Completely-Positive Non-Markovian Decoherence

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We propose an effective Hamiltonian approach to investigate decoherence of a quantum system in a non-Markovian reservoir, naturally imposing the complete positivity on the reduced dynamics of the system. The formalism is based on the notion of an effective reservoir, i.e., certain collective degrees of freedom in the reservoir that are responsible for the decoherence. As examples for completely positive decoherence, we present three typical decoherence processes for a qubit such as dephasing, depolarizing, and amplitude-damping. The effects of the non-Markovian decoherence are compared to the Markovian decoherence.

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I. INTRODUCTION

An open quantum system has been extensively studied for potential applications to quantum computation and quantum information processing [1, 2, 3, 4]. Through the interaction with a reservoir a quantum system loses its coherent information. This is a so-called decoherence process [5]. Decoherence has been regarded as a critical obstacle against quantum information processing. However, as the notion of “reservoir engineering” has been suggested for laser cooled trapped ions [6], the studies on the decoherence have moved to control it in laboratories instead of suffering from it [7]. Many of these studies have been based on Markovian reservoirs [8]. A Markovian reservoir is characterized by two essential properties: a) weak coupling with a system by which Born approximation is validated and b) rapid relaxation time such that the information of the system is diffused over the reservoir in a rather short time compared to the time scale in which the system changes. In the perspective of engineering a reservoir, it is desirable to study the decoherence caused through a non-Markovian reservoir. It is strongly believed that a solid-state realization of quantum information processing may be extremely useful [9]. The decoherence phenomena for solid state systems including a photonic band gap material and a quantum dot cannot always be understood under the Markovian assumption [10]. The evolution of a single-mode atomic cavity in a non-Markovian reservoir has been studied for an atom laser [11].

A non-Markovian reservoir has “memory effects” such that it preserves the coherent information of the system within its relaxation time. In order to analyze how many operations can be performed by preserving the coherent information of the system, one may consider three types of characteristic times: operation time $\tau_o$, for a single quantum operation on the system, decoherence time $\tau_d$ due to the interaction between the system and the reservoir, and the relaxation time $\tau_r$ within the reservoir. When a quantum system decoheres in a Markovian reservoir with $\tau_r \ll \tau_d$, the time correlation of fluctuations in the reservoir is neglected. However, in a non-Markovian reservoir with $\tau_r$ comparable to $\tau_d$, it is expected that the time correlation between fluctuations during the time interval $\Delta \tau \lesssim \tau_r$ raises a correlated influence on the system. Once perturbed by the system, the reservoir may memorize a part of the system’s information during $\tau_r$ [12, 13]. This memorized information will be fed back to the system at another perturbation within $\tau_r$. It is expected that the correlated influence significantly suppresses the decoherence of the system [14]. An extremal example for the memory effects by a non-Markovian reservoir is that two atoms, which are prepared initially in a product states, come to be an entangled state through the interaction with a common thermal reservoir [15, 16]. In Ref. [17], the reservoir consists of a single-mode field with no relaxation, $\tau_r \to \infty$. It is desirable to study decoherence processes by bridging the gap between the two limiting cases of $\tau_r/\tau_d \to 0$ and $\tau_r/\tau_d \to \infty$.

Dynamics of an open system may be obtained from a quantum Liouville equation of the total system that consists of the system and the reservoir [2, 8]. The reduced density operator of the system needs to be positive at all times. The positivity does not need to be altered by the presence of any other system, in other words, the reduced density operator of the system needs to be completely positive if the system and the reservoir are initially un-

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correlated \cite{17,18,19}. The complete positivity was intensively discussed for a Markovian reservoir \cite{20}. For a non-Markovian reservoir, it was discussed for a master equation derived by using an operator sum representation with coarse-graining approximation \cite{21}. A random telegraph signal model was studied for a system in a non-Markovian reservoir \cite{22}. However the complete positivity of the system density operator in a non-Markovian reservoir is still an important problem to be unraveled.

In this paper, we propose an effective Hamiltonian approach to investigate the decoherence of a quantum system in a non-Markovian reservoir. Instead of deriving a non-Markovian master equation, the formalism is based on the notion of an effective reservoir, \textit{i.e.}, certain collective degrees of freedom in the reservoir that is responsible for the decoherence. We show that \textit{the reduced dynamics of the open system is completely positive and the complete positivity is naturally imposed from the effective Hamiltonian approach}. As examples for completely positive decoherence, we present three typical decoherence processes for a qubit such as dephasing, depolarizing, and amplitude-damping. The effects of the non-Markovian decoherence are compared to the Markovian decoherence.

