Non-Abelian Geometric Quantum Memory with Atomic Ensemble

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We study a quantum information storage scheme based on an atomic ensemble with near (also exact) three-photon resonance electromagnetically induced transparency (EIT). Each 4-level-atom is coupled to two classical control fields and a quantum probe field. Quantum information is adiabatically stored in the associated dark polariton manifold. An intrinsic non-trivial topological structure is discovered in our quantum memory implemented through the symmetric collective atomic excitations with a hidden SU(3) dynamical symmetry. By adiabatically changing the Rabi frequencies of two classical control fields, the quantum state can be retrieved up to a non-abelian holonomy and thus decoded from the final state in a purely geometric way.

Quantum information storage is a physical process to encode the state of a quantum system into the state of another system referred to as a quantum memory [1]. Compared to the original quantum system the quantum memory should possess a large decoherence time for effective storing of quantum information. Moreover, the original state of the quantum system should be retrievable from the encoding quantum memory state. By means of quantum memory one can transport quantum information from place to place within the decoherence time. Recently ensemble of Λ-type atoms has been proposed [2–4] as a candidate for practical quantum memory. The idea is to store and transfer quantum information contained in photonic states by the collective atomic excitations. This approach is based on the phenomenon of electromagnetically induced transparency (EIT) [5]. Some experiments [6,7] have already demonstrated the central principle of this technique, namely, the reduction of the group velocity of light.

Most recently a system with quasi-spin wave collective excitations of many Λ-type atoms fixed in "atomic crystal" has been considered as a candidate for a robust quantum memory [8]. A hidden dynamical symmetry of such a system is discovered and it is observed that in certain cases [9] the quantum state can be retrieved up to a non-abelian Berry phase, i.e., a non-abelian holonomy [10–15]. This observation extends the concept of quantum information storage. Quantum information storage of photonic states with this topological character can be implemented in an atomic ensemble with off-resonance EIT. In such a case the stored state can be decoded in a purely geometric way. However, this non-abelian holonomy is in some sense trivial due to the fact that the quantum storage space splits into an orthogonal sum of invariant one dimensional subspaces.

In this letter, we shall describe a quantum information storage protocol based on a truly non-abelian holonomy. To this aim we will consider an ensemble of $N$ 4-level-atoms [13,14], where two meta-stable states are coupled to the excited state by two classical control fields respectively while the ground state is coupled to the excited state by a quantum probe field. In the large $N$ limit with low excitation, a three-exciton system is formed by the symmetric collective excitations from the ground states up to the excited state plus the two virtual excitations from the two meta-stable states to the excited state. It is easy to prove that these three collective excitations indeed behave as three bosons in the large $N$ limit with low excitation. Intertwining between the excited state and two meta-stable ones, the collective operators generate an SU(3) algebra. Based on the spectrum generating algebra theory [16] associated with this $SU(3)$, we construct the degenerate eigen-states of the three-mode exciton-photon system. In particular the collective manifold of dark states can be shown to split into dynamically invariant higher-dimensional subspaces. Using these degenerate eigen-states as a quantum memory, quantum information storage of photonic states can be implemented up to a non-abelian holonomy.

FIG. 1. Four-level atom interacting with a quantum probe field (with coupling constant $g$, frequency $\omega$, and the detuning $\Delta_p$) and two classic control fields (with frequency $\nu_k$, coupling Rabi frequency $\Omega_k$, and the detuning $\Delta_k = \omega_k - \nu_k$, $k = 1, 2$). When $\delta_k (= \Delta_k - \Delta_p)$ are very tiny, the system satisfies the near 3-photon resonance EIT condition.

Our system consists of $N$ identical 4-level-atoms [13,14], where all the atoms are coupled to two single-mode classical control fields and a quantum probe field as shown in Fig. 1. The atomic levels are labelled...
as the ground state $|b\rangle$, the excited state $|a\rangle$, and the meta-stable states $|k\rangle$ ($k = 1, 2$). The atomic transition $|a\rangle \rightarrow |b\rangle$, with energy level difference $\omega_{ab} = \omega_0 - \omega_a$, is coupled to the probe field of frequency $\omega$ ($= \omega_{ab} - \Delta_p$) with the coupling coefficient $g$; and the atomic transition $|a\rangle \rightarrow |k\rangle$ ($k = 1, 2$), with energy level difference $\omega_{ak}$, is driven by the classical control field of frequency $\nu_k$ ($= \omega_{ak} - \Delta_k$) with Rabi-frequency $\Omega_k(t)$.

