New Geometric Formalism for Gravity Equation in Empty Space

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Abstract

In this paper, a complex daor field which can be regarded as the square root of space-time metric is proposed to represent gravity. The locally complexified geometry is set up, and the complex spin connection constructs a bridge between gravity and SU(1,3) gauge field. Daor field equations in empty space are acquired, which are one-order differential equations and not conflict with Einstein’s gravity theory.

Keywords: Daor field; complex connection; gravity; empty space.

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1 Introduction

How to incorporate general relativity with quantum field theory is a big problem in modern physics. There are now several popular paradigms trying to solve this problem, among which string theory and loop quantum gravity are well-known. The string theory inherits most merits of quantum field theory. The graviton in string theory is a spin-2 bosonic string. Since string theory is constructed in 10-dimensional space-time (11-dimensional space-time for M-theory), there are enough inner freedom to accommodate all gauge interactions [1]. The most difficult question in string theory is how to compactify extra-dimensions to predict new observable phenomena. The loop quantum gravity theory originates from Ashtekar’s re-expression for general relativity [2], and inherits the geometric viewpoint from general relativity. The loop quantum gravity theory has yielded several interesting results, such as the discrete spectrum of the space volume [3].

Based on the concept of Yang-Mills field [4] and the spontaneous symmetry broken mechanism [5], Glashow, Salam and Weinberg constructed a renormalizable electroweak gauge theory [6]. Combining this theory with quantum chromodynamics yielded the so-called $SU(3)_C \times SU(2)_L \times U(1)_Y$ standard modern in particle physics. Recently, it is claimed that some great unified theories (GUTs) endowed with $SO(10)$ gauge group [7] can predict neutrino masses and mixing angles [8] which are not contradictory to modern experiments [9]. If physicists believe that the standard model or GUTs should be a low-energy approximation of a high energy unified quantum theory which incorporate gravity with gauge fields, then those inner symmetry such as $SU(3)_C$ and $U(1)_Y$ should be reasonably accommodated in the higher energy theory. In this paper we propose a possibility to give those inner symmetries without introducing extra-dimensions. Then in our framework there are no difficulties on compactification. After proposing the basic principles we give the locally complexified geometry and study the doar field equations in a simple case—gravity in empty space. Then we
construct new geometric formalism for gravity equation in empty space.

In general relativity Einstein made the assumption that gravity equation in empty space is \[ R_{\mu\nu} = 0, \] (1)
where \( R_{\mu\nu} \) is Ricci tensor, which is symmetrical.

In Minkowski space-time, Dirac equation is usually written as \((\hbar = c = 1)\) \[ (i\gamma^a \frac{\partial}{\partial x^a} - m) \psi(x) = 0. \quad a = 0, 1, 2, 3. \] (2)
Where \( \gamma \)'s are Dirac matrices, which satisfy \[ \gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}. \] (3)
Where \( \eta^{ab} \) is the metric tensor of Minkowski space-time, i.e.
\[ \eta^{00} = +1, \quad \eta^{11} = \eta^{22} = \eta^{33} = -1, \quad \eta^{ab} = 0 \quad \text{for} \quad a \neq b. \] (4)
Eq.(2) describes a free massive spinor field.

Physicists will find many advantages in Eq.(2) when he tries to realize the quantization of fields. Dirac equation is a one-order differential equation. The spinor \( \psi(x) \) is complex, which is consistent with the wave function in Schrödinger equation in quantum mechanics. Comparing Einstein equation (1) with Dirac equation (2), we believe that quantizing gravitation needs to reformulate gravity theory such that new formalism at least includes two properties: (1)the complexified field; (2)the reduction of the order of the field equation.

In this paper we propose a basic principle that inner symmetries arise from the extended local symmetry of space-time. The complexified vierbeins (or tetrads) should be treated as the fundamental field. Then we give the locally complexified geometry and find that SU(1,3) Yang-Mills field appears as a natural result without introducing

\(^1\)Here “empty” means that there is no matter present and no physical fields except the gravitational field.
extra-dimension. Furthermore we give a new geometric formalism for gravity equation in empty space which looks more like Dirac equation. At the end we conclude the idea on daor field and the main results of this paper.

2 Basic Principles

At first we propose two basic principles: (1)Our space-time is a 3 + 1 manifold, which looks like a Minkowski space-time around each point. (2)The intrinsic distance

\[ ds^2 = dx^\mu g_{\mu\nu} dx^\nu \]  

(2.1)

is invariant under any physical transformations. One kind of these local transformations is corresponding to one kind of interactions.

