Spin-orbit coupling and magnetic spin states in cylindrical quantum dots

C. F. Destefani, Sergio E. Ulloa, and G. E. Marques

1Department of Physics and Astronomy, Ohio University, Athens, Ohio 45701-2979
2Departamento de Física, Universidade Federal de São Carlos, 13565-905, São Carlos, São Paulo, Brazil

We make a detailed analysis of each possible spin-orbit coupling of zincblende narrow-gap cylindrical quantum dots built in a two-dimensional electron gas. These couplings are related to both bulk (Dresselhaus) and structure (Rashba) inversion asymmetries. We study the competition between electron-electron and spin-orbit interactions on electronic properties of 2-electron quantum dots.

PACS numbers: 71.70.Ej, 73.21.La, 78.30.Fs

The creation and manipulation of spin populations in semiconductors has received great attention since the Datta-Das proposal of a spin field-effect transistor based on Rashba spin-orbit coupling of electrons in a bidimensional electron gas and the possibility for quantum computation devices using quantum dots (QDs). Thus, it is important that every spin-orbit (SO) effect be clearly understood for a full control of spin-flip mechanisms in nanostructures.

There are two main SO contributions in zincblende materials. In addition to the structure inversion asymmetry (SIA) caused by the 2D confinement (the Rashba SO), there is a bulk inversion asymmetry (BIA) term in those structures (the Dresselhaus SO). An additional lateral confinement defining a dot introduces another SIA term with important consequences, as we will see in detail. Although the relative importance of these two effects depends on the material and on sample design (via interfacial fields), only recently have authors begun to consider the behavior of spins under the influence of all effects.

The goal of this work is to show how important different types of SO couplings are on the spectra of parabolic QDs built with narrow-gap zincblende materials. We consider both Rashba and a diagonal SIA, as well as all Dresselhaus BIA terms in the Hamiltonian, in order to study features of the spectrum as function of magnetic field, dot size, and electron-electron interaction.

Consider a heterojunction or quantum well confinement potential $V(z)$ such that only the lowest $z$-subband is occupied. The Hamiltonian for a cylindrical QD, in the absence of SO interactions, is given by $H_0 = (h^2/2m)c_0^2 + V(r) + gμ_B B \cdot σ/2$, where $c_0 = -i\nabla + eA/hc$, $A = Bρ(-\sin θ, \cos θ, 0)/2$ describes a magnetic field $B = Bz$, $m$ is the effective mass in the conduction band, $g$ is the bulk $g$-factor, $μ_B$ is Bohr’s magneton, $V(r) = m_\omega^2 r^2/2$ is the lateral dot confinement, and $σ$ is the Pauli spin vector. The analytical solution of $H_0$ yields the Fock-Darwin (FD) spectrum, $E_{n\ell m} = (2n + |\ell| + 1)\hbar\omega_L + \hbar\omega_C/2 + gμ_B B\sigma/2$, with effective (cyclotron) frequency $Ω = \sqrt{\omega_L^2 + \omega_C^2}/4$ ($\omega_C = \omega_L/m\mu_B$). The FD states are given in terms of Laguerre polynomials. The magnetic and effective lengths are $l_\omega = \sqrt{\hbar/(m\omega_C)}$, $l_B = \sqrt{\hbar/(m\omega_L)}$, and $λ = \sqrt{\hbar/(mΩ)}$, respectively.

The SIA terms for the full confinement potential, $V(r) = V(z) + V(\rho)$, and coupling parameter $α$ are decomposed as $H_{SIA} = H_R + H_{SIA}^D + H_K$. These three forms are: i) $H_K = \alpha_λ \hbar \omega_L / l_\omega^2 λ x [σ_x L_– − σ_– L_x] (k_z) + k_z / 2$ gives zero contribution when $|k_z| = 0$ (pure state parity); ii) $H_{SIA}^D = \alpha \sigma_λ \hbar \omega_L / l_\omega^2 [λ x σ_x L_– - σ_– L_x] + l_\omega^2 λ x σ_y L_–]$ is the diagonal contribution due to the lateral confinement and $x = ρ/λ$ is the dimensionless radial coordinate, $L_– = -iθ/σ_0$ is the z-orbital angular momentum; iii) $H_R = -2α/λ dz [σ_+ L_+ + σ_+ L_+ + σ_– L_–] + z$ is the Rashba term for the perpendicular confinement $dV/\partial z$, $l_\omega = e^{z/\theta}$, $σ_± = (σ_x ± iσ_y)/2$ and $A_± = [π\partial/\partial x] L_± + e^{z/\theta}/2$. In principle these terms can be tuned since $H_{SIA}^D$ depends on the confinement frequency $ω_0$ while $H_R$ depends on the interfacial field $dV/\partial z$.

