Energy Conservation and Hawking Radiation

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Abstract

The conservation of energy implies that an isolated radiating black hole cannot have an emission spectrum that is precisely thermal. Moreover, the no-hair theorem is only approximately applicable. We consider the implications for the black hole information puzzle.

1. Introduction

Stephen Hawking’s astounding discovery that black holes radiate thermally set up a disturbing and difficult problem: what happens to information during black hole evaporation? Taken literally, Hawking’s result implies the loss of unitarity or, to put it more dramatically, the breakdown of quantum mechanics. Although derivations in string theory support the idea that Hawking radiation can be described within a manifestly unitary theory, it remains a mystery how information is returned. One suggestion is that nonlocality should play an important role; indeed, one expects on general grounds that canonical commutation relations should be modified in the presence of gravity. Here we will explore an alternative hope: that perhaps energy conservation or, more generally, gravitational back-reaction provides a loophole.

At a macroscopic level, the claim of information loss in black hole radiance rests on two pillars: numerous derivations showing that black holes have an exactly thermal emission spectrum, and the validity of the no-hair theorem. A thermal spectrum is entirely determined by specifying a single number, the temperature. So, if exact thermality holds, the outgoing radiation does not have any information. Meanwhile, the no-hair theorem asserts that the geometry outside a stationary black hole is entirely specified by a small handful of parameters: the mass, the charge, the angular momentum, and any other Noether charges. (Indeed, these charges also determine the Hawking temperature). So the spacetime geometry

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doesn’t carry much information either. But if neither the geometry nor the radiation carries any information, then, once the black hole has evaporated, there are no macroscopic signatures of the collapsed matter left. (Of course, a loss of macroscopic information is not in conflict with quantum mechanics. But if we could show that there are coarse-grained features of the outgoing radiation that correlate with the configuration of the collapsed matter, it would demonstrate that at least some information is returned without having to solve the full quantum gravity problem.)

Upon a moment’s reflection, however, we see that both thermality and hairlessness cannot be taken at face value. Either condition, if strictly true, would violate the conservation of energy. A thermal spectrum contains a tail of arbitrarily high energies, but an isolated black hole obviously cannot emit a particle with more energy than the mass of the hole. So energy conservation guarantees that as one goes to higher energies, the spectrum must start deviating from thermality. Put another way, in a microcanonical ensemble, temperature is only a low-energy approximation. Moreover, the conservation of energy demands that, as a quantum of radiation is emitted, the left-over mass of the black hole must decrease, and the hole must shrink. So the no-hair theorem, which describes the possible configurations of unchanging stationary black holes, applies only approximately.

Indeed, energy conservation is not merely a technical detail that needs to be respected. Rather it is what drives the dynamics: a black hole radiates because it can lower its mass. This supports the idea that, in quantum gravity, one should regard a nonextremal black hole as an excited, metastable state. We therefore need a derivation of Hawking radiation that is suited to enforcing energy conservation i.e. for which the spacetime geometry is dynamical. One such derivation directly implements Hawking’s heuristic picture of the radiation as particles tunneling across the horizon. (An alternate viewpoint, in which the radiation is regarded as the spontaneous emissions of a membrane living at the horizon is also possible.) Here we will show that the radiation can indeed be viewed as tunneling particles and that this leads to nonthermality. The corrected emission rate may plausibly lead to short-time correlations in the spectrum.

2. Painlevé Coordinates

To describe tunneling we need a coordinate system that, unlike Schwarzschild coordinates, is regular at the horizon; particularly convenient are Painlevé coordinates. Consider then a general static metric of the form

\[ ds^2 = -(1 - g(r))dt^2 + \frac{dr^2}{1 - g(r)} + r^2d\Omega_{D-2}^2. \] (1)

For a Schwarzschild black hole in four dimensions, \( g(r) = 1 - 2M/r \).

