|nn}}{N} - \text{tr}(K) \frac{N_{|n}}{N} \right] \]

\[+ \frac{\partial K_{ab}}{\partial n} + 2(K^2)_{ab} - g_{ab} \frac{\partial \text{tr}(K)}{\partial n} + \frac{1}{2} g_{ab} \text{tr}(K^2) \]

See, for instance, equations (21.57) and (21.63) of \(^26\). Further note that \(\partial_n N = N_{|n}\) and these forms can be used interchangeably.

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\[-\text{tr}(K) K_{ab} + \frac{1}{2} (\text{tr}K)^2 g_{ab}, \tag{39}\]

\[(3+1) G_{\hat{a}a} = K_{a} - K_{ab} N_{b} - K_{a} N_{b} \frac{\partial n N_{b}}{N}, \tag{40}\]

\[(3+1) G_{\hat{t}a} = 0, \tag{41}\]

\[(3+1) G_{\hat{n}\hat{n}} = -\frac{1}{2} R_{\parallel} - \frac{1}{2} \text{tr}(K^2) + \frac{1}{2} (\text{tr}K)^2 + \frac{(2) \Delta N}{N} - \text{tr}(K) \frac{N}{N} |_{n}, \tag{42}\]

\[(3+1) G_{\hat{t}\hat{n}} = 0, \tag{43}\]

\[(3+1) G_{\hat{t}\hat{t}} = \frac{1}{2} R_{\parallel} + \frac{\partial \text{tr}(K)}{\partial n} - \frac{1}{2} \text{tr}(K^2) - \frac{1}{2} (\text{tr}K)^2. \tag{44}\]

In this way, we have (for any arbitrary static spacetime) completely specified the (3+1)-dimensional Einstein tensor in terms of the Ricci scalar of the arbitrarily chosen 2-surface, the extrinsic geometry of this 2-surface in the 3-geometry of “space”, and the “lapse function” \(N\) (and its gradients) at the 2-surface.

It is sometimes useful to simplify \((3+1) G_{ab}\) by virtue of the fact that

\[\frac{N_{\parallel n} N_{n}}{N} = -\frac{1}{2} R_{\perp}, \tag{45}\]

where \(R_{\perp}\) is the Ricci scalar with respect to \(g_{\perp}\) — the metric in the \(t-n\)-plane perpendicular to the chosen 2-surface.

For subsequent considerations, the following combination is of particular interest:

\[(3+1) G_{\hat{t}\hat{t}} + (3+1) G_{\hat{n}\hat{n}} \quad = \quad \frac{(2) \Delta N}{N} + \frac{\partial \text{tr}(K)}{\partial n} - (\text{tr}K) \frac{\partial N}{N} - \text{tr}(K^2) \quad (46)\]

\[= \quad \frac{(2) \Delta N}{N} + N \partial_n \left[ N^{-1} \text{tr}(K) \right] - \text{tr}(K^2). \quad (47)\]

Observe that the Ricci scalar \(R_{\parallel}\) has dropped out, and so this particular combination depends only on the extrinsic curvature and the lapse. Moreover, we will ultimately demonstrate that, at a black hole horizon [static Killing horizon], this combination limits to zero!

3 The horizon limit

We will now specifically consider a black hole spacetime which, apart from being static, is allowed to be completely general. (In particular, we are not
assuming spherical symmetry nor asymptotic flatness, as is often the case in studies of this nature.) The existence of a black hole horizon tells us there must be an equipotential surface with $N = 0$ (i.e., a surface of infinite redshift). So let us choose our arbitrary 2-surface as the $N = 0$ surface, and set up a Gaussian coordinate system $(t, n, x, y)$ with $n$ now denoting the normal distance to the horizon. The $(3+1)$-metric then takes the particularly convenient form

$$g_{\mu\nu} = \begin{bmatrix} -N^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g_{ab} \end{bmatrix}.$$  \hspace{1cm} (48)

