Bell inequalities versus entanglement and mixedness
for a class of two-qubit states

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Abstract: For a class of mixed two-qubit states we show that it is not possible to discriminate between states violating or non-violating Bell-CHSH inequalities, knowing only their entanglement and mixedness. For a large set of possible values of these quantities, we construct pairs of states with the same entanglement and mixedness such that one state is violating but the other is non-violating Bell-CHSH inequality.

1. Introduction

Contradiction between quantum theory and local realism manifests by the violation of Bell-CHSH inequalities [1, 2]. It is experimental fact that those inequalities can indeed be violated in many quantum systems (for review see e.g. [3]). On the other hand, it is well known that quantum states violating Bell inequalities have to be entangled [4, 5]. Since all pure entangled states violate Bell inequalities [6], it was believed that entanglement is equivalent to such violation. After the work of Werner [7], it turned out that violation of Bell inequalities is not necessary for mixed states entanglement. Thus the relation between entanglement and Bell inequalities is not clear and is the interesting problem that should be investigated in details.

In the case of two-qubit system, there is an effective criterion for violating the CHSH inequalities [8, 9]. It enables to associate with any two-qubit state some numerical parameter ranging from 0 for "local states" to 1 for states maximally violating such inequalities. Using this criterion, one can study for example the relation between entanglement, the CHSH violation and their behaviour under the local filtering operations [10]. Another interesting question is the following: what is the connection between entanglement and mixedness of the state, and the amount of CHSH violation given by that state. It is known that to produce an equal amount of CHSH violation some states require more entanglement than others. In Ref. [11], it was suggested that if the more mixed is a state, the higher degree of entanglement is required for it to violate CHSH inequality. However, there are examples of states that counter that suggestion. One can find states with equal amount of CHSH violation and entanglement, but one of them is more mixed than that other. Moreover, one can construct such states that for fixed CHSH violation, the order of mixedness for them is always reserved with respect to the order of their entanglements [12].

In the present paper, we study another aspects of the relationship between entanglement, measured by concurrence $C(\rho)$, mixedness measured by linear entropy $S_L(\rho)$ and CHSH violation. We ask the following question: is it possible to discriminate between states violating or non-violating CHSH inequalities computing only their entanglement and mixedness? We solve the problem for some class of mixed two-qubit states. We show that there is a large set of possible values of entanglement and mixedness such that for fixed pair $(s, c)$ in that set, we can always construct states $\rho_1, \rho_2$ with $C(\rho_1) = C(\rho_2) = c$ and $S_L(\rho_1) = S_L(\rho_2) = s$, such that $\rho_1$ is violating CHSH inequality but $\rho_2$ is not violating this inequality. On the other hand, there is also a subset on the $(s, c)$ plane such that the corresponding states always violate CHSH inequalities, and the other subset to which correspond non-violating states. Our results indicate that the reason why given mixed
state violates Bell-CHSH inequality cannot be explained by their entanglement and mixedness alone.

2. **Violation of Bell inequalities for a pair of qubits**

2.1. **Entanglement.** Consider two-level system $A$ (one-qubit) with the Hilbert space $\mathcal{H}_A = \mathbb{C}^2$ and the algebra of observables $\mathfrak{A}_A$ given by $2 \times 2$ complex matrices. For a joint system $AB$ of two qubits $A$ and $B$, the algebra $\mathfrak{A}_{AB}$ is equal to $4 \times 4$ complex matrices and the Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^4$. Let $\mathcal{E}_{AB}$ be the set of all states of the compound system i.e.

\[
\mathcal{E}_{AB} = \{ \rho \in \mathfrak{A}_{AB} : \rho \geq 0 \quad \text{and} \quad \text{tr} \rho = 1 \}
\]

The state $\rho \in \mathcal{E}_{AB}$ is **separable** [7], if it has the form

\[
\rho = \sum_k \lambda_k \rho_k^A \otimes \rho_k^B, \quad \rho_k^A \in \mathcal{E}_A, \quad \rho_k^B \in \mathcal{E}_B, \quad \lambda_k \geq 0 \quad \text{and} \quad \sum_k \lambda_k = 1
\]

The set $\mathcal{E}_{sep}^{AB}$ of all separable states forms a convex subset of $\mathcal{E}_{AB}$. When $\rho$ is not separable, it is called **inseparable** or **entangled**. Thus

\[
\mathcal{E}_{ent}^{AB} = \mathcal{E}_{AB} \setminus \mathcal{E}_{sep}^{AB}
\]