II. EFFECTIVE ENVIRONMENT

Dynamics of an open quantum system has commonly been investigated by a master equation for the density operator or a Fokker-Planck equation for the quasi-probability function such as a Wigner function \cite{[5]}. It is however difficult to derive a completely positive master equation for a system interacting with a general non-Markovian reservoir. We suggest an effective Hamiltonian approach by introducing the notion of “effective variables” in the environment. It will be shown that the large (normally infinite) number of environmental degrees of freedom, which we call “environmental variables”, is reductive into a small number of \textit{effective environmental variables} in the sense that both cases result in the same master (or Fokker-Planck) equation for the system. For instance, a Markovian thermal environment is reductive into a collective single-mode boson field in a thermal state \cite{[22]}. The property of being reductive into the small effective environmental variables, that leads to the correct and equivalent description of the reduced dynamics for a system, is predicted by two observations. First, at a given time \(\tau\), the density operator \(\hat{\rho}\) of the system \(S\) can always be purified to a pure state \(|\psi\rangle\) of a larger composite system consisting of the system \(S\) and an ancillary system \(R\), such that \(\hat{\rho}\) is obtained by tracing the pure state over the ancillary system, \textit{i.e.}, \(\hat{\rho} = \text{Tr}_R(|\psi\rangle\langle\psi|)\). For the purification, an ancillary system is required to have its Hilbert space larger than or equal to that of the system \cite{[1]}. For instance, a qubit may suffice for the ancillary system if the system is a qubit. The composite system \(S + R\) provides all physical descriptions relating to the system and the ancillary system is called effective variable(s), which is in effect certain collective degrees of freedom, of the environment at the given time \(\tau\). Secondly, the effective variables may be dynamic over the whole environmental variables as the interaction time \(\tau\) passes. On the other hand, the dynamics of the effective variables can be absorbed by time-dependent coupling constants while keeping the effective variables stationary. The set of the time-dependent coupling constants and the effective variables contains all the information of the environment that governs the decoherence of the system. We call this set an effective environment.

Suppose \(\hat{H}_{\text{int}}(\tau)\) is an interaction Hamiltonian between a system \(S\) and an effective environment \(R\). We write the interaction Hamiltonian \(\hat{H}_{\text{int}}(\tau)\), in the interaction picture, in the form of

\[\hat{H}_{\text{int}}(\tau) = \sum_{\alpha} \hat{S}_\alpha \otimes \hat{D}_\alpha(\tau)\]

where \(\hat{S}_\alpha\) is a Hermitian operator for the system \(S\) and \(\hat{D}_\alpha(\tau) = \sum_\beta \lambda_{\alpha\beta}(\tau) \hat{R}_\beta\) with Hermitian operator \(\hat{R}_\beta\) is for the effective environment \(R\). The unit of \(\hbar = 1\) is used throughout the paper. Here, the time-dependent coupling constant \(\lambda_{\alpha\beta}(\tau)\) is a Hermitian matrix. The composite system \(S + R\) is assumed initially in a product state,

\[\hat{\rho}_T(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_R(0).\]

The effective environment \(R\) consists of collective degrees of freedom in the environment \(E\) \cite{[22]}. Its initial state \(\hat{\rho}_R(0)\) is not necessarily a thermal state even when the real environment is thermal. The Hilbert space of the effective environment and the time-dependent interaction Hamiltonian \(\hat{H}_{\text{int}}(\tau)\) are determined such that they result in the correct reduced dynamics of the system. They may be determined from first principle calculation such as the time-convolutionless projection-operator method \cite{[13]}. The use of the time-convolutionless projection-operator method is presented in Sec. III.

Once the effective environment is determined, one may consider and solve the quantum Liouville equation for the composite system,

\[\frac{d}{d\tau} \hat{\rho}_T(\tau) = -i[\hat{H}_{\text{int}}(\tau), \hat{\rho}_T(\tau)].\]

The evolution operator is now given as

\[\hat{U}_{\text{int}}(\tau) = \mathcal{T} \exp \left(-i \int_0^\tau d\tau' \hat{H}_{\text{int}}(\tau')\right)\]

where \(\mathcal{T}\) is the time-ordering operator. In most cases, it is convenient to let \(\lambda_{\alpha\beta}(\tau) = \lambda(\tau) g_{\alpha\beta}\) or \(\hat{D}_\alpha(\tau) = \lambda(\tau) \hat{D}_\alpha\) so that \(\hat{U}_{\text{int}}(\tau) = \exp[-i\lambda(\tau)\sum_\alpha \hat{S}_\alpha \otimes \hat{D}_\alpha]\) where \(\lambda(\tau) = \int_0^\tau d\tau' \lambda(\tau')\). This simpler form of \(\hat{U}_{\text{int}}(\tau)\) has advantages for later discussions from the calculational point of view.