Next, let us introduce an appropriate notion of “local gravity”. For this purpose, we will first define

$$\kappa \equiv \partial_n N$$  \hspace{1cm} (49)

and then

$$\kappa_H \equiv \lim_{n \to 0} \kappa.$$  \hspace{1cm} (50)

One can readily verify that $\kappa_H$ complies with the standard version of the surface gravity as defined by, for instance, Wald [20]. In fact, it can also be shown that, away from the horizon, $\kappa/N$ is simply the normal component of the 4-acceleration of an observer at fixed $n, x, y$. (See the appendix for further discussion.) This enables us (in the first instance) to write a near-horizon Taylor expansion for the lapse,

$$N(n, x, y) = \kappa_H(x, y) \ n + o(n^2),$$  \hspace{1cm} (51)

though we will soon refine the form of this expansion considerably.

To proceed, it is useful to consider the following curvature invariant:

$$(3+1)R_{\mu\nu\alpha\beta} (3+1)R^{\mu\nu\alpha\beta} = (3)R_{ijkl} (3)R^{ijkl} + 4 \frac{N_{ij}N_{ij}}{N^2}.$$  \hspace{1cm} (52)

Since we want the horizon to be regular and not possess a curvature singularity, this quantity must remain finite in the horizon limit. Furthermore, since this is a sum of squares (relative to the positive-definite 3-metric $g_{ij}$), it follows that the 3-geometry must remain regular,

$$\lim_{n \to 0} (3)R_{ijkl} (3)R^{ijkl} = \text{finite},$$  \hspace{1cm} (53)

and additionally

$$\lim_{n \to 0} \frac{N_{ij}}{N} = \text{finite}.$$  \hspace{1cm} (54)
Therefore, since the denominator is $o(n)$ for non-extremal horizons,  
\[ N_{ij} = o(n) . \]  
(55)

Now decomposing these 3-derivatives by way of the Gauss-Weingarten equations, we have

\[ N|_{nn} = o(n) , \quad (56) \]
\[ N|_{ab} = N_{ab} - K_{ab} N|_n = o(n) , \quad (57) \]
\[ N|_{na} = \partial_a N_n + K^b_a N_b = o(n) . \quad (58) \]

The first of these equations implies that we can refine the expansion for the lapse as

\[ N(n, x, y) = \kappa_H(x, y) n + o(n^3) , \quad (59) \]

Meanwhile, the second equation indicates

\[ K_{ab} = o(n) , \quad (60) \]

meaning that the extrinsic curvature limits to zero on the horizon. Finally, the third equation implies the following:

\[ \{ \kappa_H(x, y) \}_a = 0 , \quad (61) \]

which is, in fact, the zeroth law of black hole mechanics — the surface gravity is a constant over the Killing horizon. Notably, this has been accomplished on purely geometrical grounds, without invoking any additional constraints such as the dominant energy condition (DEC). This finding is compatible with the analysis of (for instance) Wald, where a careful inspection of pages 333-334 of [20] reveals that the DEC is not actually necessary for proving the "zeroth law" for a static Killing horizon (although this point is not explicitly made in the text). A further discussion that dispenses with the energy conditions, in more general situations than considered in the present article, can be found in [28].

The previous deductions enable us to write

\[ N(n, x, y) = \kappa_H n + \frac{\kappa_2(x, y)}{3!} n^3 + o(n^4) \]

(62)

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5For the time being, we will assume a non-extremal horizon or, equivalently, that $\kappa_H$ is non-vanishing. The extremal ($\kappa_H = 0$) case will be addressed in the following section.
and
\[ g_{ab}(n, x, y) = [g_H]_{ab}(x, y) + \frac{[g_2]_{ab}(x, y)}{2!} n^2 + o(n^3) . \] (63)

Indeed, the above forms are necessary and sufficient conditions for all polynomial scalar invariants in the Riemann tensor to be finite at the horizon.