As a measure of the amount of entanglement a given state contains we take the entanglement of formation [13]

\[
E(\rho) = \min \sum_k \lambda_k E(P_k)
\]

where the minimum is taken over all possible decompositions

\[
\rho = \sum_k \lambda_k P_k
\]

and

\[
E(P) = -\text{tr} [(\text{tr}_AP) \log_2 (\text{tr}_AP)]
\]

In the case of two qubits, $E(\rho)$ is the function of another useful quantity $C(\rho)$ called **concurrence**, which also can be taken as a measure of entanglement [13] [15]. $C(\rho)$ is defined as follows

\[
C(\rho) = \max (0, 2p_{\text{max}}(\hat{\rho}) - \text{tr} \hat{\rho})
\]

where $p_{\text{max}}(\hat{\rho})$ denotes the maximal eigenvalue of $\hat{\rho}$ and

\[
\hat{\rho} = (\rho^{1/2} \hat{\rho}^{1/2})^{1/2}
\]

with

\[
\rho^{\dagger} = (\sigma_2 \otimes \sigma_2) \hat{\rho} (\sigma_2 \otimes \sigma_2)
\]

The value of the number $C(\rho)$ varies from 0 for separable states, to 1 for maximally entangled pure states. For the class $\mathcal{E}_0$ of states consisting of density matrices of the form

\[
\rho = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \rho_{22} & \rho_{23} & 0 \\
0 & \rho_{32} & \rho_{33} & 0 \\
0 & 0 & 0 & \rho_{44}
\end{pmatrix}
\]

$C(\rho)$ is given by

\[
C(\rho) = |\rho_{23}| + \sqrt{\rho_{22} \rho_{33}} - |\rho_{23}| - \sqrt{\rho_{22} \rho_{33}}
\]

By positive-definiteness of $\rho$, $|\rho_{23}| \leq \sqrt{\rho_{22} \rho_{33}}$, thus

\[
C(\rho) = 2 |\rho_{23}|
\]
2.2. Bell - CHSH inequalities. Let \( a, a', b, b' \) be the unit vectors in \( \mathbb{R}^3 \) and \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \). Consider the family of operators on \( \mathcal{H}_{AB} \)

\[
B_{CHSH} = a \cdot \sigma \otimes (b + b') \cdot \sigma + a' \cdot \sigma \otimes (b - b') \cdot \sigma
\]

(13)

Then Bell - CHSH \cite{2} inequalities are

\[
|\text{tr} (\rho B_{CHSH})| \leq 2
\]

(14)

If the above inequality is not satisfied by the state \( \rho \) for some choice of \( a, a', b, b' \), we say that \( \rho \) violates Bell inequalities (\( \rho \) is VBI). In the case of two-qubit system, the violation of Bell - CHSH inequalities by mixed states can be studied using simple necessary and sufficient condition \cite{8, 9}. Any state \( \rho \in \mathcal{E}_{AB} \) can be written as

\[
\rho = \frac{1}{4} \left( I_2 \otimes I_2 + r \cdot \sigma \otimes I_2 + I_2 \otimes s \cdot \sigma + \sum_{n,m=1}^{3} t_{nm} \sigma_n \otimes \sigma_m \right)
\]

(15)

where \( I_2 \) is the identity matrix in two dimensions, \( r, s \) are vectors in \( \mathbb{R}^3 \) and \( r \cdot \sigma = \sum_{j=1}^{3} r_j \sigma_j \). The coefficients

\[
t_{nm} = \text{tr} (\rho \sigma_n \otimes \sigma_m)
\]

(16)

form a real matrix \( T_\rho \). Define also real symmetric matrix

\[
U_\rho = T_\rho^T T_\rho
\]

(17)

where \( T_\rho^T \) is the transposition of \( T_\rho \). Violation of inequality (14) by the density matrix (15) and some Bell operator (13) can be checked by the following criterion: Let

\[
m(\rho) = \max_{j < k} (u_j + u_k)
\]

(18)

and \( u_j, j = 1, 2, 3 \) are the eigenvalues of \( U_\rho \). As was shown in \cite{8, 9}

\[
\max_{B_{CHSH}} \text{tr} (\rho B_{CHSH}) = 2 \sqrt{m(\rho)}
\]

(19)

