A Scenario for the Dimensional Compactification in Eleven-Dimensional Space-Time

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PACS 04.20.Jb, 98.80.Dr

Abstract

We discuss the inhomogeneous multidimensional mixmaster model in view of appearing, near the cosmological singularity, a scenario for the dimensional compactification in correspondence to an 11-dimensional space-time.

Our analysis candidates such a collapsing picture toward the singularity to describe the actual expanding 3-dimensional Universe and an associated collapsed 7-dimensional space. To this end, a conformal factor is determined in front of the 4-dimensional metric to remove the 4-curvature divergences and the resulting Universe expands with a power-law inflation.

Thus we provide an additional peculiarity of the eleven space-time dimensions in view of implementing a geometrical theory of unification.
The attempt for a geometrical unification of the fundamental interactions present in Nature leads, in the classical [1,2], as well as supergravity [3] Kaluza-Klein theories, to represent the space-time as the direct product of a generic 4-dimensional manifold and an internal compact space; such a compact space must be homogeneous (to reflect, via its isometries, a gauge group) and have size of some orders the Planck length. Though the existence of such an (actually) unobservable internal compact space can be introduced as an intrinsic feature, due to a spontaneous compactification process (i.e. a spontaneous breaking of the Poincaré symmetry), nevertheless it require a cosmological justification on the base of a suitable dynamics of dimensional compactification.

Over the years many efforts were done (see for instance [3,4]) in order to obtain a multidimensional cosmology in which takes place the dynamical decomposition in terms of an expanding 3-dimensional space phenomenologically compatible with the Friedmann-Robertson-Walker model and an extra-dimensional one collapsing to Planckian scales. Here we show how, in correspondence to an 11-space-time, the appearance of a compactification process acquires very general character, no longer related to the choice of particular models.

We start by a brief review of the basic dynamical features characterizing, in vacuum, the asymptotic evolution to a spacelike singularity proper of a generic multidimensional cosmology (see [3-13]).

Let us consider a $(d+1)$-dimensional space-time $(d \geq 3)$, whose associated metric tensor obeys to a dynamics described by an Einstein-Hilbert action, i.e. by the following system of field equations

$$(d+1)R_{ik} = 0; \quad (i, k = 0, 1, ..., d),$$

where the $(d+1)$-dimensional Ricci tensor takes its natural form in terms of the metric components $g_{ik}(x^l)$.

In [9] it is shown that, within the framework of the Einstein theory, the inhomogeneous mixmaster behavior derived in [7], with respect to generic 3-dimensional cosmologies, finds a direct generalization in correspondence to any value of $d$.

In a synchronous reference (described by usual coordinates $(t, x^\gamma)$, $\gamma = 1, 2, 3$), the time evolution of the $d$-dimensional spatial metric tensor $\gamma_{\alpha\beta}(t, x^\gamma)$ singles out, near the cosmological singularity $(t = 0)$, an iterative structure; each single stage consists of intervals of time (Kasner epochs) during which tensor $\gamma_{\alpha\beta}$ takes the generalized Kasner form

$$\gamma_{\alpha\beta}(t, x^\gamma) = \sum_{i=1}^{d} t^{2p_i} x^{\alpha} x^{\beta},$$

where the Kasner index functions $p_i(t, x^\gamma)$ have to satisfy the conditions

$$\sum_{i=1}^{d} p_i(t, x^\gamma) = \sum_{i=1}^{d} p_i(t, x^\gamma)^2 = 1$$

(3)
and $1^1(x^\gamma), ..., 1^d(x^\gamma)$ denote $d$ linear independent vectors, whose components are arbitrary functions of the spatial coordinates.

In each point of the space, the conditions (3) define a set of ordered indices $\{p_i\}$ ($p_1 \leq p_2 \leq \ldots \leq p_d$) which, from a geometrical point of view, fix one point in $R^d$, lying on a connected portion of a $(d - 2)$-dimensional sphere. We note that the validity of the conditions (3) requires $p_1 \leq 0, p_{d-1} \geq 0$; where the equality takes place only for the values $p_1 = ... = p_{d-1} = 0, p_d = 1$.