In the literature, decomposing the curvilinear metric into vierbeins or tetrads \( e^a_\mu(x) \) has been used extensively [12, 13]. But the vierbein decomposition

\[ g_{\mu\nu} = \eta_{ab} e^a_\mu(x) e^b_\nu(x) \]  

(2.2)

just keeps \( ds^2 \) invariant under local SO(1, 3) group transformation, which is not the largest symmetry group transformation as we will show. Now we will discuss the larger symmetry transformation which keeps \( ds^2 \) invariant. To do so, we introduce a complex vierbein field \( h^a_\mu \) or \( H^a_\mu \), which satisfies

\[ 2g_{\mu\nu} = \bar{h}^a_\mu \eta_{ab} h^b_\nu + h^a_\mu \eta_{ab} \bar{h}^b_\nu \]  

(2.3)

\[ 2g^{\mu\nu} = \bar{H}_a^\mu \eta^{ab} H_b^\nu + H_a^\mu \eta^{ab} \bar{H}_b^\nu \]  

(2.4)

\[ g_{\mu\nu} g^{\nu\lambda} = g^{\lambda\nu} g_{\nu\mu} = \delta^\lambda_\mu \]  

(2.5)

where bar denotes complex conjugation. It is stressed that \( g_{\mu\nu} \) is still real and symmetrical in above equations. About half a century ago, Einstein and Strauss constructed their theory under a complexified geometry where the metric is hermitian and thus

\[ x^\mu's(\mu = 0, 1, 2, 3) \] are a system of curvilinear coordinates of space-time manifold.

\[ ^\text{§} \] In this paper, using Roman suffixes to refer to the bases of local Minkowski frame; using Greek suffixes to refer to the space-time coordinates.
asymmetry [14]. After them Schrödinger explained this theory explicitly in his famous book [15]. Following the method of complexified geometry, Penrose proposed the method of twistor to argue the quantization of space-time [16]. The main development on this way before 1980s was reviewed by Israel [17]. In the process of preparing this paper, we found that Ali H. Chamseddine had used complex vierbein to construct gravity theory under noncommutative geometry [18].

At the first blush, Eq.(2.3) looks like the following formula\footnote{Since Fock first used this relation to study the Dirac equation on Riemannian space-time [19], this equation had been always adopted by many physicists.}

\begin{equation}
2g_{\mu\nu}(x) = \gamma_{\mu}(x)\gamma_{\nu}(x) + \gamma_{\nu}(x)\gamma_{\mu}(x).
\end{equation}

Similar to Eq.(3) it is obvious that $\gamma_{\mu}(x)$ in Eq.(2.6) should be matrices in the spin space. But $h^b_{\nu}(x)$ in Eq.(2.3) don’t relate to the spin space. The complex vierbein and complex spin connection have appeared in twistor theory [16] or in some formulae in loop quantum gravity [3]. We directly use the complexification of vierbein to reformulate the gravity equation in empty space and argue the possibility of accommodating SU(1,3) Yang-Mills field in our framework in this paper.

To embody the linking character of $h^a_{\mu}(x)$ between matter fields and the space-time structure, also between gravitation and gauge interactions, here we intend to give the complex vierbein (or tetrad) field $h^a_{\mu}(x)$ a new name. “Dao” is a basic and important concept in ancient Chinese philosophy. “Dao” is used to refer not only the unobservable existence from which everything originate but also the laws which dominate the doom of everything. “Dao” is also used to demonstrate the abstract relationship between the dual things such as “Yin” and “Yang”, nihility and existence. Since $h^a_{\mu}(x)$ plays such a similar role in physics, which will be discussed in the following, we suggest calling $h^a_{\mu}(x)$ the “daor field”.

The metric is symmetrical in general relativity. Thus, there are at most 10 free-parameters in curvilinear metric $g_{\mu\nu}$. Eq.(2.3) and Eq.(2.4) demonstrate that there are too much free-parameters in the daor field. To cancel nonphysical freedom, we require
that the following covariant constraint should be satisfied

\[ \bar{h}_\mu^a \eta_{ab} h^b_{\nu} = h_\mu^a \eta_{ab} \bar{h}^b_{\nu} . \]  

(2.7)

It is obvious that Eq.(2.7) can be rewritten equivalently

\[ \bar{H}_\mu^a \eta^{ab} H^b_{\nu} = H_\mu^a \eta^{ab} \bar{H}^b_{\nu} . \]  

(2.8)

Then Eq.(2.3) and Eq.(2.4) become

\[ g_{\mu\nu} = \bar{h}_\mu^a \eta_{ab} h^b_{\nu}, \quad g^{\mu\nu} = \bar{H}_\mu^a \eta^{ab} H^b_{\nu} . \]  

(2.9)

Define a tensor \( N_{\mu\nu} \) to be

\[ 2N_{\mu\nu} = \bar{h}_\mu^a \eta_{ab} h^b_{\nu} - h_\mu^a \eta_{ab} \bar{h}^b_{\nu} . \]  

(2.10)

Eq.(2.10) shows that the tensor \( N_{\mu\nu} \) is antisymmetrical, namely \( N_{\mu\nu} = -N_{\nu\mu} \). Thus the covariant constraint \( N_{\mu\nu} = 0 \) only provides 6 independent constraint equations to the components of the daor field. That is to say, the daor field can at most have 26 independent free-parameters.