Thus (14) is violated by some choice of \( a, a', b, b' \) iff \( m(\rho) > 1 \). We can also introduce another parameter

\[
n(\rho) = \max(0, m(\rho) - 1)
\]

ranging from 0 for non VBI states to 1 for state maximally VBI. For the class \( \mathcal{E}_0 \) we obtain the following expression for \( m(\rho) \)

\[
m(\rho) = \max (2 C^2(\rho), (1 - 2 \rho_{44})^2 + C^2(\rho))
\]

(20)

where \( C(\rho) \) is the concurrence of the state \( \rho \). Notice that all states \( \rho \in \mathcal{E}_0 \) with concurrence greater then \( \frac{1}{\sqrt{2}} \) are VBI. In the next section we focus on states with \( C(\rho) \leq \frac{1}{\sqrt{2}} \).

3. The main result

Consider now the relation between mixedness, entanglement and violation of Bell inequalities for mixed states from the class \( \mathcal{E}_0 \). Since for \( C(\rho) > \frac{1}{\sqrt{2}} \) every mixed state is VBI, consider \( \rho \) such that \( C(\rho) \leq \frac{1}{\sqrt{2}} \). Then \( m(\rho) > 1 \) when

\[
(1 - 2 \rho_{44})^2 + C^2(\rho) > 1
\]

(21)

The above inequality is equivalent to

\[
|\rho_{23}|^2 > \rho_{44}(1 - \rho_{44})
\]

Let us introduce the normalized linear entropy of the state \( \rho \)

\[
S_L(\rho) = \frac{4}{3} (1 - \text{tr} \rho^2)
\]
as a measure of its mixedness. We see that $S_L(\rho) = 0$ for pure states and $S_L(\frac{1}{4}I_4) = 1$. For states from the class $\mathcal{E}_0$

$$S_L(\rho) = \frac{8}{3}(\rho_{22}\rho_{33} + \rho_{22}\rho_{44} + \rho_{33}\rho_{44} - |\rho_{23}|^2)$$

On the other hand

$$|\rho_{23}|^2 - \rho_{44}(\rho_{22} + \rho_{33}) = |\rho_{23}|^2 - \rho_{44}(1 - \rho_{44}) > 0$$

so

$$\rho_{22}\rho_{33} - \frac{3}{8}S_L(\rho) = |\rho_{23}|^2 - \rho_{44}(\rho_{22} + \rho_{33}) > 0$$

Thus inequality (21) is satisfied iff [16]

$$\rho_{22}\rho_{33} > \frac{3}{8}S_L(\rho)$$

Inequality (22) indicates that states $\rho$ with sufficiently small mixedness and non-zero entanglement should be VBI. On the other hand, large mixedness should lead to non-violation of any Bell inequality. Below we show that there is also another possibility. For the intermediate values of mixedness, there exist states with the same linear entropy and concurrence and such that one of them is VBI, but the other is not VBI. To study this problem, introduce the subset $\Lambda_{\mathcal{E}_0} \subseteq \mathbb{R}^2$

$$\Lambda_{\mathcal{E}_0} = \{(S_L(\rho), C(\rho)) : C(\rho) > 0 \text{ and } \rho \in \mathcal{E}_0\}$$

Theorem 3.1.

$$\Lambda_{\mathcal{E}_0} = \{(s, c) \in \mathbb{R}^2 : 0 < c \leq 1, 0 \leq s \leq S_{\max}(c)\}$$

where

$$S_{\max}(c) = \begin{cases} \frac{8}{9} - \frac{2}{3}c^2, & c < \frac{2}{3} \\ \frac{8}{9}c(1 - c), & c \geq \frac{2}{3} \end{cases}$$

Proof: We parametrize the states $\rho \in \mathcal{E}_0$ as follows

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & \frac{1}{2}ce^{i\theta} & 0 \\ \frac{1}{2}ce^{-i\theta} & b & 0 \\ 0 & 0 & 0 & 1 - a - b \end{pmatrix}, \ a, b \geq 0, \theta \in [0, 2\pi]$$

Then positive definiteness of $\rho$ is equivalent to

$$ab \geq \frac{c^2}{4} \text{ and } a + b \leq 1$$

On the other hand,

$$S_L(\rho) = \frac{4}{3}\left(1 - a^2 - b^2 - (1 - (a + b))^2 - \frac{c^2}{2}\right)$$