In the present work we consider the case of $\delta_k$ ($= \Delta_k - \Delta_p$) being very small, that is, those three fields have almost the same detuning with respect to the upper level $|a\rangle$. In view of the physical intuition, each meta-stable state with its relevant control field would constitute a near two-photon resonance EIT if another meta-stable state and its relevant control field do not exist. With the case of two-photon resonance EIT [17–19] (where the control and probe fields have the same detuning) in mind, we would refer to our case of $\Delta_p \simeq \Delta_k$ as a near “$\beta$-photon resonance” EIT.

Under the rotating wave approximation the interaction Hamiltonian can be written as (let $\hbar = 1$) [8]

$$H_I = \Delta_p S + g\sqrt{N}aA^\dagger + \Omega_1 \exp[i\phi_1(t)]T_+^{(1)} + \Omega_2 \exp[i\phi_2(t)]T_+^{(2)} + h.c.,$$  

(1)

where

$$S = \sum_{j=1}^{N} \sigma_{j0}^{(j)}, \quad A = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \sigma_{b0}^{(j)},$$

$$T_+^{(k)} = \sum_{j=1}^{\nu} \sigma_{ka}^{(j)}, \quad T_+^{(k)} = (T_+^{(k)})^\dagger, \quad k = 1, 2,$$  

(2)

are symmetrized collective atomic operators. Here $\sigma_{j\nu}^{(j)} = |j\rangle \mu \langle \nu|$ denotes the flip operator of the $j$-th atom from state $|\nu\rangle_j$ to state $|\mu\rangle_j$ ($\mu, \nu = a, b, 1, 2$); $a^\dagger$ and $a$ the creation and annihilation operators of quantum probe field respectively; and $\delta_k(t) = \delta_k t$. The coupling coefficients $g$ and $\Omega_{1,2}$ are real and assumed to be identical for different atoms in the ensemble. A similar effective Hamiltonian was given in Ref. [8] for the case of an “atomic crystal”, in terms of quasi-spin-wave type collective atomic operators and a hidden dynamical symmetry was there discovered. The symmetrized operators (2) are just a special instance of the quasi-spin-wave operators discussed in [8].

Let us first consider a similar dynamical symmetry in the low excitation regime of the atomic ensemble where most of $N$ atoms stay in the ground state $|b\rangle$ and $N \rightarrow \infty$. It is obvious that $T_+^{(k)}$ and $T_+^{(k)}$ ($k = 1, 2$) generate two mutually commuting $SU(2)$ subalgebras of $SU(3)$ [20]. To form a closed algebra containing $SU(3)$ and $\{A, A^\dagger\}$, we need to introduce two additional collective operators

$$C_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \sigma_{bk}^{(j)}, \quad k = 1, 2$$  

(3)

along with their hermitian conjugates. These operators have the non-vanishing commutation relations $C_k = [A, T_+^{(k)}]$ and $[C_k, T_+^{(k)}] = A$ ($k = 1, 2$). As a special case of quasi-spin wave excitation with zero varying phases, the above three mode symmetrized excitations defined by $A$ and $C_{1,2}$ behave as three independent bosons. Indeed one can check that the operators (3), in the large $N$ limit with low excitation, satisfy the bosonic commutation relations [20]. The commutation relations between the $SU(3)$ algebra and the Heisenberg-Weyl algebra $h$ generated by $A, A^\dagger, C_k$, and $C_k^\dagger$ imply that the dynamical symmetry of evolution governed by $H_I$ can be described by the semi-direct product algebra $SU(3) \otimes h$.