Consider general real coordinate transformations \( x \rightarrow x'(x) \), since

\[ dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu, \quad h'^h_{\mu} = h^b_{\nu} \frac{\partial x'^\mu}{\partial x^\nu}, \]  

(2.11)

the intrinsic distance is thus invariant under general coordinate transformations.

Under the rotation of the locally complexified Minkowski frame, the daor field \( h^b_{\nu}(x) \) transforms as follows

\[ h'^{a}_{\nu}(x) \rightarrow h'^{a}_{\nu}(x) = S^a_b(x)h^b_{\nu}(x) . \]  

(2.12)

if the matrix \( S^a_b(x) \) satisfies

\[ \bar{S}_a^c(x) \eta_{cd} S^d_b(x) = \eta_{ab} , \]  

(2.13)

namely, \( S^a_b(x) \) being the element of \( U(1,3) \) group, then the intrinsic distance is invariant under the rotation of the local complexified Minkowski frame.
Hence we can draw a conclusion: by introducing the complex daor field, we find that the intrinsic distance and the covariant constraint are invariant under two kinds of transformations: One is the general coordinate transformation \( x \rightarrow x'(x) \); The other is the U(1,3) transformation of the Roman suffixes.

3 Daor Geometry

It is well known that the aim of geometry is to study the invariant properties of the continuous point set \( E \) (called space) under some symmetry transformations. So, from the geometric point of view, there are also a kind of geometry in which the distance defined by Eq.(2.1) is invariant under two kinds of symmetry transformations discussed in the last section.

By defining the Hermitian conjugate of the daor field \( h^\mu_a \) or \( H^\mu_a \)

\[
\begin{align*}
\hat{h} = (h^\mu_a)^\dagger = (\bar{h}^\mu_a) , & \quad \hat{H} = (H^\mu_a)^\dagger = (\bar{H}^\mu_a) , \\
\end{align*}
\]

(3.1)

from Eq.(2.5) and Eq.(2.9) we can easily acquire the following relations

\[
\begin{align*}
g = \hat{h} \eta h , & \quad g^{-1} = \hat{H} \eta H , & \quad \hat{H} = h^{-1} , & \quad H^{-1} = h^\dagger . \\
\end{align*}
\]

(3.2)

To provide an algebraic preparation for an advanced study in daor manifold, we will study multilinear complex algebra without considering the covariant constraint first. Now let’s define \( l\)

\[
\begin{align*}
h^a = h^\mu_a dx^\mu , & \quad \hat{h}^a = dx^\mu \bar{h}^\mu_a , & \quad H^\mu_a = \bar{H}^\mu_a \partial_\mu , & \quad H_a = H^\mu_a \partial_\mu , \\
\end{align*}
\]

(3.3)

where \( h^a \) and \( \hat{h}^a \) are daor field 1-forms. Here we choose the set \( \{ H^\mu_0, H^\mu_1, H^\mu_2, H^\mu_3 \} \) to be the basis of locally complexified Minkowski frame because they are linearly independent everywhere. Thus any vector of locally complexified Minkowski space-time \( M \) can be uniquely expressed by a linear combination of \( H^\mu_a \)'s. Suppose \( f : M \rightarrow C \) is a \( C \)-valued** function on \( M \). If for any \( Z_1, Z_2 \in M \) and \( \alpha^1, \alpha^2 \in \mathbb{C} \),

\[
f(\alpha^1 Z_1 + \alpha^2 Z_2) = \alpha^1 f(Z_1) + \alpha^2 f(Z_2) ,
\]

(3.4)

\[\text{We use } \partial_\mu \text{ to denote the partial differential operator } \frac{\partial}{\partial x^\mu} \text{ for brevity.}
\]
\[\text{** } \mathbb{C} \text{ refers to the field of complex numbers in this paper.}\]
then $f$ is called a $C$-valued linear function on $M$. It is easy to prove that the set of all $C$-valued linear functions on $M$ forms a vector space over $C$, called the dual space of $M$ [20], denoted by $M^*$. The Hermitian conjugated local Minkowski space-time is denoted by $M^\dagger$, in which the set $\{H_0, H_1, H_2, H_3\}$ is a basis.

By using the same definition of inner product as in real manifold, namely

$$<\partial_\mu, dx^\nu> = \delta_\mu^\nu,$$  \hspace{1cm} (3.5)

from Eq.(3.2) and Eq.(3.5), we can obtain the inner product of $h^a$ and $H^b_a$ as follows

$$<H^b_a, h^a> = <H_a^b, h^a> = \delta_a^b.$$  \hspace{1cm} (3.6)

Eq.(3.6) demonstrates that the set $\{h^0, h^1, h^2, h^3\}$ is a basis of $M^*$. Obviously, $M^*$ and $M$ are dual spaces of each other.