It is now straightforward to obtain the horizon limit for the (3+1)-Einstein tensor:

\[ (3+1)G_{\hat{t}\hat{a}|H} = -\{[g_2]_{ab} - [g_H]_{ab} \text{tr}[g_2]\} + [g_H]_{ab} \frac{\kappa_2}{\kappa_H} , \] (64)

\[ (3+1)G_{\hat{t}\bar{a}|H} = 0 , \] (65)

\[ (3+1)G_{\bar{t}a|H} = 0 , \] (66)

\[ (3+1)G_{\hat{a}\hat{n}|H} = -\frac{1}{2} R_\parallel + \frac{1}{2} \text{tr}[g_2] , \] (67)

\[ (3+1)G_{\bar{t}\bar{n}|H} = 0 , \] (68)

\[ (3+1)G_{\bar{t}\bar{t}|H} = \frac{1}{2} R_\parallel - \frac{1}{2} \text{tr}[g_2] . \] (69)

We have also verified these expressions by symbolic computation. Using the Taylor-series expansions (for the lapse and 2-metric) and then Maple to symbolically calculate the Einstein tensor, we have evaluated the \( n \to 0 \) limit and found that it reproduces the above analytic result. Similarly, we have used a Maple calculation to verify that the horizon possesses a curvature singularity if these Taylor series are not obeyed. \(^6\)

Of particular importance, we find (at the horizon) that
\[ (3+1)G_{\bar{t}\bar{t}|H} + (3+1)G_{\bar{a}\bar{n}|H} = 0 , \] (70)

as previously advertised. Indeed, by inspection, \( G_\perp \propto g_\perp \). For the “parallel” [in-horizon] components, there is, however, no generically simple relation of this type.

Some progress may be made by first noting that, at the horizon, the transverse Ricci scalar satisfies
\[ R_\perp = -2 \frac{N_{\perp n}}{N} \to -2 \frac{\kappa_2}{\kappa_H} . \] (71)

\(^6\)To make the Maple calculation tractable, it is useful to invoke the remaining on-horizon coordinate freedom to write \( [g_H]_{ab} = \exp[2\theta(x, y)] \delta_{ab} \).
Next, let us split the tensor \((3+1)G_{ab}|_H\) into a trace and trace-free part. Finally, using capitalized Latin indices to denote the \(t\) and \(n\) directions perpendicular to the horizon, we define \(\eta_{AB} = [\text{diag}(-1, 1)]_{AB}\). Given these considerations, our result can then be expressed in the following compact form:\footnote{Moreover, this version makes it particularly simple to analyze the situation in the special case of spherical symmetry.}

\[
(3+1)G_{ab}|_H = -\frac{1}{2} \left\{ R_{\perp} - \text{tr}[g_2] \right\} [g_H]_{ab} - \left\{ [g_2]_{ab} - \frac{1}{2} [g_H]_{ab} \text{ tr}[g_2] \right\},
\]

\[
(3+1)G_{\hat{A}a}|_H = 0,
\]

\[
(3+1)G_{\hat{A}\hat{B}}|_H = -\frac{1}{2} \left\{ R_{\parallel} - \text{tr}[g_2] \right\} \eta_{\hat{A}\hat{B}}.
\]

Let us again emphasize that this “boundary condition” holds true at any static Killing horizon. Moreover, given that our analysis is purely geometrical in nature, it has quite strong repercussions on what type of matter/energy can exist near a black hole horizon. The implication is that, once Einstein’s equations have been imposed \(i.e., (3+1)G_{\mu\nu} = 8\pi G_N T_{\mu\nu}\), where \(T_{\mu\nu}\) is the stress-energy tensor and \(G_N\) is Newton’s constant, whatever quantum fluctuations might be present are forced to satisfy rather tight constraints. Which is to say, the stress tensor at the horizon must take on the following block-diagonal form:

\[
T_{\mu\nu}|_H = \begin{bmatrix}
\rho_H & 0 & 0 \\
0 & -\rho_H & 0 \\
0 & 0 & T_{\hat{a}\hat{b}}
\end{bmatrix},
\]

where \(\rho_H\) is the energy density at the horizon and, by virtue of symmetry, \(\rho_H = -\rho_H\) is the transverse component of the pressure.