The total system is described by the density operator

\[\hat{\rho}_T(\tau) = \hat{U}_{\text{int}}(\tau) \hat{\rho}_T(0) \hat{U}_{\text{int}}^\dagger(\tau),\]

(5)
and the reduced density operator of $S$ is given by partially tracing the total density operator over $R$,

$$\dot{\rho}_S(\tau) = \text{Tr}_R \dot{\rho}_T(\tau).$$

(6)

Letting $\{ |n\rangle \}$ be the orthonormal basis set of $R$ that diagonalizes $\hat{R}(0) = \sum_n p_n |n\rangle \langle n|$, the reduced dynamics of the system is described by a Kraus representation (or operator sum representation) $\mathcal{K}$:

$$\dot{\rho}_S(\tau) = \mathcal{S}(\tau) \dot{\rho}_S(0) = \sum_{n,m} K_{nm}(\tau) \dot{\rho}_S(0) \hat{K}_{nm}^\dagger(\tau)$$

(7)

where $K_{nm}(\tau) = R\langle n| \hat{U}(\tau) |m\rangle R\sqrt{p_m}$. Note that the superoperator $\mathcal{S}$ is a linear operator in the Hilbert-Schmidt space of density operators. Calligraphic letters are used for superoperators on the Hilbert-Schmidt space throughout the paper. It is remarkable that the existence of the Kraus representation $\{K_{nm}(\tau)\}$ directly implies the complete positivity of the evolution superoperator $\mathcal{S}(\tau)$ $\mathbb{E}$. This fact was already guaranteed since the Hamiltonian formalism was adopted in the present approach of effective environment. This approach is applicable to the system decohering in a non-Markovian as well as a Markovian environment.

The present approach of effective environment has several advantages in describing the reduced dynamics of a system. Firstly, it provides the correct and equivalent description, similar to the master equation. Secondly and most importantly, it guarantees the complete positivity in a general environment like a non-Markovian environment. Thirdly, it enables the analysis of the structure for effective variables, in a given environment, that are directly responsible for the decoherence. This analysis will be given in Sec. [IV]. Lastly, one may simulate the decoherence in an experiment by introducing such an effective environment and controlling the coupling to the given system.

III. DETERMINATION OF EFFECTIVE ENVIRONMENT

In the previous section we showed that the effective environment approach provides the description of the reduced dynamics of a quantum system, which decoheres through an environment, provided the Hilbert space of the effective environment, its initial state $\dot{\rho}_R(0)$ and the time-dependent interaction Hamiltonian $\hat{H}_{int}(\tau)$ are determined. In this section we suggest a scheme which determines them by a first principle theory, in particular, the time-convolutionless projection-operator method $^{12}$ $^{24}$ $^{25}$ which has been employed to study non-Markovian environments for quantum information processing $^{13}$.

A. Time-convolutionless projection-operator scheme

Consider the total system of a quantum system and an environment. The total system is assumed to have the Hamiltonian,

$$\hat{H}_T = \hat{H}_S + \hat{H}_E + \hat{H}_{int},$$

(8)

where $\hat{H}_S$, $\hat{H}_E$, and $\hat{H}_{int}$ are the Hamiltonians for the system, the environment, and their interaction respectively. It is straightforward to generalize that $\hat{H}_S$ may contain time-dependent external fields to control a quantum operation on the system. For a typical form of the interaction Hamiltonian, we consider a Caldeira-Leggett-type model $^{26}$ given by

$$\hat{H}_{int} = \sum_{\alpha} \hat{S}_\alpha \otimes \hat{E}_\alpha$$

(9)

where $\hat{S}_\alpha$ is a Hermitian operator acting on the system and $\hat{E}_\alpha = \sum_k (g_{\alpha k} \hat{e}_k^\dagger + g_{\alpha k}^* \hat{e}_k)$ is a fluctuating boson field due to perturbation arising from the system. Here, $\hat{e}_k$ ($\hat{e}_k^\dagger$) is an annihilation (creation) operator for a boson mode $k$. The unperturbed boson fields of the environment are governed by $\hat{H}_E = \sum_k \omega_k \hat{e}_k^\dagger \hat{e}_k$. The set of the operators $\{ \hat{E}_\alpha \}$ describes various decoherence processes.

The quantum Liouville equation for the density operator $\dot{\rho}_T$ of the total system is given by

$$\frac{d}{dt} \dot{\rho}_T(\tau) = -i [\hat{H}_T, \dot{\rho}_T(\tau)] = -i \mathcal{L}_T \dot{\rho}_T(\tau),$$

(10)

where $\mathcal{L}_T = \mathcal{L}_S + \mathcal{L}_E + \mathcal{L}_{int}$ is the Liouville operator. Here, the symbol tilde is given to indicate a density operator in the Schrödinger picture. Each Liouville operator $\mathcal{L}$ is a superoperator on the Hilbert-Schmidt space of density operators. The Liouville operators are in one-to-one correspondence to the Hamiltonians of the same subscriptions. Before the system starts to decohere, the environment is assumed initially in a thermal state

$$\dot{\rho}_E = \frac{1}{Z} \exp \left(-\hat{H}_E/k_B T\right)$$

(11)

where $Z = \text{Tr} \exp(-\hat{H}_E/k_B T)$, $T$ is the temperature and $k_B$ the Boltzmann constant. The assumption may be released to all the time-independent states that commute with the non-interacting Hamiltonian of the environment, i.e., $\mathcal{L}_E \dot{\rho}_E = 0$.