Furthermore, the inner product of the vector $U = U^\mu \partial_\mu$ and the covector $V = V_\nu dx^\nu$ can be expressed as follows

$$<U, V> = <V, U> = U^\mu V_\mu = V_\mu U^\mu = V_a U_a = \bar{U}_a \bar{V}_a.$$  \hspace{1cm} (3.7)

Where $\bar{U}_a$, $U^a$, $\bar{V}_a$ and $V_a$ are given by

$$U^a = h^a_\mu U^\mu, \quad \bar{U}^a = U^\mu \bar{h}_a^\mu, \quad V_a = V_\mu \bar{H}_a^\mu, \quad \bar{V}_a = H_a^\mu V_\mu.$$  \hspace{1cm} (3.8)

For brevity, we introduce the signs $U^\bar{a}$ and $V_\bar{a}$ defined by

$$U^\bar{a} = \bar{U}^a = U^\mu \bar{h}_a^\mu, \quad V_\bar{a} = \bar{V}_a = H_a^\mu V_\mu.$$  \hspace{1cm} (3.9)

The vector $U = U^a H^b_a$ and the covector $V = V_a h^a$ are obviously invariant under two kinds of transformations.

Next let’s give the notion of tensors. The elements in the tensor product

$$M^r_s = \bigotimes^{r \mathrm{ terms}} M \otimes \cdots \otimes M \otimes \bigotimes^{s \mathrm{ terms}} M^* \otimes \cdots \otimes M^*,$$  \hspace{1cm} (3.10)

are called $(r, s)$-type tensors, where $r$ is the contravariant order and $s$ is the covariant order. The elements in $M^r_0$ are called contravariant tensors of order $r$, and those in $M^0_s$
are called covariant tensors of order \( s \). We also use the following conventions: \( M_0^0 = C \), \( M_0^1 = M \), \( M_1^0 = M^* \). Since \( \{H_i^\dagger\}_{0 \leq i \leq 3} \) and \( \{h^i\}_{0 \leq i \leq 3} \) are dual bases in \( M \) and \( M^* \) respectively, then

\[
H_{i_1}^\dagger \otimes \cdots \otimes H_{i_r}^\dagger \otimes h^{j_1} \otimes \cdots \otimes h^{j_s}, \quad 0 \leq i_1, \ldots, i_r, j_1, \ldots, j_s \leq 3,
\]

form a basis of \( M_r^s \). Therefore an \((r,s)\)-type tensor \( T \) can be uniquely expressed as

\[
T = T_{i_1 \cdots i_r}^{i_1 \cdots i_r} H_{i_1}^\dagger \otimes \cdots \otimes H_{i_r}^\dagger \otimes h^{j_1} \otimes \cdots \otimes h^{j_s},
\]

where the \( T_{i_1 \cdots i_r}^{i_1 \cdots i_r} \) are called the components of the tensor \( T \) under the basis (3.11).

In the curvilinear coordinate a real \((r,s)\)-type tensor \( P \) is uniquely expressed as

\[
P = P_{\mu_1 \cdots \nu_s}^{\mu_1 \cdots \nu_s} \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_r} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_s}.
\]

From Eq.(3.3) and Eq.(3.8) we therefore acquire

\[
P_{\nu_1 \cdots \nu_s}^{\mu_1 \cdots \mu_r} = P_{j_1 \cdots j_s}^{i_1 \cdots i_r} H_{i_1}^\dagger \cdots H_{i_r}^\dagger h^{j_1}_{\nu_1} \cdots h^{j_s}_{\nu_s}.
\]

Suppose \( J \) is an \((r_1, s_1)\)-type tensor and \( K \) is an \((r_2, s_2)\)-type tensor. Then their tensor product \( J \otimes K \) is an \((r_1 + r_2, s_1 + s_2)\)-type tensor, and the components of \( J \otimes K \) are the products of the components of \( J \) and \( K \), i.e.,

\[
(J \otimes K)_{i_1 \cdots i_{r_1 + r_2}}^{j_1 \cdots j_{s_1 + s_2}} = J_{i_1 \cdots i_{r_1}}^{j_1 \cdots j_{s_1}} K_{j_1 + i_{r_1 + 1} \cdots j_{s_1 + s_2}}^{i_{r_1 + 1} \cdots i_{r_1 + r_2}}.
\]

The multiplication of tensors satisfies the distributive and associative laws.