As shown in [8 10] (see also [14 15]), each single step of this iterative solution results to be stable, in a given point of the space, if take place there the following conditions:

$$\forall(x^1, ..., x^d): \quad \alpha_{ijk}(x^\gamma) > 0 \quad (i \neq j, i \neq k, j \neq k) \quad (i, j, k = 1, ..., d), \quad (4)$$

where the quantities $\alpha_{ijk}(x^\gamma)$ are defined, in each space point, by expressions of the form:

$$\alpha_{ijk} = 2p_i + \sum_{l \neq i, j, k} p_l \quad (i \neq j, i \neq k, j \neq k), \quad (i, j, k = 1, ..., d) \quad (5)$$

and take values in the available domain for the ordered indices $\{p_i\}$.

It can be shown [8 10] that, for $3 \leq d \leq 9$ at least the smallest of the quantities $\alpha_{12}^{d-1,d}$ results to be always negative (excluding isolated points $\{p_i\}$ in which it vanishes); while for $d \geq 10$ there exists an open region of the $(d - 2)$-dimensional Kasner sphere where this same quantity takes positive values, the so-called Kasner Stability Region (KSR).

As a consequence, for $3 \leq d \leq 9$, the evolution of the system to the singularity consists of an infinite number of Kasner epochs; instead for $d \geq 10$, the existence of the KSR, implies a profound modification in the asymptotic dynamics. In fact the (reliable) indications presented in [9 15] in favor of the “attractivity of the KSR, imply that in each space point (excluding sets of zero measure) a final stable Kasner-like regime appears.

Finally we stress that, in correspondence to any value of $d$, the considered iterative scheme contains the right number of $(d + 1)(d - 2)$ physically arbitrary functions of the spatial coordinates, required to specify generic initial conditions (on a non-singular spacelike hypersurface); therefore this piecewise solution describes the asymptotic evolution of a generic inhomogeneous multidimensional cosmological model.

Now we show how, for $d = 10$ the Kasner stability region is characterized by a peculiar feature which has relevant dynamical implications for a dimensional reduction scenario toward the singularity.

First we rewrite, in terms of the ordered Kasner indices $\{p_i\}$, the conditions which define in $R^d$ the $(d - 2)$-dimensional allowed domain, i.e.

$$\sum_{i=1}^{d} p_i = \sum_{i=1}^{d} p_i^2 = 1; \quad p_1 \leq p_2 \leq \ldots \leq p_d. \quad (6)$$
Within such a domain, the KSR is defined by the following conditions:

\[
\alpha_{1,d-1,d} = 2p_1 + p > 0; \quad p = \sum_{i=2}^{d-2} p_i; \quad (7)
\]

in fact, the validity of this inequality ensures the positiveness of all the quantities \(\alpha_{ijk}, (i \neq j, i \neq k, j \neq k), (i, j, k = 1, \ldots, d)\).

We observe that those sets \(\{p_i\}\) which verify the condition \(\alpha_{1,d-1,d} = 2p_1 + p = 0\),
and therefore (for \(d \geq 10\)) lay at the boundary of the Kasner stability region, are fixed points with respect to the iteration of the multidimensional map (associated with replacing of Kasner indices). It is worth noting that constitutes an exception the point \(p_1 = \ldots = p_{d-1} = 0, p_d = 1\) which, although is a fixed one for any value of \(d\), nevertheless
it does not lay (for \(d \geq 10\)) at the boundary of the Kasner stability region, remaining in
this sense an isolated fixed point.