The BIA Hamiltonians for zincblende materials, after averaging along the $z$-direction, is given by $H_{BIA} = γ [σ_x k_x k^*_y - σ_y k_y k^*_x] + γ [k^2_y] [σ_y k_y - σ_x k_x] + γ [σ_x (k_x^2 - k_y^2)]$, where $γ$ is the Dresselhaus parameter. The first (second) term is cubic (linear) in the in-plane momentum. The last term will be zero for systems where $|k_z| = 0$, while $\langle k^2_y \rangle \approx (π/2)^2$, $z_0$ being the z-direction (perpendicular) confinement length. In cylindrical coordinates $H_{BIA}$ can be written as $H_{BIA} = H_{BIA}^D + H_{BIA}^C$, where $H_{BIA}^D = -i(γ/λ) [σ_+ L_+ + σ_– L_–] + k_y^2 \langle k^2_y \rangle$ is the linear term and, after long algebra manipulation, the cubic term becomes $H_{BIA}^C = i(γ/λ^3) [σ_– L_+ H_1 + σ_+ L_– H_2 + σ_– L_– H_3 + σ_+ L_– H_4]$, where $H_i = A_i + \frac{3}{2} C_i + \frac{1}{6} D_i$, with $i = 1, 2, 3, 4$. The long expressions for the sixteen functions $A_i, B_i, C_i, D_i$ are given in Ref. [3].

Finally, the electron-electron interaction $H_{ee} = e^2/|r_1 - r_2|$, with $e$ being the dielectric constant of the material, is expanded into Bessel functions $J_k(ξ)$ as $H_{ee} = (h^2/2m) \sum_k e^{|k(1 - θ_2)|} J_k(ξc) J_k(ξc) e^{-ξ_0/λ}$, where $a_B = εh^2/(me^2)$ is the effective Bohr radius. The FD basis states must be properly antisymmetrized to describe unperturbed spin eigenstates $^0$.

Summarizing, our total single-particle Hamiltonian is given by $H = H_0 + H_{SIA}^D + H_R + H_{BIA}^D + H_{BIA}^C$. For the
two-particle case, we study the states and spectrum of $H + H_{xc}$. Parameters for InSb are in Ref. 5.

We present results by analyzing the role of each SO term in the Hamiltonian. We take into account all states in the FD basis having $n \leq 4$ and $|l| \leq 9$ in our numerical diagonalization, which is equivalent to the first ten energy shells at zero field and embodies a total of 110 basis states. The sequence of FD states of $H_0$ starts at zero $B$-field with

\[ \{n, l, \sigma_Z\} \equiv \{0, 0, \pm 1\}, \]

followed by the degenerate set $\{0, -1, \pm 1\}$ and $\{0, 1, \pm 1\}$. The next energy shell is composed by $\{0, -2, \pm 1\}$, $\{1, 0, \pm 1\}$, and $\{0, 2, \pm 1\}$. Spin and orbital degeneracies are broken by $B$ and the states with negative $l$ and positive $\sigma_Z$ acquire lower energies because of the negative $g$-factor of InSb. The lowest FD crossing occurs between states $\{0, 0, -1\}$ and $\{0, -1, +1\}$, at a critical field $B_C^0 = m\hbar\omega_0/\mu_B\sqrt{m/g}(m/g + 2)$. The moderate value of $B_C^0$ in InSb ($\simeq 2.6$ T for $\hbar\omega_0 = 15$ meV) is a direct consequence of its large $g$ factor. For GaAs ($|g| = 0.44, \tilde{m} = m/m_0 = 0.067$), this level crossing would appear only at $B_C^0(GaAs) \simeq 9.5$ T for a much smaller confinement, $\hbar\omega_0 = 2$ meV, corresponding to a regime where Landau levels are well defined.

Any figure showing the structure has the panel: Panel A shows QD spectrum for the full FD basis (110 states); Panel B shows a zoom on the three lowest shells, plus inset with another zoom on the 4 levels of the second shell; Panel C and D show, respectively, the evolution of spin $\sigma_Z$ and orbital $l$ angular momenta, for the full FD basis, while their insets take into account only the lowest 7 QD levels.