In order to derive and solve a reduced equation for the system alone, we employ a projection-operator method $^{27}$. Time-independent projection operators $\mathcal{P}$ and $\mathcal{Q}$ are defined as

$$\mathcal{P} \hat{X} = \hat{P}_E \text{Tr}_E(\hat{X}), \quad \mathcal{Q} = 1 - \mathcal{P},$$

(12)

for a dynamical variable $\hat{X}$ on the total system. Here $\text{Tr}_E$ indicates a partial trace over the environment. The reduced density operator of the system is given by

$$\dot{\rho}_S(\tau) = \text{Tr}_E \dot{\rho}_T(\tau) = \text{Tr}_E [\mathcal{P} \dot{\rho}_T(\tau)].$$
The quantum Liouville equation (10) can be decomposed into two coupled equations for $\mathcal{P}\hat{\rho}_T$ and $\mathcal{Q}\hat{\rho}_T$ respectively by applying the projection operators. For the total system decoupled at $\tau = 0$, the solution for $\mathcal{Q}\hat{\rho}_T$ is substituted for the equation of $\mathcal{P}\hat{\rho}_T$ with an ansatz
\[
\mathcal{P}\mathcal{L}_{\text{int}}\mathcal{P} = 0. \quad (13)
\]
The ansatz is introduced to ignore the renormalization of the unperturbed energy [27] (otherwise it raises the Lamb shift [28]). Represented in the interaction picture, the time-convolutionless form of Eq. (14) is one of the quantum fluctuations $\hat{\chi}$
\[
\mathcal{E}_a(\tau)\hat{\rho}_S(\tau) = \sum_{\alpha\beta} \int_0^\tau d\tau' \left\{ \hat{\chi}_{\alpha\beta}(\tau - \tau') [\hat{S}_{\beta}(\tau') \hat{\rho}_S(\tau') \hat{S}_\alpha(\tau')] \right\}
\]
where $\hat{S}_\alpha(\tau) = \exp(i\tau\mathcal{L}_S)\hat{S}_\alpha$ and
\[
\hat{\chi}_{\alpha\beta}(\tau) = \text{Tr}_E \left[ \hat{E}_a(\tau) \hat{\rho}_E \right] = \hat{\chi}_{\alpha\beta}(-\tau)
\]
with $\hat{E}_a(\tau) = \exp(i\tau\mathcal{L}_E)\hat{E}_a$. The function $\hat{\chi}_{\alpha\beta}(\tau)$ characterizes all properties of the environment for the decoherence of the system. In particular, it describes the time correlation between the quantum fluctuations $\hat{E}_a(\tau)$ and $\hat{E}_\beta(0)$ perturbed by the system at the respective times. The time convolutionless form of Eq. (14) is one of the crucial advantages of using the projection-operator scheme while a time-convolution equation would be derived by a simple perturbation theory [29]. It was shown [12] that the time-convolutionless equation (14) becomes the Lindblad master equation in the Markov approximation.

B. Connection to effective environment approach

In order to make a connection to the effective environment scheme presented in Sec. II, Eq. (14) will be further analyzed. In Eq. (14) the system operators depend on the evolution time. The time-dependent operators can be expanded by a complete set of Hermitian operators, $C$, in the Hilbert-Schmidt space where the set $C$ is chosen to include the operators $\hat{S}_\alpha$ of the system:
\[
\hat{S}_\alpha(\tau) = \sum_{\beta = 1}^{d^2-1} c_{\alpha\beta}(\tau) \hat{S}_\beta. \quad (17)
\]
where $d$ is the dimension of the systems Hilbert space. The coefficient $c_{\alpha\beta}(\tau)$ can be obtained by solving the Heisenberg equation,
\[
\frac{d}{d\tau} \hat{S}_\alpha(\tau) = i[\hat{H}_S, \hat{S}_\alpha(\tau)]. \quad (18)
\]
Eqs. (17) and (18) give the following equation, which is equivalent to Eq. (14),
\[
\mathcal{C}(\tau)\hat{\rho}_S(\tau) = \sum_{\alpha\beta = 1}^{d^2-1} \left\{ \gamma_{\alpha\beta}(\tau)[\hat{S}_{\beta}(\tau) \hat{\rho}_S(\tau) \hat{S}_\alpha(\tau)] + \gamma_{\beta\alpha}^*(\tau)[\hat{S}_\beta(\tau) \hat{\rho}_S(\tau) \hat{S}_\alpha(\tau)] \right\}
\]
where
\[
\gamma_{\alpha\beta}(\tau) = \int_0^\tau d\tau' \gamma_{\alpha\beta}(\tau, \tau') \quad (20)
\]
\[
\gamma_{\beta\alpha}^*(\tau) = \int_0^\tau d\tau' \gamma_{\beta\alpha}(\tau', \tau) \quad (21)
\]
\[
\chi_{\alpha\beta}(\tau, \tau') = \sum_{\gamma} \gamma_{\alpha\beta}(\tau) \chi_{\gamma\delta}(\tau - \tau') c_{\delta\beta}(\tau'). \quad (22)
\]
In the form of Eq. (17), the matrix $\gamma_{\alpha\beta}(\tau)$ determines the master equation (14) and the reduced dynamics of the system. The matrix $\gamma_{\alpha\beta}(\tau)$ may be represented by a sum of Hermitian and anti-Hermitian matrices, i.e., $\gamma_{\alpha\beta}(\tau) = \gamma^H_{\alpha\beta}(\tau) + i\gamma^A_{\alpha\beta}(\tau)$. The Hermitian term $\gamma^H_{\alpha\beta}(\tau)$ involves in the decoherence while the anti-Hermitian term $i\gamma^A_{\alpha\beta}(\tau)$ contributes to the Hamiltonian dynamics of the system. We call the Hermitian matrix $\gamma^H(\tau)$ a decoherence rate matrix. Now, the system operators in Eq. (19) are time-independent which enables the connection to the effective environment approach.