With clear reference to our leading idea, let us now search for points \(\{p_i\}\) in the
allowed domain having the following structure:

\[
p_1 = p_2 = p_3 = -X; \quad p_4 = p_5 = \ldots = p_d = Y; \quad (X \geq 0, Y > 0); \quad (d > 3). \quad (8)
\]

By the conditions \((8)-\underline{10})\) we obtain the following simple algebraic system in the variables \(X, Y\):

\[
-3X + (d - 3)Y = 1; \quad 3X^2 + (d - 3)Y^2 = 1; \quad (X \geq 0, Y > 0); \quad (d > 3), \quad (9)
\]

whose solution gives us the explicit expressions of \(X\) and \(Y\) as functions of \(d\); so we
define, in the allowed domain, a point \(\{p_i(d)\}\) having the form \(^1\):

\[
p_1 = p_2 = p_3 = \frac{1}{d} \left[1 - \frac{1}{3} \sqrt{(d-1)(d-3)}\right]; \quad \quad (10a)
\]

\[
p_4 = p_5 = \ldots = p_d = \frac{1}{d(d-3)} \left[d - 3 + \sqrt{3(d-1)(d-3)}\right]. \quad (10b)
\]

In correspondence to these “special” points the quantity \(\alpha_{1,d-1,d}\) takes the following
explicit expression as function of \(d\):

\[
\alpha_{1,d-1,d}[p_i(d)] \equiv \alpha_{1,d-1,d}^*(d) = \frac{1}{d} \left[d - 1 - \frac{d+3}{d-3} \sqrt{\frac{1}{3}(d-1)(d-3)}\right]. \quad (11)
\]

It can be verified the validity of the following statements:

a) For \(d = 4, \alpha_{1,3,4}^* = 0\) in \((p_1 = p_2 = p_3 = 0, p_4 = 1)\)

b) For \(5 \leq d \leq 8, \alpha_{1,d-1,d}^* < 0\)

\(^1\)We note that the existence of such points in the case \(d = 9\) and \(d = 10\) was first pointed out in \(\underline{1}\)
with reference to a different purpose.
c) For $d = 9$, $\alpha^*_1,9,9 = 0$ in $(p_1 = p_2 = p_3 = -1/3; p_4 = p_5 = ... = p_9 = 1/3)$
d) For $d \geq 10$, $\alpha^*_1,d-1,d > 0$

Thus we find the key result that, for $d \geq 10$, the points of the form \textbf{[10]} belong always to the KSR. This fact acquires particular relevance for the dimensional reduction since it is clear that, due to the continuity, in a small enough neighborhood of such points (interely within the KSR) the Kasner indices $\{p_i\}$ have to conserve the same structure with three negative values and all the other positive ones (although, in general, we have no longer any equality condition in each of these two groups of values).

Let us now show that for $d = 10$ there exists a whole connected domain in the KSR which possesses such a structure, i.e. is constituted by points $\{p_i\}$ for which always three indices are negative and the remaining seven all positive ones.

For $d = 10$, the point of the form \textbf{[10]} reads explicitly

$$p_1 = p_2 = P_3 = (1 - \sqrt{21})/10; \quad p_4 = p_5 = ... = p_{10} = (7 + 3\sqrt{21})/70$$ \hspace{1cm} (12)

and the quantity $\alpha^*_1,9,10$ takes the positive value $(63 - 13\sqrt{21})/70$.

In a small neighborhood of the point \textbf{[12]}, the KSR have to contain points with three negative indices. We observe that, as far as we deal with a connected region, the presence of points $\{p_i\}$ having two or four (in case till to $d - 2$) negative indices, implies that there exist curves (interely inside such a region), which joint points of such a kind to the one \textbf{[12]}. However, any of such curves has to necessarily cross the hyperplanes $p_3 = 0$ or $p_4 = 0$ in correspondence to points of the form