A special class of tensors, the totally skew-symmetric covariant tensors have played an important role in the study of manifolds. We begin by defining Cartan’s exterior product as the antisymmetric tensor product of cotangent space basis elements \( h^a \). For instance, the exterior product of covector fields, which is a skew-symmetric linear mapping, called 2-form and constitute a space \( \Lambda^2(M) \). The bases of the 2-form space \( \Lambda^2(M) \) are

\[
h^a \wedge h^b = (h^a \otimes h^b - h^b \otimes h^a).
\]
Then any 2-form $\alpha_2 \in \Lambda^2(M)$ can be written as

$$
\alpha_2 = \frac{1}{2} f_{ab} h^a \wedge h^b, \quad f_{ab} = -f_{ba} .
$$

Let $\Lambda^r(M)$ ($r = 0, \ldots, 4$) be the set of skew-symmetric $M^0_r$ tensors. This is a vector space of dimension $\frac{4!}{r!(4-r)!}$. Therefore, any $r$-form $\alpha_r \in \Lambda^r(M)$ can be expressed as

$$
\alpha_r = \frac{1}{r!} f_{i_1 \cdots i_r} h^{i_1} \wedge \cdots \wedge h^{i_r} ,
$$

(3.18)

where function $f_{i_1 \cdots i_r}$ is completely skew-symmetric with respect to its subscripts, and $\Lambda^1(M) = M^*$ is the set of complex cotangent vector fields. Denote the formal sum

$$
\sum_{r=0}^{4} \Lambda^r(M)
$$

by $\Lambda^*(M)$. Then $\Lambda^*(M)$ is a 24-dimensional vector space. Let

$$
\alpha = \sum_{r=0}^{4} \alpha_r , \quad \beta = \sum_{s=0}^{4} \beta_s ,
$$

(3.19)

where $\alpha_r \in \Lambda^r(M), \beta_s \in \Lambda^s(M)$. Define the exterior product of $\alpha$ and $\beta$ by

$$
\alpha \wedge \beta = \sum_{r,s=0}^{4} \alpha_r \wedge \beta_s .
$$

(3.20)

Then $\Lambda^*(M)$ becomes an algebra with respect to the exterior product, and is called the exterior algebra or Grassman algebra of $M$. Obviously, $\alpha_r$ and $\beta_s$ satisfy $\alpha_r \wedge \beta_s = (-1)^{rs} \beta_s \wedge \alpha_r$.

Using the tool of exterior product, we then rewrite the covariant constraint as

$$
\eta_{ab} \tilde{h}^a \wedge h^b = 0 , \quad \eta_{ab} H^a \wedge H^b = 0 .
$$

(3.21)

Another useful tool for manipulating differential forms is the exterior differentiation $d$, which is a differential operator within Cartan exterior algebra $\Lambda^*(M)$, and defined by:

$$
d : \Lambda^r(M) \longrightarrow \Lambda^{r+1}(M) .
$$

(3.22)

For a complex function $f(x^\mu)$, its exterior differentiation is expressed as

$$
d f = f_\alpha h^\alpha \quad (f_\alpha \equiv \frac{\partial f}{\partial h^\alpha}) .
$$

(3.23)
For a $p$-form $\alpha_p = \frac{1}{p!} f_{a_1 \cdots a_p} h^{a_1} \wedge \cdots \wedge h^{a_p}$, its exterior differentiation is defined as follows

$$d\alpha_p = \frac{1}{p!} f_{a_1 \cdots a_p, k} h^k \wedge h^{a_1} \wedge \cdots \wedge h^{a_p}.$$  
(3.24)

Let $\alpha_p \in \Lambda^p(M)$, $\beta_q \in \Lambda^q(M)$. Making use of Leibniz rule for differentiation of functions, it can be proved that

$$d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q.$$  
(3.25)

The inner product Eq.(3.6) can be extended to the inner product between a vector field $Y$ and a $k$-form $\alpha_k$ as follows

$$i_Y \alpha_k \equiv \langle Y, \alpha_k \rangle \in \Lambda^{k-1}(M).$$  
(3.26)

Where the operator $i_Y$ acts on the differential forms only. Express the vector field $Y$ as $Y = \zeta^i H_i^\dagger$, it is easy to obtain

$$i_Y \alpha_k = \langle Y, \alpha_k \rangle = \frac{1}{(k-1)!} \zeta^a f_{a_1 \cdots a_k} h^{a_2} \wedge \cdots \wedge h^{a_k}.$$  
(3.27)

having constructed fundamental algebraic tools, we will study the geometric properties of the locally complexified manifold in the next section.

4 Daor Connection

In the following, we will consider a special subset of the daor field $h^a_\mu(x)$ in which an element $k^a_\mu(x)$ is expressed in terms of the real vierbeins as

$$k^a_\mu(x) = l^a_b(x) e^b_\mu(x),$$  
(4.1)

where the matrix $l^a_b(x)$ satisfies Eq.(2.13), namely, $l^\dagger \eta l = \eta$. It is obvious that the matrix $k^a_\mu(x)$ satisfies the covariant constraint Eq.(2.7).