That a black hole horizon enforces boundary conditions on the curvature (and, consequently, the stress-energy) can be viewed as a natural extension of the “membrane paradigm” of black hole mechanics \cite{29}. Indeed, one can simplify many calculations in classical black hole physics by treating the horizon as a boundary and \(i.e.,\) placing specific boundary conditions, on say, the electric and magnetic fields \cite{29}. In this article, we have extended the notion of horizon boundary conditions — going beyond the test field limit — to an arbitrary static Killing horizon with
an arbitrary matter distribution near the horizon. More to the point, we have demonstrated that the absence of curvature singularities at the horizon enforces a very specific boundary condition on the curvature.

4 Extremal horizons

At an extremal horizon — for which $\kappa_H = 0$ — we require a different analysis. Let us start by assuming that the surface gravity has an order $m$ degeneracy:

$$N(n, x, y) = \frac{\kappa_m(x, y)}{(m + 1)!} n^{m+1} + o(n^{m+2}),$$

so that

$$\kappa(n, x, y) = \frac{\kappa_m(x, y)}{m!} n^{m} + o(n^{m+1}).$$

Then repeating the arguments that banned curvature singularities from a non-extremal horizon, we find that the finiteness of $(3+1)G_{na}$ requires that

$$N(n, x, y) = \frac{\kappa_m}{(m + 1)!} n^{m+1} + \frac{\kappa_{m+2}(x, y)}{(m + 3)!} n^{m+3} + o(n^{m+4})$$

and

$$\kappa(n, x, y) = \frac{\kappa_m}{m!} n^{m} + \frac{\kappa_{m+2}(x, y)}{(m + 2)!} n^{m+2} + o(n^{m+3}).$$

Unfortunately, there is now a problem with $(3+1)G_{ab}$:

$$(3+1)G_{ab} = g_{ab}\frac{N_{nm}}{N} + o(1) = \frac{m(m + 1)}{n^2}g_{ab} + o(1),$$

indicating that the horizon has a curvature singularity unless $m(m + 1) = 0$. Now $m = 0$ corresponds to the non-extremal horizon considered earlier, while $m = -1$ corresponds to the lapse being finite and no horizon forming.

Thus if an extremal horizon is located at any finite value of the normal coordinate $n$ (so that it makes sense to shift the location to $n = 0$), we must conclude that the extremal horizon possesses a curvature singularity. Conversely, any extremal horizon that is not simultaneously a curvature singularity must be located at infinite proper distance $n = -\infty$. It is gratifying to see this well-known result regarding extremal horizons arising naturally in this new context.
5 Conformal deformations

Let us now investigate what happens to the on-horizon structure of the Einstein-tensor under a conformal deformation of the spacetime metric, 

\[ g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \exp[2\omega(n, x, y)] g_{\mu\nu} , \tag{81} \]

with 

\[ \omega(n, x, y) = \omega_H(x, y) + \omega_1(x, y) n + \frac{1}{2} \omega_2(x, y) n^2 + O(n^3) . \tag{82} \]

For any such deformation, Jacobson and Kang [31] have demonstrated that the notion of a horizon and the value of the surface gravity, \( \kappa_H \), are invariants. Indeed, if \( \omega \) is time independent (as assumed here), then the deformed geometry \( \tilde{g}_{\mu\nu} \) has the same timelike Killing vector as the original geometry \( g_{\mu\nu} \). Consequently, the form of the Einstein tensor (and, in fact, all of the curvature tensors) will be unaltered. The specific value, however, will generally change. (That is, the form of the on-horizon curvature is a conformal invariant, while the value of the on-horizon curvature is conformally covariant.)