Based on this hidden dynamical symmetry of the interaction Hamiltonian, we can introduce a dark-state polariton operator

$$D = a \cos \theta - C \sin \theta,$$  

(4)

where

$$C = C_1 \exp[i\phi_1(t)] \cos \kappa + C_2 \exp[i\phi_2(t)] \sin \kappa$$  

(5)

is a coherent mixing of two collective atomic excitations $C_1$ and $C_2$, and $\kappa = \arctan \frac{2\sqrt{N}}{\Omega_1}, \quad \theta = \arctan \frac{\sqrt{N}}{\Omega_2}$. In terms of a new operator by

$$T_+ = T_+^{(1)} \exp[i\phi_1(t)] \cos \kappa + T_+^{(2)} \exp[i\phi_2(t)] \sin \kappa,$$  

(6)

we can then rewrite the interaction Hamiltonian as

$$H_I = \Delta_p S + g\sqrt{N}aA^\dagger + \Omega T_+ + h.c..$$  

(7)

Since $[C, T_+] = A$ and $[A, T_+] = C$, one can readily verify that $[D, H_I] = 0$. To generate the full eigen-space of $H_I$ with zero eigenvalue, i.e., the dark-polariton manifold, we need to consider another dark-state polariton operator complementary to $D$:

$$E = C_2 \exp[i\phi_2(t)] \cos \kappa - C_1 \exp[i\phi_1(t)] \sin \kappa.$$  

(8)

It is worthwhile to point out that $E$ satisfies the bosonic commutation relation as well and it is independent of $D$ since $[E, D^\dagger] = 0$. Moreover we have $[E, H_I] = 0$ by construction. Our instantaneous quantum storage subspace $\mathcal{V}(t)$ is given by the linear span of the following family of instantaneous dark-states, i.e., the eigen-states of $H_I(t)$ with vanishing eigenvalues

$$|D_{m,n}(t)\rangle = \frac{1}{\sqrt{mn!}} D^m E^n |0\rangle,$$  

(9)

where $|0\rangle = |0\rangle_p \otimes |b\rangle \equiv |0\rangle_p \otimes b, b, ..., b$ represents the ground state of the total coupled system with each atom being in the ground state $|b\rangle$ and the quantum probe field being in the vacuum state $|0\rangle_p$. It is easy to prove that any other dark-state polariton operator can be expressed as a linear combination of $D$ and $E$. Notice
that one can introduce the bright-state polariton operator: \( B = a \sin \theta + C \cos \theta \), which can be used to generate eigenstates involving the excited state |\( a \rangle \). However, as shown in Ref. [8], under the adiabatic manipulations, these states will not get coupled to the above constructed dark states (9). Of course, the states obtained by applying \( D \) are not absolutely dark since the excited state can spontaneously decay.

To study the geometric quantum information storage based on the above zero-eigenvalue dark states (9), we now consider the quantum evolution in \( V(t) \) caused by the adiabatic change of the parameters \( \Omega_k(t) \). The adiabatic condition is here given by [21,22]

\[
g \sqrt{N} x_k \ll 1, \quad x_k = |\Omega_k|, \Omega_k \delta_k \tag{10}\]

for \( k = 1, 2 \). Let us consider a state vector \( |\Phi(t)\rangle = \sum_m c_{mn}(t)|D_{m,n}(t)\rangle \) belonging to \( V(t) \). A straightforward calculation gives the matrix equation [21,22] for the coefficients \( c_{mn}(t) \): \( \partial_t C(t) = K(t)C(t) \), where the vector \( C(t) \) of coefficients and the connection matrix \( K(t) \) are respectively defined by \( C(t) = (c_{00}(t), c_{01}(t), \ldots, c_{10}(t), c_{11}(t), \ldots)^T \), and \( K(t)_{m,n'} = -\langle D_{m',n'}(t)|\partial_t D_{m,n}(t)\rangle (m, m', n, n' = 0, 1, 2, \ldots) \).

The quantum storage space \( V(t) \) is, in the considered limit, an infinite dimensional one. Thus in general it is difficult to write down the relevant connection matrix \( K(t) \) explicitly. On the other hand the adiabatic quantum evolution in \( V(t) \) can be reduced, i.e., this space splits into dynamically invariant finite-dimensional sectors. Let us explain this point now.