We are looking for maximal value of (27) for fixed $c$ and $a, b$ such that conditions (26) are satisfied. It turns out that for $c \in \left(0, \frac{4}{3}\right)$, maximal value of $S_L$ is attained at $a = b = \frac{4}{3}$ and is given by

$$S_{\max}(c) = \frac{8}{9} - \frac{2}{3}c^2, \ c \in \left(0, \frac{2}{3}\right)$$

For $c \in \left[\frac{2}{3}, 1\right]$, $S_{\max}(c)$ is attained at $a = b = \frac{2}{3}$, thus

$$S_{\max}(c) = \frac{8}{3}c(1 - c), \ c \in \left[\frac{2}{3}, 1\right]$$
Remark 3.1. Notice that $S_{\text{max}}(c)$ is realized by states

$$\rho_1(c) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} e^{i\theta} & 0 \\ 0 & \frac{1}{3} e^{-i\theta} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad c \in \left(0, \frac{2}{3}\right)$$

end

$$\rho_2(c) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} e^{-i\theta} & \frac{1}{3} e^{i\theta} & 0 \\ 0 & \frac{1}{3} e^{i\theta} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 - c \end{pmatrix}, \quad c \in \left[\frac{2}{3}, 1\right)$$

The states (30) and (31) are locally equivalent to maximally entangled mixed states discovered in [17]. We have obtained the same result starting from different class of states.

Now consider the structure of the set $\Lambda_{E_0}$.

Theorem 3.2. $\Lambda_{E_0}$ is a sum of disjoint subsets $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$ with the properties:

1. If $(s, c) \in \Lambda_1$, then every state $\rho \in E_0$ such that $S_L(\rho) = s$ and $C(\rho) = c$ is VBI.
2. If $(s, c) \in \Lambda_2$, then there exist states $\rho_1, \rho_2 \in E_0$ such that $S_L(\rho_1) = S_L(\rho_2) = s$, $C(\rho_1) = C(\rho_2) = c$ and $\rho_1$ is VBI, but $\rho_2$ is not VBI.
3. If $(s, c) \in \Lambda_3$, then every state $\rho \in E_0$ such that $S_L(\rho) = s$ and $C(\rho) = c$ is not VBI.

The sets $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$ can be described as follows (Fig. 1):

$$\Lambda_1 = \{(s, c) : 0 < c \leq \frac{1}{\sqrt{2}}, \quad 0 \leq s < S_1(c)\} \cup \{(s, c) : \frac{1}{\sqrt{2}} < c \leq 1, \quad 0 \leq s \leq S_{\text{max}}(c)\}$$

$$\Lambda_2 = \{(s, c) : 0 < c \leq \frac{1}{\sqrt{2}}, \quad S_1(c) \leq s < S_2(c)\}$$

$$\Lambda_3 = \{(s, c) : 0 < c \leq \frac{1}{\sqrt{2}}, \quad S_2(c) \leq s \leq S_{\text{max}}(c)\}$$

with

$$S_1(c) = \frac{2}{3} c^2, \quad S_2(c) = \frac{2 - c^2 + 2\sqrt{1 - c^2}}{6}$$
**Fig. 1.** The set $\Lambda_{E_0}$ of admissible pairs $(S_L(\rho), C(\rho))$ for $\rho \in E_0$

**Proof:** Consider the parametrization (25) and introduce new variables

$$x = \frac{1}{\sqrt{2}} (a - b), \quad y = \frac{1}{\sqrt{2}} \left( a + b - \frac{2}{3} \right).$$

Then conditions (26) can be rewritten as

$$\frac{y^2}{2} + \frac{\sqrt{2}y}{3} - \frac{x^2}{2} - \frac{c^2}{4} + \frac{1}{9} \geq 0 \quad (32)$$

and

$$y \leq \frac{1}{3\sqrt{2}} \quad (33)$$

Thus every point $(x, y) \in X_+$, where

$$X_+ = \{(x, y) : \frac{y^2}{2} + \frac{\sqrt{2}y}{3} - \frac{x^2}{2} - \frac{c^2}{4} + \frac{1}{9} \geq 0, \ y \leq \frac{1}{3\sqrt{2}} \}$$

defines the state $\rho \in E_0$. We see also that

$$S_L = -\frac{8}{3} \left( \frac{x^2}{2} + \frac{3y^2}{2} + \frac{c^2}{2} - \frac{1}{3} \right) \quad (34)$$

and the level set $S_L = s$ is the ellipse

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1 \quad (35)$$

with

$$A = \sqrt{6D}, \ B = \sqrt{2D}, \ \text{and} \ D = -\frac{c^2}{12} - \frac{s}{8} + \frac{1}{9}$$