Figure 1 shows the simultaneous addition of both SIA terms, $H_{SIA}^D$ and $H_R$, to $H_0$. The diagonal term $H_{SIA}^D$ causes small splittings on the zero-field spectrum and readjust the sequence of states according to total angular momentum $l = l + \sigma_Z/2$. For example, the highest (lowest) zero-field energy level in the second shell has $j = 3/2$ ($j = 1/2$). Since $H_{SIA}^D$ does not induce shift on the accidental degeneracy points of the FD spectrum at finite fields, the first level crossing occurs at a critical value $B_C \simeq B_0^c \simeq 2.6$ T. Also, this term does not induce any level mixture on the FD states. The Rashba term $H_R$ introduces a strong state mixture for any magnitude of $\alpha$. This is evident for any pair of FD levels satisfying $\Delta l = -\Delta \sigma_Z/2 = \pm 1$ that show a crossing at given accidental degeneracy of the FD spectrum. The induced mixture converts this crossing at $B_0^c$ to an anticrossing (AC), at a shifted critical field $B_C \simeq 2.5$ T $\lesssim B_0^c$, with an energy minigap. Higher energy crossings, also satisfying this selection rule, present ACs around the same value of $B_C$, and gives origin to the observed collapse in both $\sigma_Z$ and $l$ quantum numbers at $B \simeq 2.5$ T, shown in Panels C and D.

The range of critical fields (between 2.1 and 2.6 T), and the size of the minigaps opened at those AC regions, are proportional to the magnitude of $\alpha$. $H_R$ also induces small splittings in the zero-field spectrum and slightly shifts the accidental degeneracy points at finite fields. After adding both SIA terms simultaneously (full spectrum in Panel A), one can see in the inset of Panel B that the ordering of states is the one determined by $H_{SIA}^D$. However, the energies of states $j = 3/2$ at 30 meV and $j = 1/2$ at a slightly smaller value, as well as the value of field ($B > 0.1$ T) where the normal state ordering (the one existing in the absence of SO terms) for $g < 0$ material is restored, are determined by $H_R$. For increasing energy, the ordering for this second shell is $\{0, -1, +1\}, \{0, -1, -1\}, \{0, 1, +1\}, \{0, 1, -1\}, \{0, 1, -1\}$, $\{0, -1, +1\}$. The width of critical fields becomes wider, between 2.2 and 3.6 T, as seen in Panel C. Furthermore, Panel D shows that orbitals having $l < 0$ ($l > 0$) present ACs at fields smaller (larger) than the field $B_C \simeq 2.55$ T where occurred the first AC (see insets of Panels C and D). For a future comparison notice, in Panel B, that the first AC near 50 meV involving $n = 1$ states $\{0, 1, -1\}$ and $\{1, 0, +1\}$ occurs at that same $B_C$ value. In general, ACs between states with any $n$ value occur inside the same unique range of critical fields, as shown in Panels C and D. Finally, observe that both SIA terms can be reduced by decreasing $\omega_0 (H_{SIA}^D)$ or $dV/dz (H_R)$, and that all even and negative $l$ states ($l = -2, -4, -6, -8$) show anticrossings.