We first observe that the following dynamical commutation relations hold \( f_D(t) := [D, \partial_t D^\dagger] = -i \sin^2 \theta (\delta_1 \cos^2 \kappa + \delta_2 \sin^2 \kappa) \); \( f_E(t) := [E, \partial_t D^\dagger] = -i \sin \theta + i (\delta_2 - \delta_1) \sin \theta \cos \kappa \sin \kappa \); \( f_{DE}(t) = [-f_{ED}(t)]^* := [D, \partial_t E] \); \( f_{EE}(t) := [E, \partial_t E^\dagger] = -i (\delta_1 \sin^2 \kappa + \delta_2 \cos^2 \kappa) \). Using these relations for \( l' \geq m \geq 0 \) and \( l, l' \geq 0 \), we obtain

\[
\langle D_{l'-m,n}(t)|\partial_t D_{l,n}(t)\rangle = \delta_{l',m,n}(l-m)f_{DD}(t) + mf_{EE}(t) + \delta_{l',m,n-1}\sqrt{(m+1)(l-m)}f_{DE}(t) + \delta_{l',m,n+1}\sqrt{m(l+m+1)}f_{ED}(t). \tag{11}\]

Now it is clear from this expression that the total space \( V(t) \) can be decomposed into a direct sum of sub-spaces: \( V(t) = \oplus_{l,m=0}^\infty V_l(t) \), where \( V_l(t) = \text{span}\{|D_{l,-m}(t)|m = 0, 1, \ldots\} \) has dimension \( l + 1 \). Notice that each \( V_l(t) \) is an invariant sub-space under the adiabatic manipulation, i.e., if \( |\Phi(t)\rangle \in V_l(t) \), then \( |\Phi(t)\rangle = \sum_m c_m(t)|D_{l,-m}(t)\rangle \in V_l(t) \). The restricted dynamics in \( V_l(t) \) is governed by the restricted dynamic equation \( \partial_t C_l(t) = K_l(t)C_l(t) \), where the sub-coefficient vector \( C_l(t) \) and the reduced connection matrix \( K_l(t) \) are respectively given by \( C_l(t) = (c_0^l(t), c_1^l(t), \ldots, c_l^l(t))^T \), and \( K_l(t) = -\langle D_{l'-m}(t)|\partial_t D_{l,-m}(t)\rangle \) \( m = 0, 1, 2, \ldots \).

The solution \( C_l(t) = W_l(t)C_l(0) \) formally determines the non-abelian holonomy \( W_l(t) = T \exp\{ \int K_l(t) dt \} \), where \( T \) is the time-ordering operator. This non-abelian holonomy is non-diagonal and thus can mix different instantaneous eigenstates \( |D_{l,-m}(t)\rangle \ (m = 0, \ldots, l) \) inducing in this way a truly non-abelian gauge structure.

In the following discussion, we consider the simplified model related to the above system as shown in Fig. 1: \( \delta_{l'2} = 0 \). Such a system has only two controllable parameters \( \Omega_{1,2} \) and can be readily realized experimentally. Mathematically the sub-connection can be simplified as

\[
K_l(t) = -\langle D_{l'-m}(t)|\partial_t D_{l,-m}(t)\rangle \quad \text{if } m = 0, 1, 2, \ldots \tag{12}\]

\[
\equiv \dot{k} \sin \theta K_l^0, \tag{14}\]

where \( K_l^0 \) is a constant matrix whose \((m,n)\) entry is \( \delta_{m,n-1}\sqrt{(m+1)(l-m)} - \delta_{m,n+1}\sqrt{m(l+m+1)} \). (13)

In this case the time-ordering becomes irrelevant and the non-abelian holonomy can be explicitly computed. In fact we have

\[
W_l(t) = \exp\{ \phi(t)K_l^0(t) \}, \tag{14}\]

where \( \phi(t) = \int \dot{k}\sin \theta dt \). By noticing that \( K_l^0 \) is proportional to \((E^\dagger D - \text{h.c.})\) restricted \( V_l(t) \), we find that this non-abelian holonomy can be rather easily cast in a diagonal form by introducing a new instantaneous basis. Let us introduce the new set of dark-state polaritons operators

\[
D' = \frac{1}{\sqrt{2}}(iD + E), E' = \frac{1}{\sqrt{2}}(-iD + E), \tag{15}\]

and the associated dark states \( |D'_{l,-m,n}(t)\rangle = D'_{l,-m,n}(t)|E'_{l,-m,n}(t)\rangle \). A straightforward calculation then gives a diagonal connection matrix \( K'_l(t) = -i \kappa \sin \theta \text{diag}(l, l - 2, \ldots, -l) \) and the corresponding holonomy

\[
W'_l(t) = \text{diag}(e^{-i\phi(t)}), e^{-i(l-2)\phi(t)}, \ldots, e^{l\phi(t)}). \tag{16}\]