Thus the set of states with fixed concurrence $C(\rho) = c$ and linear entropy $S_L(\rho) = s$ is determined by the intersection of the ellipse (35) and $X_+$. On the other hand, the condition (21) equivalent
to \( m(\rho) > 1 \) now reads

\[
8y^2 + \frac{4\sqrt{2}}{3}y + c^2 - \frac{8}{9} > 0
\]

The above inequality can be satisfied by admissible variables \( y \) only when

\[
y > y_+ = \frac{-1 + 3\sqrt{1 - c^2}}{6\sqrt{2}}
\]

Similarly, \( m(\rho) \leq 1 \) for \( y \leq y_+ \). Now the idea of the proof is simple. For fixed concurrence \( c \), the intersection of the level set of the function \( S_L \) with \( X_+ \) can lie below or above the line \( y = y_+ \) or can intersect this line, depending on the value of \( s \) (Fig. 2). The ellipse (35) can intersect the line \( y = y_+ \) when \( B > y_+ \), thus for

\[
s < \frac{2 - c^2 + 2\sqrt{1 - c^2}}{6}
\]

there are VBI states. The part of ellipse above the line \( y = y_+ \) represents VBI states, whereas the remaining part corresponds to states with the same \( c \) and \( s \), which are not VBI. For

\[
s \geq \frac{2 - c^2 + 2\sqrt{1 - c^2}}{6}
\]

all states are not VBI. In the case when the ellipse (35) intersects hyperbola (32) above the line \( y = y_+ \), all states are VBI. This can be achieved when

\[
s < \frac{2}{3} c^2
\]

Fig. 2. Intersections of \( X_+ \) with level sets \( S_L = s \) (dotted lines) for different values of \( s \).
4. Examples

We can use parametrization of the ellipse (35) to construct examples of states with properties listed in Theorem 3.2. If

\[ x = A \cos \varphi, \quad y = B \sin \varphi \]

then

\[ \rho(\varphi, \theta) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{3} + \sqrt{D} (\sin \varphi + \sqrt{3} \cos \varphi) & \frac{2}{3} e^{i\theta} & 0 \\ \frac{2}{3} e^{-i\theta} & \frac{1}{3} + \sqrt{D} (\sin \varphi - \sqrt{3} \cos \varphi) & 0 \\ 0 & 0 & 0 & \frac{1}{3} - 2 \sqrt{D} \sin \varphi \end{pmatrix} \]

where \( \theta \in [0, 2\pi] \) and \( \varphi \in I_+ \) (\( I_+ \) will depend on specific values of \( c \) and \( s \)), defines two parameter family of states with fixed concurrence and linear entropy. The set \( I_+ \) is defined as follows. Let \( c \leq \frac{1}{\sqrt{2}} \). Then:

a. for \( \{(s, c) : 0 \leq s < S_1(c)\} \cup (A_2 \cap \{(s, c) : s < \frac{2}{3}(1 - c^2)\}) \)

\( I_+ = [\varphi_1, \varphi_2] \cup [\pi - \varphi_2, \pi - \varphi_1] \)

where

\( \varphi_1 = \arcsin \left[ \frac{1}{\sqrt{D}} \left( \frac{1}{4} \sqrt{1 - 3s/2 - 1/12} \right) \right], \quad \varphi_2 = \arcsin \frac{1}{6\sqrt{D}} \)

b. for \( (A_2 \cap \{(s, c) : s \geq \frac{2}{3}(1 - c^2)\}) \cup (A_3 \cap \{(s, c) : s \in [0, \frac{2}{3}], c \leq \frac{1}{2} \sqrt{2 + 2 \sqrt{1 - \frac{2}{3}s - \frac{3}{2}s}}\}) \)

\( I_+ = [\varphi_1, \pi - \varphi_1] \)

c. for \( A_3 \setminus \{(s, c) : s \in [0, \frac{2}{3}], c \leq \frac{1}{2} \sqrt{2 + 2 \sqrt{1 - \frac{3}{2}s - \frac{1}{2}s}}\} \)

\( I_+ = (0, 2\pi] \)