Actually, a direct application of our prior formalism is a little tricky since we would first need to construct [to suitable accuracy] the Gaussian normal coordinate patch \((t, \tilde{n}, \tilde{x}, \tilde{y})\) appropriate to the metric \( \tilde{g}_{\mu\nu} \), and such a construction is a tedious exercise that is quite prone to error. Instead, we will start with the well-known transformation law for the Einstein tensor under an arbitrary conformal transformation [20], 

\[ (3+1)\tilde{G}_{\mu\nu} = (3+1)G_{\mu\nu} - 2\nabla_\mu \nabla_\nu \omega + 2\nabla_\mu \omega \nabla_\nu \omega + g_{\mu\nu} \left\{ 2 \Delta \omega + (\nabla \omega)^2 \right\} , \tag{83} \]

and confront it with the symmetries (as previously established) of the Einstein tensor at any static Killing horizon.

First of all, the preservation of these symmetries [in particular, equations (73) and (74)] immediately implies that 

\[ \lim_{n \to 0} \left\{ \omega|_{nn} - \omega|_{n} \omega|_{n} \right\} = 0 \tag{84} \]

and 

\[ \lim_{n \to 0} \left\{ \omega|_{na} - \omega|_{n} \omega|_{a} \right\} = 0 . \tag{85} \]
One effect of these constraints is to enforce $\omega_2 = (\omega_1)^2$. But if $\omega_1 \neq 0$, then the quantity

$$^{(3+1)}\Delta \omega = \frac{1}{N \sqrt{\text{det}[g_{ab}]}} \partial_\mu \left( N \sqrt{\text{det}[g_{ab}]} g^{\mu \nu} \partial_\nu \omega \right)$$

(86)

simplifies to

$$^{(3+1)}\Delta \omega = (2) \Delta \omega + \frac{1}{\kappa_H n} \partial_n \left( \kappa_H n [\omega_1 + \omega_2 n] + o(n) \right) + (\omega_1 n + o(1)) \rightarrow \infty ;$$

(87)

thus leading to an undesirable curvature singularity at the horizon. Therefore, we must have $\omega_1 = 0$ and, consequently, $\omega_2 = 0$. [In contrast, $\omega_H(x, y)$ remains an arbitrary unconstrained function.] Hence,

$$\omega(n, x, y) = \omega_H(x, y) + O(n^3),$$

(88)

and the on-horizon limit now yields

$$^{(3+1)}\Delta \omega = (2) \Delta \omega + \frac{1}{\kappa_H n} \partial_n \left( o(n^3) \right) + o(n) \rightarrow (2) \Delta \omega_H ,$$

(89)

which is finite. Similarly,

$$\lim_{n \to 0} \nabla_a \nabla_b \omega \equiv \lim_{n \to 0} \omega_{|ab} = \{\omega_H\}_{:ab} ,$$

(90)

and

$$\lim_{n \to 0} \nabla_a \omega \nabla_b \omega = \{\omega_H\}_{:a} \{\omega_H\}_{:b} ,$$

(91)

so that, for the on-horizon Einstein tensor, we have

$$^{(3+1)}\tilde{G}_{ab}|_H = (^{(3+1)}G_{ab}|_H - 2\{\omega_H\}_{:ab} + 2\{\omega_H\}_{:a} \{\omega_H\}_{:b}$$

$$+ g_{ab} \left\{ 2 \left( ^{(2)}\Delta \omega_H + g^{cd} \{\omega_H\}_{:c} \{\omega_H\}_{:d} \right) \right\} ,$$

(92)

$$^{(3+1)}\tilde{G}_{Aa}|_H = 0 ,$$

(93)

$$^{(3+1)}\tilde{G}_{A\dot{B}}|_H = \exp(-2\omega_H) \left[ ^{(3+1)}G_{A\dot{B}}|_H \right.$$}

$$+ \eta_{\dot{A}\dot{B}} \left\{ 2 \left( ^{(2)}\Delta \omega_H + g^{cd} \{\omega_H\}_{:c} \{\omega_H\}_{:d} \right) \right\} \right] .$$

(94)

Let us be clear about what has transpired here. The form of the conformal deformation near the horizon is constrained by the need to avoid curvature
singularities at the horizon. Once this has been done, the change in the Einstein tensor at the horizon depends only on the conformal deformation at the horizon itself.