Figure 2 shows the simultaneous addition of both BIA terms, $H_{BIA}^D$ and $H_{BIA}^F$, to $H_0$. The cubic term $H_{BIA}^F$, under the present QD parameters, has small influence over the $H_0$ spectrum. Small state mixtures is induced at $B_0 \simeq 1$ and $\simeq 5$ T, both involving ACs satisfying $\Delta l = \mp 3$ and $\Delta \sigma_Z = \pm 1$. The first one, at 1 T (5 T), occurs between states $\{0, 1, -1\}$ and $\{0, -2, +1\}$ ($\{0, 0, -1\}$ and $\{0, 3, +1\}$). This term also induces zero-field splittings on the FD spectrum and also a shift due to matrix elements for $\Delta l = \pm 1 = \Delta \sigma_Z/2$. However, such splittings and opened minigaps at the ACs regions are very small. Therefore, the simultaneous addition of both BIA terms (full spectrum in Panel A), where the linear term $H_{BIA}^D$ is the most important, drastically changes the general features of the FD spectrum, inducing strong zero-field splittings and shifting its accidental degeneracies to higher fields. Yet, the respective matrix elements ($\Delta l = \pm 1 = \Delta \sigma_Z/2$) do not introduce ACs on the lowest energy levels. As seen in Panels C and D, the mixing induced by the linear term is so strong that the QD states are not anymore pure states even at zero field. Notice in Panel C that at $B = 0$ values of $|\sigma_Z| < 0.5$ are found for high energy states, while in its inset one finds $\sigma_Z \simeq 0.7$ for the ground state. As an example of level crossings displaced to higher fields, observe in Panel B that the first one has moved to $B_C \simeq 3.3$ T, there is only one crossing present in the second shell at about 0.45 T (see inset), and the second one occurs a higher field around 3.5 T. Thus, contrary to the observed for SIA case, the normal ordering of state is no longer restored. We will come back to this fact later. As a final note, in the same inset and at zero field, the highest (lowest) energy state has $j = 3/2$ with eigenvalue equal to energy of 30 meV of the pure $H_0$ ($j = 1/2$ at smaller energy near 27 meV). The influence of $H_{BIA}^F$ on the spectrum changes with $\omega_0$.

Figure 3 shows the one-particle QD spectrum for full $H$ or when all SO terms are simultaneously taken into account. From the previous discussions, one may identify which SO mechanism is dominant in each of the main
FIG. 1: Spectrum when $H_{\text{SIA}}^D$ and $H_R$ are added to $H_0$ (A and B). Critical field range for ACs is seen on C and D. The lowest one ($B_C \simeq 2.55$ T, insets) occurs near $B_C^0$. ACs involving $l < 0$ ($l > 0$) orbitals are shifted to lower (higher) fields (D).

signatures present on the spectrum. An enormous state mixture, even at small magnetic fields (Panels C and D and their insets), as also splittings, position and ordering of states (Panels A, B and its inset) are dominated by $H_L^D$, although with contributions from both SIA terms. The small influence of $H_C^D$ remains around 6 T. The lowest ACs in the QD spectrum are caused by the selection rules of $H_R$, the presence of $H_D^C$ and $H_D^C$, in the full $H$, displaces and regroup all states with same $n$ value that contribute to the minigap near a fixed critical field value.

In Fig. 4 we simulate the cancellation of zero-field splittings even in the presence of all SO terms, what is reasonably obtained by taking an interfacial field $dV/dz$ four times stronger than that one considered before (see Ref. [7], other parameters remained unchanged). This is equivalent to increase the influence of the Rashba term $H_R$. Notice,
FIG. 3: Spectrum of full $H$, where $H_R$ induces minigap regions that are shifted to higher fields by $H_{D}^{B}$. $H_{D}^{C}$ has small influence but induces state mixtures. The zero-field splittings produced by $H_{SIA}^{D}$ are dominated by those from $H_{B}^{D}$.

in Panels $A$ and $B$, that not only the zero-field splittings are nearly vanished, but also the Zeeman splittings are practically suppressed at low fields ($B_0 \lesssim 1.5$ T). At zero magnetic field, an energy shell structure identical to the pure $H_0$ and with the same level separation of 15 meV is formed at displaced energies. In the inset of Panel $B$ one sees that the energy of $j = 3/2$ level is pushed near $j = 1/2$ level, going from 30 (in Fig.3 B) to 26.5 meV (in Fig.4 B). While the zero-field splittings nearly vanish the energy minigaps increases, as seen in Panel $B$. The rearrangement of electronic levels is so remarkable that ACCs related to the cubic BIA term at 1.2 T become visible (Panel $B$ and insets of Panels $C$ and $D$). The minigaps at 33 (44) meV involves states $\{0, 1, -1\}$ and $\{0, -2, +1\}$ ($\{1, 0, -1\}$ and $\{0, -3, +1\}$). Even though the electronic levels are less disperse here than in Fig. 3, the SO-induced state mixture is much more intense, as can be seen in Panels $C$ and $D$. Between 0 and 4 T, most of the QD levels have $|\sigma_Z| < 0.5$ and only the ground state has $\sigma_Z \simeq 0.7$. As mentioned before, the insets of Panels $C$ and $D$ show that a strong Rashba term enlarges the spin-flip region near $B_C$.