Being similar with Eq.(3.3), It can be defined that

$$e^a = e^a_\mu dx^\mu, \quad k^a = l^a_b e^b, \quad k^a = e^b_l^a, \quad K^a = \bar{K}^a_\mu \partial_\mu, \quad K_a = K^a_\mu \partial_\mu.$$  
(4.2)
where $\bar{K}^\mu_a$ and $K^\mu_a$ are given by

\[
K^\dagger = k^{-1}, \quad K^{-1} = k^\dagger. \quad (4.3)
\]

In order to “differentiate” vector fields on the space-time manifold, we need to introduce a structure called the connection on a vector bundle. When the linear connection is given on the cotangent bundle, there is a continuous linear mapping between sections of the tensor bundle, namely:

\[
\nabla : \, \mathbf{M}_s^r \rightarrow \mathbf{M}_{s+1}^r, \quad J \in \mathbf{M}_s^r \rightarrow \nabla J \in \mathbf{M}_{s+1}^r. \quad (4.4)
\]

$\nabla J$ is called the covariant differentiation of tensor field $J$ [21]. The mapping $\nabla$ satisfies

1) Linearity:

\[
\nabla(aJ + bJ') = a\nabla J + b\nabla J', \quad a, b \in \mathbb{C}. \quad (4.5)
\]

2) Leibniz rule:

\[
\nabla(J \otimes J') = (\nabla J) \otimes J' + J \otimes (\nabla J'), \quad (4.6)
\]

\[
\nabla <Y, \alpha_r> = <\nabla Y, \alpha_r> + <Y, \nabla \alpha_r>. \quad (4.7)
\]

3) For function $f \in \mathbf{M}_0^0$,

\[
\nabla f = df. \quad (4.8)
\]

4) For cotangent vector field $\alpha_1 = f_a(x)k^a(x)$, when the daor field is chosen,

\[
\nabla \alpha_1 = df_a \otimes k^a + f_a \nabla k^a. \quad (4.9)
\]

From Eq.(4.9), $\nabla \alpha_1$ can be calculated if covariant differentiation $\nabla k^a$ of a daor field is given. $\nabla k^a$ denote the infinitesimal variance of the daor field $k^a$ at the neighborhood of a point and can be expressed as

\[
\nabla k^a = (\nabla l^a_b) e^b + l^a_b (\nabla e^b) = -l^a_c B^c_b e^b - l^a_b \theta^b c e^c = -l^a_b \omega^b c e^c, \quad (4.10)
\]

where

\[
\omega^a_b = B^a_b + \theta^a_b = - <K^\dagger_b, \nabla k^a> = \omega^a_b k^i. \quad (4.11)
\]
Because $\omega^a_b$ are complex matrix valued 1-forms, we suggest calling $\omega^a_b$ “daor connection” 1-forms. From Eq.(4.9) and Eq.(4.10), we obtain

$$\nabla \alpha_1 = df_a l^a_b \ e^b - f_a l^a_b \omega^b_c \ e^c = l^a_b \left( df_a \delta^b_c - f_a \omega^b_c \right) e^c .$$  \hspace{2cm} (4.12)

Using Eq.(3.6) and Eq.(4.7), daor connections on the tangent bundle can be induced from daor connections on the cotangent bundle. Since $\{K^\dagger_a\}$ are the dual bases of $\{k^a\}$, it is easy to prove that

$$< \nabla K^\dagger_b, k^a >= - < K^\dagger_b, \nabla k^a >= \omega^a_b ,$$ \hspace{2cm} (4.13)

or equivalently, $\nabla K^\dagger_b = (l^{-1})^b_a \omega^d_c \ (e^{-1})^a_d \partial_\mu$. Therefore, the covariant differentiation of a tangent field $Y = \zeta^a(x) K^\dagger_a(x)$ is

$$\nabla Y = (l^{-1})^b_a \left( d\zeta^a_\delta^b_c + \zeta^a_\omega^b_c \right) (e^{-1})^a_d \partial_\mu .$$ \hspace{2cm} (4.14)

Similarly, covariant differentiation of any $(r,s)$-type tensor fields can be carried out yielding $(r,s + 1)$-type tensor fields.

It is well known that the $\theta^a_b$ defined in Eq.(4.10) is the spin connection introduced first by Cartan. From the viewpoints of Yang-Mills gauge field, the daor connection $\omega^a_b$ is the field strength of $\text{SU}(1,3) \times \text{SO}(1,3)$ gauge field. Or equivalently, in the language of differential geometry, the $\omega^a_b$ is the connection on $\text{SU}(1,3) \times \text{SO}(1,3)$ principal bundle.

The curvature of this principal bundle thus is expressed as

$$\Omega^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b .$$ \hspace{2cm} (4.15)