The evolution superoperator $S'(\tau)$, i.e., the solution to the master equation (14), is not necessarily completely positive in the long-time limit. On the other hand, the method of the effective environment guarantees the complete positivity of the evolution superoperator $S(\tau)$ at all times. In the Appendix we show how to examine the complete positivity for a given superoperator. In the effective environment approach one needs to determine the parameters of the effective environment. To determine them we require that in the short-time limit the master equation derived by the effective environment be approximately equal to the master equation (14) in conjunction with Eq. (19). The effective environment approach may have different higher-order terms from the time-convolutionless projection-operator method and this difference leads to the complete positivity of the reduced dynamics in the effective environment approach.

Using the interaction Hamiltonian (1) and following the similar procedure leading to Eq. (14), we can derive the master equation in the effective environment approach. This has to be approximate to Eq. (14) in conjunction with Eq. (19) in the short-time limit. This comparison leads to the following two conditions
\[
\text{Tr}_R \left[ \hat{D}_\alpha(\tau) \hat{\rho}_R(0) \right] = 0, \quad (23)
\]
which is comparable to Eq. (13), and
\[
\int_{0}^{\tau} d\tau' \text{Tr}_{R} \left[ \hat{D}_{\alpha}(\tau)\hat{D}_{\beta}(\tau')\hat{\rho}_{R}(0) \right] \approx \gamma_{\alpha\beta}(\tau) \tag{24}
\]

where \( \hat{D}_{\alpha}(\tau) \) is given in Eq. (1). The initial density operator \( \hat{\rho}_{R}(0) \) and the time-dependent coupling matrix \( \lambda_{\alpha\beta}(\tau) \) can be chosen by varying the operators \( \hat{R}_{\alpha} \) so as to satisfy the coupled linear equations (23) and (24). The Hilbert space of the effective environment needs to be enlarged with respect to the dimensionality unless any solution is found to Eqs. (23) and (24). As a result, the chosen set of operators \( \{ \hat{D}_{\alpha}(\tau) \} \) are responsible for the type of decoherence which the system undergoes. In this sense the operator \( \hat{D}_{\alpha}(\tau) \) is called a decoherence channel to the environment.

In most cases, an effective environment has the same Hilbert-space dimensionality as the quantum system. For instance, this is the case for Markovian environments. Nonetheless, an effective environment can be of larger dimensionality in order to properly describe the reduced dynamics of a system. An effective environment may be classified into a “Q-qubit environment” if the effective environment consists of \( Q \) qubits, “M-mode bosonic environment” if it consists of \( M \) bosonic modes, and so on.

We shall consider the asymptotic form of the time-correlation function \( \chi(\tau) = \frac{2}{\tau} \gamma(\tau) \). The time-correlation function is in general complex. However, being interested in the decoherence, we concentrate on the real part of the time-correlation function. The time-correlation function becomes real-valued when the system Hamiltonian is re-defined so that it includes Lamb shifts. Noting that the time-correlation disappears at a long time, \( \chi(\tau) \) approaches zero in the limit of \( \tau \to \infty \). Thus, \( \chi(\tau) \) is assumed to have the form of
\[
\chi(\tau) = f(\tau)e^{-g(\tau)} \tag{25}
\]

where \( f(\tau) \) and \( g(\tau) \) are polynomial (or trigonometric) functions and further \( g(\tau) \to \infty \) as \( \tau \to \pm \infty \). A simple form of \( \chi(\tau) \) will be given by
\[
\chi(\tau) = \frac{\kappa}{4\tau_{c}}e^{-|\tau|/\tau_{c}}, \tag{26}
\]

where \( \kappa = 1/\tau_{d} \) is a decoherence rate with the decoherence time \( \tau_{d} \) and \( \tau_{c} \) is a relaxation time (or memory time) in which the injected information is diffused over the environment. In the Markovian limit of \( \tau_{c} \ll \tau_{d} \), the time correlation function becomes a delta function so that \( \chi(\tau) \to \kappa \delta(\tau)/2 \) and \( \gamma(\tau) \to \kappa/4 \). The time-correlation function in Eq. (26) may be regarded as the first extension from Markovian to non-Markovian decoherence.