FIG. 4: Full $H$ spectrum with four times stronger $dV/dz$. Notice the cancellation of zero-field and Zeeman splittings at low fields ($A, B$ and inset). New ACCs due to $H_{D}^{B}$ selection rules occur near 1.2 T ($B$ and insets in $C$ and $D$). The lowest ACC is shifted back to $B_C = 2.7$ T. Notice the enormous state mixture in $C$ and, at zero field, most states are displaying $|\sigma_Z| < 0.5$.

One can further appreciate the intricate balance between SO terms by analyzing how some quantities are affected by changes on the lateral and vertical sizes, $l_0$ and $z_0$, or on Rashba field, $dV/dz$, as shown in Fig. 5. Curves with squares, circles and triangles refer to a QD, respectively, with parameters of Ref. 8, with $z_0$ doubled (smaller linear BIA contribution) and with four times stronger $dV/dz$, while the dotted curve, in middle Panel, indicates the $B_C^0$ field where the first FD level crossing occurs. The zero-field splitting (left Panel) for states $j = 3/2$ and $j = 1/2$ of
the second shell is dominated by the linear BIA contribution for any value of $l_0$. An increase on $z_0$ strongly reduces the splittings because the Dresselhaus contribution becomes weaker. The reduction is even more drastic by increasing $dV/dz$, which makes $H_R$ larger and, thus, may cancel or suppress zero-field splittings produced by $H^D_{SIA}$.

The critical fields $B_C$, where ACs determined by $H_R$ occur (middle Panel, for the lowest minigap between levels \(\{0,0,-1\}\) and \(\{0,-1,+1\}\)), decrease with increasing QD size, once $B^D_{C(L)} \approx \omega_0 \approx 1/\sqrt{\ell_0}$. Its value is close to $B^C_{1L}$, when BIA terms are not present and the inclusion of $H^D_L$ shifts $B_C$ to higher value. Increasing $z_0$ or $dV/dz$ decreases $B_C$, once they will decrease the effects due to $H_{BIA}$. The value $B_C \approx 2.1$ T, for $l_0 = 270$ Å ($\hbar \omega_0 = 7.5$ meV) is displaced to 1.8 (1.5 T) if $dV/dz (z_0)$ is four times larger (doubled). This last value can be compared to reported $B_C \approx 1.7$ T in Ref. [8], where BIA terms are absent. The small difference $\Delta B = 0.2$ T can be attributed to the inclusion of non-parabolicity effects. Anticrossings at such low fields may be interesting for applications due to easier access.

Finally, the minigap opened at $B_C$ (right Panel) has their main origin in the $H_R$ term, while the inclusion of $H_{BIA}$ causes a substantial reduction. If the value of $z_0$ is doubled the splitting is enhanced slightly. A yet larger $z_0$ produces no significant changes. However, the splitting can be drastically enhanced by increasing the Rashba field as, for example, changing from 1 to 4.2 meV at $l_0 = 190$ Å, when interfacial field is increased four times. Measurement of those three quantities should yield important information on the relative strength of SO parameters $\alpha$ and $\gamma$.

After having studied the one-particle QD problem we show, in Fig. 6, the two-electron QD spectrum under magnetic field (parameters in [8]). On the construction of Slater determinant for two-particle states we use the 20 lowest one-particle orbitals ($|| \leq 3$ and $n \leq 1$), which amounts to 190 possible two-particle states that can be labeled, in the absence of SO interactions, by the projections of orbital ($M_L$) and spin ($M_S$) total angular momenta. If no SO term is present in $H$ (Panel A), we verified that the singlet ground state is located at 35 meV, while the first excited shell at zero-field is splitted by the exchange interaction, being composed by a triplet (at 47.5 meV) and a singlet (at 50 meV) states. At very small magnetic field ($\approx 0.1$ T), the normal sequence of QD states is restored. For increasing energy and using the notation \(\{M_L,M_S\}\), the ordering is: \{0,0\} for the ground state, \{-1,1\}, \{-1,0\}T, \{-1, -1\}, \{1,1\}, \{1,0\}T, \{1, -1\} for the first excited triplet (T), and \{-1,0\}S, \{1,0\}S for the first excited singlet (S). The crossing between ground singlet and first excited triplet states occurs at $B^{(2e)}_{C} = 2.1$ T.