Since Eq.(4.11), the curvature $\Omega^a_b$ can be written as

$$\Omega^a_b = R^a_b + F^a_b + \theta^a_c \wedge B^c_b + B^a_c \wedge \theta^c_b ,$$ \hspace{2cm} (4.16)

where $R^a_b$, $F^a_b$ are the curvature of $\text{SO}(1,3)$ principal bundle and the curvature of $\text{SU}(1,3)$ principal bundle respectively. The definitions of $R^a_b$ and $F^a_b$ are

$$R^a_b = d\theta^a_b + \theta^a_c \wedge \theta^c_b ,$$ \hspace{2cm} (4.17)

$$F^a_b = dB^a_b + B^a_c \wedge B^c_b .$$ \hspace{2cm} (4.18)
Introducing SU(1,3) gauge field $\tilde{B} = \frac{1}{i\lambda} B$ and its field strength $\tilde{F} = \frac{1}{i\lambda} F$, we acquire the well-known relation

$$\tilde{F}^a_b = d\tilde{B}^a_b + i\lambda \tilde{B}^a_c \wedge \tilde{B}^c_b , \quad (4.19)$$

where $\lambda$ is the coupling constant of the SU(1,3) gauge field. In quantum field theory the freedom of gauge field is too much to describe the physical system. Theorists must introduce some kind of gauge fixing condition, such as Coulomb gauge in QED or Landau gauge in QCD, to give the observable physical results. In our framework we propose the gauge fixing condition as follows

$$\theta^a_c \wedge \tilde{B}^c_b + \tilde{B}^a_c \wedge \theta^c_b = 0 , \quad (4.20)$$

then the curvature $\Omega^a_b$ reduces to

$$\Omega^a_b = R^a_b + i\lambda \tilde{F}^a_b . \quad (4.21)$$

Eq.(4.10) shows that there are two categories of local gauge transformations: One is the local SU(1,3) group transformation. The covariant principle naturally leads to the necessary input of SU(1,3) gauge field; The other is the local SO(1,3) group transformation. The spin connection represents the effect of gravitation.

First, let us consider an intrinsic rotation of the daor field

$$k^a \rightarrow k'^a = l^a_b e^b = S^a_b k^b , \quad (4.22)$$

where $S^a_b$ satisfies Eq.(2.13), namely, $S^a_b$ is a faithful representation of SU(1,3) group. $l^a_b$ and $e^b$ are defined by $l^a_b = S^a_c l^c_d (S^{-1})^d_b$ and $e^b = S^b_i e^i$ respectively. From reference [22], it is known that under the intrinsic SU(1,3) rotation of daor field the daor connection 1-form $\omega^a_b$ transforms as follows

$$\omega'^a_b = S^a_c \omega^c_d (S^{-1})^d_b + S^a_c (dS^{-1})^c_b , \quad (4.23)$$

Since $B^a_b$ is the connection of SU(1,3) principal bundle, under the SU(1,3) gauge rotation of the daor field, the $B^a_b$ transforms into

$$B'^a_b = S^a_c B^c_d (S^{-1})^d_b + S^a_c (dS^{-1})^c_b , \quad (4.24)$$
and $\theta^a_b$ satisfies
\[ \theta^a_b = S^a_c \theta^c_d (S^{-1})^d_b. \] (4.25)

Furthermore, it is easy to prove that under this rotation $F^a_b$ and $\Omega^a_b$ become
\[ F^a_b = S^a_c F^c_d (S^{-1})^d_b, \quad \Omega^a_b = S^a_c \Omega^c_d (S^{-1})^d_b. \] (4.26)

Secondly, consider an orthogonal rotation of the real orthonormal vierbein
\[ e^a \rightarrow e'^a = \Phi^a_b e^b, \] (4.27)

where $\Phi^a_b$ satisfies
\[ \Phi^a_c \eta_{ab} \Phi^b_d = \eta_{cd}. \] (4.28)

Eq.(4.28) demonstrates that the $\Phi^a_b(x)$ is a representation of SO(1,3) group. The daor field transforms as
\[ k^a \rightarrow k'^a = l'^a_c e'^c = \Phi^a_b k^b, \] (4.29)

where $l'^a_c = \Phi^a_b l^b_c (\Phi^{-1})^c_e$. After this transformation, the new daor connection is
\[ \omega^a_b = \Phi^a_c \omega^c_d (\Phi^{-1})^d_b + \Phi^a_c (d\Phi^{-1})^c_b, \] (4.30)

Similarly, we acquire
\[ \theta^a_b = \Phi^a_c \theta^c_d (\Phi^{-1})^d_b + \Phi^a_c (d\Phi^{-1})^c_b, \] (4.31)

and
\[ B^a_b = \Phi^a_c B^c_d (\Phi^{-1})^d_b. \] (4.32)

The transformation law for the curvatures $R^a_b$ and $\Omega^a_b$ under the orthogonal rotation of the real vierbein is given by
\[ R^a_b = \Phi^a_c R^c_d (\Phi^{-1})^d_b, \quad \Omega^a_b = \Phi^a_c \Omega^c_d (\Phi^{-1})^d_b. \] (4.33)
5 Daor Field Equations in Empty Space

We discuss the daor field formalism for gravity equation in empty space in this section. Considering the conditions of complexified field and of reducing the order of the field equations, we propose the daor field equations in empty space as follows

\[(\delta^a_b d + \omega^a_b \wedge)k^b = 0 , \quad (5.1)\]
\[\omega_{ab} = \pm \frac{1}{2} \epsilon_{abcd} \omega^{cd} . \quad (5.2)\]

Where \(\epsilon_{abcd}\) is the totally antisymmetric tensor in 4-dimensions [22]. By defining the torsion operator 1-form

\[\hat{T} \equiv \delta^a_b d + \omega^a_b \wedge , \quad (5.3)\]
we can rewrite Eq.(5.1) as \(\hat{T}k = 0\). Multiplying both sides of Eq.(5.1) by the torsion operator \(\hat{T}\) yields

\[\hat{T}\hat{T}k = (d\omega^a_b + \omega^a_c \wedge \omega^c_b) \wedge k^b = \Omega^a_b \wedge k^b = 0 . \quad (5.4)\]

This equation shows that the operator \(\hat{T}\) can be regarded as the square root of the curvature 2-form.