To obtain the new line element, define a new time coordinate, \( \tau \), by \( \tau = t + f(r) \). The function \( f \) is required to depend only on \( r \) and not \( t \), so that the metric remains stationary, i.e. time-translation invariant. Stationarity of the metric automatically
implements the desirable property that the time direction be a Killing vector. Now, our key requirement is that the metric be regular at the horizon. We can implement this as follows. We know that a radially free-falling observer who falls through the horizon does not detect anything abnormal there; we can therefore choose as our time coordinate the proper time of such an observer. As a corollary, we demand that constant-time slices be flat. We then obtain the condition
\[
\frac{1}{1 - g(r)} - (1 - g(r))(f'(r))^2 = 1.
\] (2)

There is no need to integrate this; from \(dt_s = dt + f'(r)dr\), we can read off the Painlevé line element:
\[
ds^2 = -\left(1 - 2M/r\right)dt^2 + 2\sqrt{2M/r}dt \, dr + dr^2 + r^2d\Omega^2.
\] (3)

Similar coordinate systems have been found for de Sitter space\(^7\) and for black holes in anti-de Sitter space\(^8\). The Painlevé metric has a number of attractive features. First, and crucially, none of the components of either the metric or the inverse metric diverge at the horizon. Second, by construction, constant-time slices are just flat Euclidean space. Third, the generator of \(t\) is a Killing vector. “Time” becomes spacelike across the horizon, but is nevertheless Killing. Finally, an observer at infinity does not make any distinction between these coordinates and static coordinates; the function \(f\) that distinguishes the two time coordinates vanishes there. These coordinates cover the inside and outside of the black hole, or half the maximally extended space.

The radial null geodesics in these coordinates obey
\[
\frac{dr}{dt} = \pm 1 - \sqrt{\frac{2M}{r}},
\] (4)

where the plus (minus) sign corresponds to rays that go towards (away from) infinity. When the particle is inside the black hole, both ingoing and outgoing trajectories correspond to decreasing \(r\), and the particle cannot (classically) cross the horizon. For massive particles with worldline tangent \(U^a\) we find, using \(U_t = -1\), that
\[
U^t = 1 \Rightarrow \tau = t + c,
\] (5)

and we see that the Painlevé time coordinate is precisely the proper time, \(\tau\), for a radially free-falling observer.

These equations are modified when the particle’s self-gravitation is taken into account\(^9\). Consider a particle in the s-wave i.e. a shell. If the shell has energy \(E\), then the geometry inside and outside the shell are both Schwarzschild spacetimes, but with different mass parameters. One can now ask which geometry determines the motion of the self-gravitating shell. It turns out that, when the total energy is held fixed, it is the interior \(E\)-dependent metric that determines the motion. That is, we should replace \(M\) with \(M - E\) in the geodesic equation.
3. Tunneling Across the Horizon

The advantage of having a coordinate system that is well-behaved at the horizon is that one can study across-horizon physics. Here we will consider the tunneling of massless shells. The purpose of truncating to the s-wave is that it is then possible to integrate out gravity. For spherical gravity, Birkhoff’s theorem states that the only effect on the geometry that the presence of a shell has, is to provide a junction condition for matching the total mass inside and outside the shell. In other words, the outgoing shell obeys Eq. (4) with the plus sign, and with $M$ replaced by $M - E$ to account for self-gravitation.

Now, because of the infinite blueshift near the horizon, the characteristic wavelength of any wavepacket is always arbitrarily small there, so that the geometrical optics limit becomes an especially reliable approximation. The geometrical optics limit allows us to obtain rigorous results directly in the language of particles, rather than having to use the second-quantized Bogolubov method. In the semi-classical limit, we can apply the WKB formula. This relates the tunneling amplitude to the imaginary part of the particle action at stationary phase. The emission rate, $\Gamma$, is the square of the tunneling amplitude:

$$\Gamma \sim \exp(-2 \text{Im } I) \approx \exp(-\beta E).$$

On the right-hand side, we have equated the emission probability to the Boltzmann factor for a particle of energy $E$. To the extent that the exponent depends linearly on the energy, the thermal approximation is justified; we can then identify the inverse temperature as the coefficient $\beta$.

To calculate the action, first observe that we can formally write it as

$$\text{Im } I = \text{Im } \int_{r_i}^{r_f} p_r \ dr = \text{Im } \int_{r_i}^{r_f} \int_0^{p_r} dp_r \ dr,$$

where $p_r$ is the radial momentum. We expect the initial radius, $r_i$, to correspond roughly to the site of pair-creation, which should be slightly inside the horizon, $r_i \approx 2M$. We expect the final radius, $r_f$, to be slightly outside the final position of the horizon, else the particle would not be able to propagate classically to infinity. So $r_f \approx 2(M - E)$. Because the horizon shrinks, $r_f$ is actually less than $r_i$. Note how self-gravitation is essential to the tunneling picture. Without self-gravitation, particles created just inside the horizon would only have to tunnel just across – an infinitesimal separation – so there wouldn’t be any barrier. But back-reaction results in a shift of the horizon radius; the finite separation between the initial and final radius is the classically-forbidden region, the barrier.