The results of this section have, once again, been checked by symbolic computation. To elaborate, we have used the Taylor series for $\omega(n, x, y)$ to set up a Maple computation of the Einstein tensor in the vicinity of the horizon and then symbolically taken the $n \to 0$ limit. An independent computation that employs the $(t, \tilde{n}, \tilde{x}, \tilde{y})$ Gaussian coordinate patch appropriate to the metric $\tilde{g}_{\mu\nu}$ is too tedious to be worth presenting in any detail. It is, however, relatively simple to work backwards from

$$R_{\parallel}(\tilde{g}_H) = \exp(-2\omega_H) \left\{ R_{\parallel}(g_H) - 2 (2) \Delta \omega_H \right\}$$

(95)

to find that [cf, equations (74) and (94)]

$$\tilde{\text{tr}}[\tilde{g}_2] = \exp(-2\omega_H) \left\{ \text{tr}[g_2] + 2 (2) \Delta \omega_H + 2 g^{cd} \{ \omega_H \}_c \{ \omega_H \}_d \right\}.$$  

(96)

Furthermore, from the trace-free part of $(3+1)\tilde{G}_{ab}|_H$, we can extract [using equations (72) and (92)]

$$\left\{ [g_2]_{ab} - \frac{1}{2}[g_H]_{ab} \tilde{\text{tr}}[\tilde{g}_2] \right\} = \left\{ [g_2]_{ab} - \frac{1}{2}[g_H]_{ab} \text{tr}[g_2] \right\}$$

$$+ 2 \left[ \{ \omega_H \}_c \{ \omega_H \}_d - \{ \omega_H \}_c \{ \omega_H \}_d \right].$$

(97)

As a consequence, it can also be shown that [cf, equation (74)]

$$[\tilde{\kappa}_2] = \exp(-2\omega_H) \left\{ [\kappa_2] + \kappa_H g^{cd} \{ \omega_H \}_c \{ \omega_H \}_d \right\}.$$  

(98)

This last equation can equivalently be written [via equation (71)]

$$R_{\perp}(\tilde{g}_H) = \exp(-2\omega_H) \left\{ R_{\perp}(g_H) - 2 g^{cd} \{ \omega_H \}_c \{ \omega_H \}_d \right\}.$$  

(99)

The (3+1)-dimensional conformal transformation is actually changing the location of the transverse surfaces at the order $o(n^2)$, and so this transformation law for $R_{\perp}$ is what one might have naively expected. In any event, we have confronted this calculation with an explicit coordinate computation that employs Gaussian coordinates, again verifying these formulae to be correct.
To summarize, we have demonstrated that the Einstein tensor must take on a particularly simple form [see equations (72)–(74)] at any non-extremal static Killing horizon. In particular, we have shown that the on-horizon Einstein tensor block-diagonalizes and, moreover, that $(3+1)G_{tt}|_H + (3+1)G_{nn}|_H = 0$; where $\hat{t}$ and $\hat{n}$ are the (normalized) spacetime coordinates in the directions perpendicular to the horizon (timelike and spacelike respectively). Although this symmetry had already been established for static spherically symmetric horizons [16, 17], this is — to the best of our knowledge — the first time that it has been rigorously demonstrated to hold for generic static geometries.

A direct implication of our analysis is that, by way of Einstein’s equation, the matter/energy near a static Killing horizon (including any quantum fluctuations) must be highly constrained; cf, equation (75). Furthermore, we can now make the following observation: Given that the “interesting” near-horizon physics can be anticipated to take place in the $n$–$t$ plane [7] and $T_\perp \propto g_\perp$ at the horizon, the near-horizon stress tensor is effectively that of a collection of world-sheet conformal field theories. More precisely, a collection of two-dimensional conformal theories, each of which is defined at a point on the horizon and constrained to act in the $n$–$t$ plane. [And with interactions between these conformal field theories possibly being responsible for the in-horizon portion of the $(3+1)$-stress-energy tensor.] One can now see how two-dimensional conformal theories could play such a prominent role in calculations of black hole entropy, as has been central to the program of Carlip [12, 13, 14], as well as Solodukhin [15]. Moreover, we would suggest that it is these geometrical constraints on the stress tensor that underlie Carlip’s notion of black hole entropy being controlled by an asymptotic conformal symmetry near the horizon.