Define also

\( \varphi_3 = \arcsin \left[ \frac{1}{\sqrt{D}} \left( \frac{1}{4} \sqrt{1 - c^2 - 1/12} \right) \right] \)

If \( \varphi > \varphi_3 \), then the points on the ellipse (35) corresponding to \( \varphi \) lie above the line \( y = y_+ \). Thus if \( \varphi \in I_+ \cap I_B \) where \( I_B = (\varphi_3, \pi - \varphi_3) \)

all states (41) with such \( \varphi \) are VBI. On the other hand, if

\( \varphi \in I_+ \setminus I_B \)

all states (41) with such \( \varphi \) are not VBI. So we have:

1. If \( (s, c) \in A_1 \) then \( \varphi_3 < \varphi_1 \), and

\( I_+ \cap I_B = I_+ \quad \text{and} \quad I_+ \setminus I_B = \emptyset \)

so every state (41) with \( \varphi \in I_+ \) is VBI.

2. If \( (s, c) \in A_2 \), both sets \( I_+ \cap I_B \) and \( I_+ \setminus I_B \) are nonempty. Thus the states (41) with \( \varphi \in I_+ \cap I_B \) are VBI, whereas states with \( \varphi \in I_+ \setminus I_B \) are not VBI.

3. If \( (s, c) \in A_3 \), then \( \varphi_3 \) is not defined and \( I_B = \emptyset \), so every state (41) with \( \varphi \in I_+ \) is not VBI.
Consider now the concrete example. Let \( c = \frac{1}{2} \) and take the points

\[
\left( \frac{1}{8}, \frac{1}{2} \right) \in \Lambda_1, \quad \left( \frac{1}{2}, \frac{1}{2} \right) \in \Lambda_2 \quad \text{and} \quad \left( \frac{7}{10}, \frac{1}{2} \right) \in \Lambda_3
\]

Using the parametrization (41), we obtain three families of states (for simplicity we put \( \theta = 0 \)) with corresponding value of \( C(\rho) \) and \( S_L(\rho) \). So for \( s = \frac{1}{8} \) we have the family

\[
(42) \quad \rho_1(\varphi) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\frac{1}{3} + \frac{\sqrt{43}}{24} (\sin \varphi + \sqrt{3} \cos \varphi) & \frac{1}{4} & 0 \\
0 & \frac{1}{4} & \frac{1}{3} + \frac{\sqrt{43}}{24} (\sin \varphi - \sqrt{3} \cos \varphi) & 0 \\
0 & 0 & 0 & \frac{1}{3} - \frac{\sqrt{43}}{12} \sin \varphi
\end{pmatrix}
\]

with \( \varphi \in (0.54657, 0.65605) \). Then \( m(\rho_1(\varphi)) > 1 \) (Fig. 3).

Fig. 3. \( m(\rho) \) as the function of \( \varphi \) for the states (42)
Similarly, for $s = \frac{1}{2}$ we have the family

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{1}{3} + \frac{1}{6} (\sin \varphi + \sqrt{3} \cos \varphi) & \frac{1}{4} & 0 \\
0 & \frac{1}{4} & \frac{1}{3} + \frac{1}{6} (\sin \varphi - \sqrt{3} \cos \varphi) & 0 \\
0 & 0 & 0 & \frac{1}{3} - \frac{1}{3} \sin \varphi
\end{pmatrix}
\]

with $\varphi \in (0.25, 1.57)$. In that case $m(\rho)$ can be smaller or bigger then 1, depending on $\varphi$ (Fig. 4).

\[\text{Fig. 4. } m(\rho) \text{ as the function of } \varphi \text{ for the states } (43)\]

Finally, for $s = \frac{7}{10}$ we obtain

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{1}{3} + \frac{1}{6 \sqrt{10}} (\sin \varphi + \sqrt{3} \cos \varphi) & \frac{1}{4} & 0 \\
0 & \frac{1}{4} & \frac{1}{3} + \frac{1}{6 \sqrt{10}} (\sin \varphi - \sqrt{3} \cos \varphi) & 0 \\
0 & 0 & 0 & \frac{1}{3} - \frac{1}{3 \sqrt{10}} \sin \varphi
\end{pmatrix}
\]

with $\varphi \in (0, 2\pi)$. For this family $m(\rho) < 1$ (Fig. 5).
Fig. 5. $m(\rho)$ as the function of $\varphi$ for the states (44)

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