### IV. QUBIT SYSTEM

We shall present typical decoherence of a qubit system in Markovian and non-Markovian environments in the approach of an effective environment. In addition, we investigate the structure of an effective environment: a) the environmental variables that play a role in the respective decoherence, b) the coupling constants between the system and the effective environment, and c) the initial quantum states of the effective environment.

We shall first of all consider a dephasing process. As in Sec. III, we compare both master equations in the projection-operator method and the effective environment approach for a dephasing process. It is found that the effective environment has one qubit and the interaction Hamiltonian is given by
\[
\hat{H}_{\text{int}}(\tau) = \lambda(\tau)\hat{\sigma}_{z} \otimes \hat{\sigma}_{z}, \tag{27}
\]

\[
\lambda(\tau) = \frac{2\gamma(\tau)\exp[-4\Gamma(\tau)]}{\sqrt{1-\exp[-8\Gamma(\tau)]}} \tag{28}
\]

where \( \hat{\sigma}_{z} \) is a Pauli spin operator and \( \Gamma(\tau) = \int_{0}^{\tau} d\tau' \gamma(\tau') \). The initial states of the system and the one-qubit effective environment can be written as
\[
\hat{\rho}_{S}(0) = \frac{1}{2}(\hat{1} + s \cdot \hat{\sigma}), \tag{29}
\]

\[
\hat{\rho}_{R}(0) = \frac{1}{2}(\hat{1} + r \cdot \hat{\sigma}), \tag{30}
\]

where \( \hat{1} \) is an identity operator, \( s = (s_{x}, s_{y}, s_{z}) \) a Bloch vector of the system qubit, \( r = (r_{x}, r_{y}, r_{z}) \) a Bloch vector of the one-qubit environment, and \( \hat{\sigma} = (\hat{\sigma}_{x}, \hat{\sigma}_{y}, \hat{\sigma}_{z}) \). In connection with the time correlation function of the environment, the coupled linear equations (23) and (24) are satisfied for \( r = (r_{x}, r_{y}, 0) \) and \( \lambda(\tau) \) in Eq. (28), in the short-time limit of \( |\gamma(\tau)|\tau \ll 1 \). The initial state of the effective environment is unpolarized along the \( z \) axis while it may be polarized on the \( x-y \) plane for the dephasing process.

Let us consider the reduced dynamics of the system. Applying Eqs. (27) and (28) to Eqs. (4)–(10), we obtain the reduced dynamics in terms of its Bloch vector,
\[
s(\tau) = (s_{x}\exp[-4\Gamma(\tau)], s_{y}\exp[-4\Gamma(\tau)], s_{z}). \tag{31}
\]

The dynamics of the dephasing process is characterized by the decay function \( \exp[-4\Gamma(\tau)] \). The decay function may be regarded as a fringe visibility on the quantum interference for a given time \( \tau \). In order to compare the effect of the non-Markovian decoherence to the Markovian one, we employ the simple time-correlation function in Eq. (26). The decay function then becomes
\[
\exp[-4\Gamma(\tau)] = \exp \left\{ -\kappa \left[ (\tau - \tau_{c})(1 - e^{-\tau/\tau_{c}}) \right] \right\}. \tag{32}
\]

In the limit of no memory, \( \tau_{c} \to 0 \), the dephasing process is Markovian with the decay function, \( \exp[-4\Gamma(\tau)] = \exp(-\kappa\tau) \). When the environment keeps the information of the system within the memory time \( \tau_{c} \), the loss of the coherent information is suppressed in comparison to the Markovian case as present in Fig. 1. This fact can be clearly seen through the decoherence rate,
\[
\gamma(\tau) = \frac{d}{d\tau}\Gamma(\tau) = \frac{1}{4}\kappa \left( 1 - e^{-\tau/\tau_{c}} \right). \tag{33}
\]
for a given memory time $\tau_r$. The decoherence rate is $\kappa/4$ if $\tau_r \to 0$ and it is reduced by increasing the memory time $\tau_r$.

**FIG. 1:** Decay function, $\exp[-4\Gamma(\tau)]$, with respect to the evolving time $\tau$. The decay function may be regarded as a fringe visibility for quantum interference as the time passes. It depends on the memory time $\tau_r$ for a given decoherence rate $\kappa = 1/\tau_r$: a) $\kappa\tau_r = 0.1$ (solid line), b) $\kappa\tau_r = 1$ (dotted line), and c) $\kappa\tau_r = 10$ (dashed line). The case a) is already close to the Markovian decoherence ($\kappa\tau_r \to 0$)

Similar analysis can be applied to depolarizing or amplitude-damping processes for a qubit. For this purpose, the interaction Hamiltonian is assumed to be

$$\hat{H}_{int}(\tau) = \lambda(\tau) \sum_{\alpha,\beta=x,y,z} g_{\alpha\beta} \hat{\sigma}_\alpha \otimes \hat{\sigma}_\beta$$

where $\lambda(\tau)$ is given by Eq. (28) and $g_{\alpha\beta}$ is a real coupling matrix which determines the type of decoherence. The initial states of the system and the one-qubit effective environment are given by their Bloch vectors $s$ and $r$, similar to Eqs. (28) and (29). The coupling matrix $g_{\alpha\beta}$ and the initial state $r$ of the effective environment are summarized in Table. I. It is interesting to look at the initial state of the effective environment for each process. The initial state is completely random with $r = (0,0,0)$ for the depolarizing process and it is polarized along the $z$ axis for the amplitude-damping process.