With regard to our last (somewhat speculative) remark, it may be significant that any static conformal transformation will not alter the general form of the Einstein or stress tensor, and the conformal deformation of these tensors will be highly constrained (cf, Section 5). Coupled with the knowledge that such a transformation also preserves the causal structure of the solution [31], it becomes evident that our geometric symmetry is truly conformal in its nature.

An interesting open question is what (if any) symmetries would persist when the Killing horizon is no longer static. Although a technically difficult problem, we do anticipate that stationary Killing horizons would exhibit
similar symmetries to those found for the static case; and work along this
direction is currently underway. Meanwhile, a truly time-dependent geom-
etry raises serious issues that go beyond mere technical difficulties. On the
other hand, one might expect that, if a horizon is evolving slowly enough, it
should have a viable interpretation as being quasi-static (or, at least, quasi-
stationary). Under such circumstances, our current formalism could still be
applied to time-dependent scenarios with some degree of accuracy.

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Appendix: Defining the surface gravity

Here, we will verify that our version of the surface gravity \[cf, \text{equations (49) and (50)}\] is indeed compatible with the standard one (see for instance Wald \[20\]). Let us start by noting that, if \(\xi^\mu\) is the Killing vector for our
static spacetime, then
\[
\xi^\mu \xi_\mu = -N^2. \tag{100}
\]
Hence, the 4-velocity of the static fiducial observers [FIDOS] can be defined
by
\[
V^\mu = \frac{\xi^\mu}{N}. \tag{101}
\]
Consequently, the 4-acceleration of the FIDOS is given by
\[
A^\mu = (V^\nu \nabla_\nu)V^\mu = \frac{1}{N}(\xi^\nu \nabla_\nu) \left[ \frac{\xi^\mu}{N} \right] = \frac{1}{N^2}(\xi^\nu \nabla_\nu)\xi^\mu - \frac{\xi^\mu}{N^3}(\xi^\nu \nabla_\nu)N = \frac{(\xi^\nu \nabla_\nu)\xi^\mu}{N^2}, \tag{102}
\]
where only at the last step has the fact that \(\xi^\mu\) is a Killing vector been
invoked. Next, let us consider that
\[
A^\mu = \frac{(\xi^\nu \nabla_\nu)\xi^\mu}{N^2} = -\frac{(\xi^\nu \nabla_\nu)\xi^\mu}{N^2} = \frac{1}{2} \frac{\nabla_\mu(\xi^\nu \xi_\nu)}{N^2}. \tag{103}
\]
Therefore,
\[
A^\mu = \frac{1}{2} \frac{\nabla_\mu(N^2)}{N^2} = \frac{\nabla_\mu N}{N}, \tag{104}
\]
or, to put it another way,

\[ ||A|| = \frac{||\nabla N||}{N} . \] (105)

Let us now recall equation (49) for our definition of the “local gravity”. In view of this definition, as well as the acceleration being normal to surfaces of constant \( N \), it follows that

\[ \kappa = \partial_n N = N \ ||A|| , \] (106)

and so

\[ \kappa_H = \lim_{z \to H} \partial_n N = \lim_{z \to H} \{ N \ ||A|| \} , \] (107)

in agreement with equation (12.5.18) of \[20\]. Moreover, since equation (12.5.18) was derived directly from equation (12.5.2) of \[20\] (the latter being Wald’s starting-point definition of the surface gravity), it is clear that our definition of \( \kappa_H \) complies with the standard version.

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