Panel B shows the QD spectrum for full two-particle Hamiltonian, $H_{ee} + H$. We may identify some similar features to the single-particle case. For example, the linear Dresselhaus term almost destroys the energy shell structure at zero field by shifting level crossings and inducing new zero-field splittings, while the Rashba term introduces energy minigaps in the spectrum. Panel B shows details on the competition between Coulomb and spin-orbit interactions in narrow-gap cylindrical dots. Observe that the SO interaction, at zero field, acts against the direct Coulomb energy and, in a sense, favoring the exchange term. For example: i) The ground state is shifted back from 35 to 31 meV, which is close to the non-interacting value 30 meV; ii) The first excited shell states have energies from 43 to 47 meV, values even smaller than the non-interacting energy 45 meV. Other important feature in the first excited shell is the observation that the original triplet is broken on its three possible terms according to the projection of the total angular momentum, $M_J = M_L + M_S$. For increasing energy order, these terms are composed, at zero field, by the
InSb parameters: $C$ ($M_S$) and $D$ ($M_L$) show the lowest ACs as induced by $H_R$ and shifted by $H_L^D$. Critical field is decreased by increased occupation. The first excited triplet, at zero field, is split by the possible $M_L$ values as explained in text. $C$ ($M_S$) and $D$ ($M_L$) show the lowest ACs as induced by $H_R$ and shifted by $H_L^D$. For the lowest $AC,$ $\Delta$ values are taken from the right Panel of Fig. 5, from where one sees that $\Delta$ is completely changeable by the SO occupation, although $BC$ itself is decreased by increased occupation. Yet, the $H_L^D$ selection rule becomes $\Delta M_L = \mp 3$ and $\Delta M_S = \pm 1$, and its first minigap between states $\{ -2, 1 \}$ and $\{ -1, 0 \}^{S/T}$ would be visible if a higher $dV/dz$ had been considered on the solution for the two-electron problem.

A very special difference between the one- and two-particle problems is that a strong intrinsic (non phonon-assisted) singlet-triplet transition (qubit) at low magnetic fields involving the ground state becomes possible in the two-electron problem and, in principle, could be explored in implementations of quantum computing devices. As mentioned, the critical field is decreased by $\Delta$ occupation (from $BC = 3.3$ to $BC^{(2\pi)} = 2.7$ T), and this reduction may be increased by decreasing the $BC$ confinement energy. At these critical fields where the intrinsic state mixture is enhanced, the SO-induced spin relaxation rate ($\Gamma$) can be estimated from the minigap energy ($\Delta$), as $\Gamma = \hbar/\Delta$. For the lowest $AC,$ $\Delta$ values are taken from the right Panel of Fig. 5, from where one sees that $\Delta$ is completely changeable by the $BC$ parameters and, consequently, the intrinsic rate $\Gamma$ can be changed according to those parameters.

We showed that inclusion of all SO terms is essential in order to obtain a complete picture of the electronic structure of narrow-gap QDs, and discussed the role played by each BIA and SIA terms on QD spectra and on spin polarization of states. The combination of strong SO coupling in $H_R$ and large $g$-factor introduces strong intrinsic mixtures and low excitations on the single-particle spectrum; the position of critical fields where minigaps occur is affected by $H_L^D.$ We observed that the two-particle spectrum exhibits strong singlet-triplet coupling involving QD ground state at moderate fields, which may have significant consequences like possible use in qubits designs.

Work supported by FAPESP-Brazil, US DOE grant no. DE-FG02-91ER45334, and CMSS Program at OU.

7. InSb parameters: $m = 0.014 \, m_0$, $g = -51$, $\varepsilon = 16.5$, $a_B = 625 \, \text{Å}$, $\alpha = 500 \, \text{Å}^2$, $\gamma = 160 \, \text{eVÅ}^3$. Dot characteristics: $\hbar\omega_0 = 15 \, \text{meV}$ ($l_0 = 190 \, \text{Å}$), $\omega_0 = 40 \, \text{Å}$, $dV/dz = -0.5 \, \text{meV/Å}$. Prefactors (meV) at zero $B$-field: $E_{SIA}^D = \alpha \hbar \omega_0 / l_0^3 = 0.2$, $E_R = - (\alpha / \lambda) dV/dz = 1.3$, $E_{SIA}^D = \gamma / \lambda^3 = 0.02$, $E_{B2}^D = \gamma (k_2^2) / \lambda = 5.2$, and $E_{ce} = \hbar \Omega / a_B = 4.5$.