The reduce dynamics of the qubit is described in terms of the time dependence of its Bloch vector $\hat{s}(\tau)$. For the depolarizing process, the time-dependent Bloch vector is given by

$$s_{\text{pol}}(\tau) = s \exp[-4\Gamma(\tau)]$$

For the amplitude damping process, it is given by

$$[s_{\text{amp}}(\tau)]_x = s_x \exp[-4\Gamma(\tau)]$$

$$[s_{\text{amp}}(\tau)]_y = s_y \exp[-4\Gamma(\tau)]$$

$$[s_{\text{amp}}(\tau)]_z = r_z + (s_z - r_z) \exp[-8\Gamma(\tau)].$$

It is well known that a system slowly decoheres in a non-Markovian environment due to the memory effects.

**TABLE I:** The coupling matrix $g_{\alpha\beta}$ and the initial state $r$ of the effective environment for dephasing, depolarizing, and amplitude damping processes.

<table>
<thead>
<tr>
<th>Process</th>
<th>$g_{\alpha\beta}$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>dephasing</td>
<td>$\delta_{\alpha z} \delta_{\beta z}$</td>
<td>($r_x, r_y, 0$)</td>
</tr>
<tr>
<td>depolarizing</td>
<td>$\delta_{\alpha z}$</td>
<td>(0, 0, 0)</td>
</tr>
<tr>
<td>amplitude damping</td>
<td>$\delta_{\alpha z} (\delta_{\beta x} + \delta_{\beta y})$</td>
<td>(0, 0, $r_z$)</td>
</tr>
</tbody>
</table>

However, we stress that the result should be derived from the completely positive reduced dynamics and our analysis of the effective Hamiltonian approach strongly supports the result.

**V. REMARKS**

The effective Hamiltonian approach was proposed to investigate the non-Markovian decoherence of an open system and further to understand the characteristics of its environment. The formalism is based on the notion of an effective environment, i.e., certain collective degrees of freedom in the environment that are responsible for the decoherence. The present approach naturally imposes the complete positivity on the reduced dynamics for the system. We applied the approach to the dephasing, depolarizing, and amplitude-damping processes. It was found that the non-Markovian environment suppresses the decoherence of the qubit, due to the memory effect of the non-Markovian environment.

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**APPENDIX A: COMPLETE POSITIVITY OF A SUPEROPERATOR**

Consider a set of bounded operators acting on the state vector in the Hilbert space. The set of bounded operators forms a vector space $V$ with the inner product defined by Hilbert-Schmidt norm $\langle \langle \cdot, \cdot \rangle \rangle$. The inner product of two bounded operators $\hat{v}$ and $\hat{w}$ in $V$ is given by

$$\langle \langle \hat{w}, \hat{v} \rangle \rangle = \text{Tr} \left( \hat{v}^\dagger \hat{w} \right).$$  \hspace{1cm} (A1)

In the analogy of the ket and bra states, $|\hat{v}\rangle$ is called a "ket" vector of an operator $\hat{v}$ and $\langle \langle \hat{v} |$ a "bra" vector.
The set of the bra vectors forms the dual vector space. Let \( \{ \hat{e}_{ab} = |a\rangle\langle b| \} \) be an orthonormal basis for the vector space with the completeness relation,

\[
\sum_{ab} |\hat{e}_{ab}\rangle \langle \hat{e}_{ab}| = \mathcal{I}
\]

(A2)

where \( \mathcal{I} \) is an identity superoperator, i.e., a linear map of the ket vector (or ket operator) to itself. Any ket vector \( |\hat{v}\rangle \) is expanded as

\[
|\hat{v}\rangle = \sum_{ab} v_{ab}|\hat{e}_{ab}\rangle
\]

(A3)

where \( v_{ab} = \langle \hat{e}_{ab}|\hat{v}\rangle \).

A superoperator \( S \) is a linear map of \( V \) onto itself

\[
S: \hat{v} \rightarrow \hat{v}' = S(\hat{v}).
\]

(A4)

The matrix elements of \( S \) in the orthonormal basis \( \{ \hat{e}_{ab} \} \) is obtained as

\[
S_{ab,cd} = \langle \hat{e}_{ab}|S|\hat{e}_{cd}\rangle = \langle \hat{e}_{ab}|S(\hat{e}_{cd})\rangle.
\]

(A5)

The superoperator \( S \) is called “Hermitian” when

\[
S_{ab,cd} = S^\dagger_{cd,ab}.
\]

(A6)

A superoperator \( S \) is called “positive” when \( \langle \hat{\eta}|S|\hat{\eta}\rangle \) is real and positive for all \( \hat{\eta} \in V \).