In Cartan’s vierbein method, the Levi-Civita affine connection is obtained by requiring that the real spin connection \(\theta_{ab}\) satisfies the following conditions

\[no \ torsion : \quad de^a + \theta^a_b \wedge e^b = 0 , \quad (5.5)\]
\[metricity : \quad \theta_{ab} = -\theta_{ba} . \quad (5.6)\]

\(\theta^a_{b\mu}\) is then determined in terms of the vierbeins and inverse vierbeins and is related to the Levi-Civita affine connection by

\[\theta^a_{b\mu} = -(e^{-1})^\nu_b e^a_{\nu\mu} = -(e^{-1})^\nu_b (\partial_\mu e^a_\nu - \Gamma^\lambda_{\nu\mu} e^a_\lambda) . \quad (5.7)\]

Where \(\Gamma^\lambda_{\nu\mu}\) is Levi-Civita affine connection, which is real and uniquely determined by the space-time metric

\[\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^\lambda_\psi (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) . \quad (5.8)\]
Similarly, we extend the metricity and no torsion conditions to the cases where the total connection is complex. Namely the complex spin connection $\omega_{ab}$ satisfies $\omega_{ab} = -\omega_{ba}$ and $T^a = dk^a + \omega^a_b \wedge k^b = 0$. Hence $\omega^a_{b\mu}$ can be expressed in terms of the daor field

$$\omega^a_{b\mu} = k^a_\nu \bar{K}^\nu_{b\mu} = -\bar{K}^\nu_{b}(\partial k_a^\nu - \Gamma^\lambda_{\nu\mu} k^a_\lambda).$$  \hspace{1cm} (5.9)

It is obvious that Eq.(5.1) is generalized torsion-free condition $T^a = 0$ and Eq.(5.2) implies generalized metricity condition $\omega_{ab} = -\omega_{ba}$.

Einstein’s empty space equation (1) may be rationally generalized as follows [22]

$$\tilde{\Omega}^a_b \wedge k^b = 0,$$  \hspace{1cm} (5.10)

where $\tilde{\Omega}_{ab}$ is the dual of $\Omega_{ab}$, which is defined by

$$\tilde{\Omega}_{ab} = \frac{1}{2} \epsilon_{abcd} \Omega^{cd}.$$  \hspace{1cm} (5.11)

The dual of the complex connection is defined by

$$\tilde{\omega}_{ab} = \frac{1}{2} \epsilon_{abcd} \omega^{cd}.$$  \hspace{1cm} (5.12)

From Eq.(4.15) we notice that $\Omega_{ab}$ is (anti-)self-dual, namely, $\tilde{\Omega}_{ab} = \pm \Omega_{ab}$, if $\omega_{ab}$ is (anti-)self-dual $\omega_{ab} = \pm \tilde{\omega}_{ab}$. Since Eq.(5.2) shows that $\omega_{ab}$ is (anti-)self-dual, then Eq.(5.10) can be deduced from Eq.(5.4).

We have proved that the daor field equations (5.1) and (5.2) are equivalent to Eq.(5.10). If the coupling between the SU(1,3) gauge field and the daor field must be ignored, then Eq.(5.10) reduces to Eq.(1). In this case spin connection $\omega_{ab}$ is real. This demonstrates that there are no observable physical phenomena on SU(1,3) gauge fields. But in more general cases, $\omega_{ab}$ must be complex. Adding the stress-energy tensor of gauge fields in Einstein’s equation should give the couplings between the daor field and gauge fields. More results on this problem will be given in the forthcoming paper [23].

It is stressed that only daor field can embody all the symmetries of space-time. As the complex spin connection unifies gauge fields and the Cartan’s real spin connection, the daor field reflects the gravitational effect of gauge fields also. Furthermore, the daor field will be a powerful tool to realize the quantization of gravity.
6 Conclusion

In this paper, the daor field which represent gravity is suggested. There are two kinds of symmetry transformations keeping $ds^2$ invariant. Upon these local symmetries we set up the locally complexified geometry. In the complex connection, the real spin connection and the SU(1,3) field strength are unified. Hence we incorporate the SU(1,3) gauge field with the gravitation. In the last one-order differential equations of the daor field in empty space are acquired, being proven to be consistent with Einstein’s empty space equation.

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References