We now eliminate the momentum in favor of energy by using Hamilton’s equation

$$\left. \frac{dH}{dp} \right|_r = \frac{\partial H}{\partial p} = \frac{dr}{dt},$$

where $H$ is the Hamiltonian.
where the Hamiltonian, $H$, is the generator of Painlevé time. Hence within the integral over $r$, one can trade $dp$ for $dH$. The imaginary part of the action is then

$$\text{Im } I = \text{Im } \int_{r_i}^{r_f} \int_0^H \frac{dH'}{dr} dr = -\text{Im } \int_{r_i}^{r_f} \int_0^E \frac{dr}{1 - \frac{2(M - E')}{r}} dE', \quad (9)$$

where the Hamiltonian, $H$, is just $M - E$, and we have substituted the self-gravitating radial geodesic for $dr/dt$. Substituting $u = \sqrt{r}$, and using the Feynman prescription to displace the energy from $E'$ to $E' - i\epsilon$, we have

$$\text{Im } I = -\text{Im } \int_{u_i}^{u_f} \int_0^E \frac{2u^2 du}{u - \sqrt{2(M - E' + i\epsilon)}} dE'. \quad (10)$$

and we see that there is a pole in the upper-half $u$-plane. The integral can be evaluated by deforming the contour around the pole. Note that all real parts, divergent or not, can be discarded since they only contribute a phase. For example, the second member of the pair contributes nothing to the tunneling rate, since it is always classically allowed and therefore has real action. Doing the $u$ integral first we find

$$\text{Im } I = +4\pi \int_0^E dE' (M - E'), \quad (11)$$

where we used $u_i > u_f$ to obtain the right sign. The tunneling rate is therefore

$$\Gamma \sim \exp \left( -8\pi M E \left( 1 - \frac{E}{2M} \right) \right) = \exp(\Delta S). \quad (12)$$

To linear order in $E$, we find that the rate is a Boltzmann factor $\exp(-\beta E)$ with inverse temperature $\beta = 8\pi M$. This is the familiar result. But note that at higher energies the spectrum cannot be approximated as thermal. The precise expression, Eq. (12), can be written as the exponent of the difference in the Bekenstein-Hawking entropy, $\Delta S$, before and after emission.

Note also that Eq. (12) is consistent with an underlying unitary theory. For quantum mechanics tells us that the rate must be expressible as

$$\Gamma(i \rightarrow f) = |M_{fi}|^2 \cdot \text{(phase space factor)}, \quad (13)$$

where the first term on the right is the square of the amplitude for the process. The phase space factor is obtained by summing over final states and averaging over initial states. But the number of final states is just the final exponent of the final entropy, while the number of initial states is the exponent of the initial entropy. Hence

$$\Gamma \sim \frac{e^{S_{\text{final}}}}{e^{S_{\text{initial}}}} = \exp(\Delta S), \quad (14)$$

in agreement with our result. This suggests that the formula we have is actually exact, up to a prefactor.
We have found that energy conservation not only supplies the barrier through which the particle tunnels but also, as anticipated, causes the spectrum to deviate from exact thermality at higher energies. However, the form of the correction is not sufficient by itself to relay information. Consider the emission of two particles $E_1$ and $E_2$, and the emission of one particle with their combined energies, $E_1 + E_2$. We find that

$$\ln (\Gamma_{E_1} \Gamma_{E_2}) = -8\pi \left[ E_1 \left( M - \frac{E_1}{2} \right) + E_2 \left( M - E_1 - \frac{E_2}{2} \right) \right] = \ln \Gamma_{E_1+2},$$

so there is no correlation, at least at late-times. It would be very interesting to see if there are any short-time correlations. In particular, when a particle is emitted there is a relaxation time for the black hole to equilibrate. If another particle is emitted during this time, there might be a correlation that falls off as a function of the time between the two emissions.

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