If \( S \) is Hermitian, it has the right eigenvector \( |\hat{v}_s\rangle \) with the “real” eigenvalue \( \lambda_s \) such that

\[
S|\hat{v}_s\rangle = \lambda_s|\hat{v}_s\rangle.
\]

(A7)

and \( \langle \hat{v}_s| \) is the corresponding left eigenvector with the same eigenvalue. Further, if and only if \( S \) is positive, the eigenvalues are all positive. When \( S \) is not Hermitian, it may have complex eigenvalues and \( \langle \hat{v}_s| \) is no longer the corresponding left eigenvector. Instead, there exists a left eigenvector \( \langle \hat{w}_s| \) such that

\[
\langle \hat{w}_s|S = \lambda_s\langle \hat{w}_s|.
\]

(A8)

where \( \lambda_s \) is a complex number. The right and left eigenvectors satisfy the orthogonality

\[
\langle \hat{w}_s|\hat{v}_s'| \rangle = \delta_{ss'} \delta_{ss'}.
\]

(A9)

The completeness relation is given as the following

\[
\sum_s |\hat{v}_s\rangle \langle \hat{w}_s| = \mathcal{I}
\]

(A10)

and \( S \) can be represented as

\[
S = \sum_s \lambda_s \frac{|\hat{v}_s\rangle \langle \hat{w}_s|}{\langle \hat{w}_s|\hat{v}_s\rangle}.
\]

(A11)

Definition: Partial transposition, denoted by “\(^\#\)”, on a superoperator \( S \) is defined as

\[
S^\#_{ab,cd} = \langle \hat{e}_{ab}|S^\#|\hat{e}_{cd}\rangle = \langle \hat{e}_{ac}|S|\hat{e}_{bd}\rangle = S_{ac,bd}.
\]

(A12)

A superoperator \( S \) is said to be Hermitian-preserving when \( S(\hat{x}) = S(\hat{x})^\dagger \) for any \( \hat{x} \in V \). The partial transposed superoperator \( S^\# \) is Hermitian if and only if \( S \) is Hermitian-preserving. In addition, \( S^\# \) is closely related to the complete positivity of \( S \).

Theorem: The following three conditions for a superoperator \( S \) are all equivalent:

(a) \( S \) is completely positive

(b) \( S \) has a Kraus representation, i.e.,

\[
S(\hat{x}) = \sum_{\mu} \hat{K}_\mu \hat{x} \hat{K}_\mu^\dagger
\]

(A13)

(c) \( S^\# \) is positive

Proof: It is well known \[13\] that (a) and (b) are equivalent. Consider (b) \( \Rightarrow \) (c). When \( S \) has a Kraus representation, its matrix elements are given as

\[
S_{ab,cd} = \sum_{\mu} K_{\mu ab}^c K_{\mu d}^b\ast.
\]

(A14)

The partial transposition \( S^\# \) becomes

\[
S^\#_{ab,cd} = S_{ac,bd} = \sum_{\mu} K_{\mu ab}^c K_{\mu d}^b\ast.
\]

(A15)

It is clear that for all \( \hat{\eta} \in V \)

\[
\langle \hat{\eta}|S^\#|\hat{\eta}\rangle = \sum_{abcd} \eta^\ast_{ab} S^\#_{ab,cd} \eta_{cd} = \sum_{\mu} X_{\mu} X_{\mu}^\ast > 0 \quad (A16)
\]

where \( X_{\mu} = \sum_{b} \eta_{ab} K_{\mu b}^{ab} \) Thus \( S^\# \) is positive.

The converse (c) \( \Rightarrow \) (b) is now considered. Since \( S^\# \) is positive, it is also Hermitian and all the eigenvalues \( d_\nu \) are positive:

\[
S^\#_{ab,cd} = \sum_{\nu} d_\nu \langle \hat{e}_{ab}|\hat{v}_\nu\rangle \langle \hat{v}_\nu|\hat{e}_{cd}\rangle
\]

(A17)

where \( |\hat{v}_\nu\rangle \) are normalized eigenvectors. Since \( d_\nu \) is positive, we can define the following matrix

\[
\tilde{K}_\nu^{ab} = \sqrt{d_\nu} \langle \hat{e}_{ab}|\hat{v}_\nu\rangle.
\]

(A18)

Thus the superoperator \( S \) has a Kraus representation as

\[
S_{ab,cd} = S^\#_{ac,bd} = \sum_{\nu} (\tilde{K}_\nu^{ac})(\tilde{K}_\nu^{bd})\ast.
\]

(A19)

The set of Kraus operators are not unique \[18\] and all equivalent sets of Kraus operators for the given superoperator can be generated by “unitary remixing” of the canonical set with the eigenvalue vector \( \mathbf{d}^\prime \) extended by some arbitrary number of zeros, i.e. \( \mathbf{d}^\prime = (d, 0, \ldots, 0) \) \[31\].
[28] Here, the Lamb shift is undistinguished from the Stark shift, although the Lamb shift results from the vacuum fluctuation, whilst Stark shift is from the thermal fluctuation (see Ref. [3]).