Generally in EIT based quantum information storage protocols, the Rabi frequencies \( \Omega_{1,2} \) of the two classical control fields are initially set to a very large value compared to \( g \sqrt{N} \) and then decreased independently and adiabatically (e.g., as shown in Fig. 2). Thus \( \theta(t = 0) \rightarrow 0 \) and \( D(0) \rightarrow 0 \). The initial state \( |\Phi(0)\rangle = \sum_t \phi_t^0|0\rangle \), \( |0\rangle \) can be written as \( |\Phi(0)\rangle = \sum_m c_m^0(t)|0\rangle \). \( D'_{l,-m,m}(t) \), relative to the new basis \( |D'_{l,-m,m}(t)\rangle \) with the coefficients

\[
c_m^0(t) \equiv \frac{(-1)^{l-m}\nabla_0^m(t)}{(i\sqrt{2})^{l-m}l!}\tag{17}\]

and \( |\Phi(t)\rangle = \sum_m c_m^0(t)|D'_{l,-m,m}(t)\rangle \), where \( c_m^0(t) = \exp[-i(l-2m)\phi(t)]c_m^0(0) \). When \( \Omega_{1,2} \) become negligible compared to \( g \sqrt{N} \) at time \( \tau \), \( \theta(\tau) \rightarrow \pi/2 \) and
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\[ D(\tau) \to -C_1 \cos \kappa - C_2 \sin \kappa. \]  
This means that the quantum information, initially encoded in photonic states, has been transferred and written to atomic collective excitations. This accomplishes the quantum information storage protocol.

In order to recover the stored information one needs to drive adiabatically the system parameters \( \Omega_{1,2} \) along a cyclic evolution such that at the time \( T \) the condition \( \Omega_{1,2} \gg g \sqrt{N} \) is satisfied in order to guarantee \( \theta(T) \to 0 \) (see the Fig. 2). At the intermediate times \( t \in (0, T) \) quantum information is encoded in a combination of photonic and atomic collective excitations. In general, if one wants to recover exactly the initial state after the adiabatic loop has been completed, she has to perform a unitary transformation to get rid of the effect of the non-abelian Berry phase factor. In particular, for a cyclic evolution of the parameters \( \Omega_{1,2} \) if

\[
\phi(T) = \int_0^T \kappa \sin \theta dt = 2j\pi
\]

\((j \text{ is an integer})\), then it follows that \( c_{m}^{(l)}(T) = c_{m}^{(l)}(0). \) In this case the system state at the final time \( T \) coincides with the initial state \( |\Phi(0)\rangle.\)

We are now in the position, before concluding, to make a few comments on the relations between the results presented in this paper and the general holonomic approach to quantum information processing \([12,14]\). In that approach information is encoded in degenerate eigenstates of a parametric family of Hamiltonians, and in the generic case universal quantum computation \([23]\) can be achieved by resorting to non-abelian holonomies only \([12]\). By regarding the non-trivial holonomy one gets after an adiabatic loop as a designed quantum state transformation, rather than something one wants to get rid of, it should be then evident that the EIT-based scheme here discussed represents an instance of such general strategy. For example the one exciton space \( V_1 \) can encode one qubit: \( |0\rangle := E^{\dagger}|0\rangle, |1\rangle := D^{\dagger}|0\rangle. \) In this language the transformation (16) is nothing but a diagonal phase-shift \([23]\). In order to get non-diagonal single-qubit operations one would have to relax the condition \( \delta_1 = \delta_2 = 0. \) Encoding many-qubit states and enacting a controllable geometric coupling between them – as required for realizing universal computations – along with the robustness of the scheme against the various sources of errors is a more complex problem that calls for further investigations.

\[ FIG. 2. \text{Cyclic evolution of the parameters } \Omega_{1,2}. \text{ At time } t = \tau, \Omega_{1,2} \ll g \sqrt{N}; \text{ at time } t = 0 \text{ or } t = T, \Omega_{1,2} \gg g \sqrt{N}. \]