1) $[(p_1 < 0); (p_2 \leq 0); (p_3 = 0); (p_4, ..., p_8 \geq 0); (p_9, p_{10} > 0)]$
2) $[(p_1 < 0); (p_2, p_3 \leq 0); (p_4 = 0); (p_5, ..., p_8 \geq 0); (P_9, P_{10} > 0)]$

where all the above non-vanishing values have to be regarded as unspecified generic ones. Of course sets of values of the form 1) and 2) define points belonging to the KSR, only if the conditions \textbf{[9]} and \textbf{[7]} take place simultaneously. But (at least) one index, $p_3$ or $p_4$, must be zero and therefore such conditions reduce exactly to those ones relative to a 9-dimensional space. Since, as discussed before, in correspondence to $d = 9$ there exist no points in the allowed domain satisfying the condition \textbf{[7]} and then we can conclude that for $d = 10$ it exists a whole connected portion of the KSR which does not contain points of the form 1) or 2).

Thus, in agreement to the previous reasoning, it follows that, for $d = 10$, such a connected region is really constituted only by sets $\{p_i\}$ having three indices always negative and all the remaining seven positive ones, i.e. of the form:

3) $[(p_1, p_2, p_3 < 0); (p_4, ..., p_{10} > 0)]$

It is easy to recognize that such a connected region should have a boundary constituted of fixed points. In analogy to the above proof, we can show that, for $d = 10$, also these fixed points possess the same structure 3), with the only exception of the special point

$$p_1 = p_2 = p_3 = -1/3; \quad p_4 = 0; \quad p_5 = p_6 = ... = p_{10} = 1/3$$ \hspace{1cm} (13)
However, we stress that, to the 10-dimensional KSR belongs the point

\[ p_1 = \ldots = p_4 = -(\sqrt{27/2} - 1)/10 \quad p_5 = \ldots = p_1 = (1 + \sqrt{6})/10 , \]

which, in fact, corresponds to

\[ \alpha_{1,9,10} = (9 - 7/2\sqrt{6})/10 > 0 . \]  

Here, the relevant feature relies on that, the region having the property above outlined is an isolated one and the space regions where three and four indices are negative (i.e. the corresponding number of dimensions expand), are separated by the 2-dimensional surface defined via the conditions

\[ \{ p_1(x^\gamma) = p_2(x^\gamma) = p_3(x^\gamma) = -1/3 \quad p_4(x^\gamma) = 0 \quad p_5(x^\gamma) = \ldots = p_8(x^\gamma) = 1/3 ; \]

indeed the remaining indices \( p_9 \) and \( p_{10} \) are obliged to the value 1/3 by the conditions \( 2 \).

We conclude this analysis by observing that the above properties of the KSR, as derived for \( d = 10 \), do not hold for higher dimensional cases.

In fact, for \( d = 11 \), the KSR still never meets the hyperplane \( p_4 = 0 \), but now it contains points of the hyperplane \( p_4 = 0 \) in correspondence to a small enough neighborhood of the point

\[ p_1 = p_2 = p_3 = (1 - \sqrt{21})/10 ; \quad p_4 = 0 , \quad p_5 = p_6 = \ldots = p_{11} = (7 + 3\sqrt{21})/70 , \]

for which \( \alpha_{1,10,11} \) coincides with \( \alpha_{1,9,10} (> 0) \).

Thus, the iteration of our discussion, for increasing values of \( d \), allows to say that the KSR can never contain points \( \{ p_i \} \) with only two negative indices; but a whole connected portion of the KSR, having three negative Kasner indices, no longer exists. Thus for \( d > 11 \) the portions of the space where three indices are negative correspond to open sets which lay directly in contact with those ones where four indices become negative, via the \( d - 1 \)-dimensional hypersurface \( p_4(x^\gamma) = 0 \).

Now we describe the dynamical implications of the results above obtained about the KSR, with respect to the asymptotic evolution toward the singularity.

We see how, for \( d = 10 \), due to the structural feature of the KSR, during the asymptotic evolution, in each space point (of the spatial domains, bounded by 2-dimensional surfaces), a stable Kasner-like regime take place. The dynamics is characterized by a natural decomposition in terms of three expanding directions (associated in that point to the three negative Kasner indices) and the collapse of the remaining seven ones (associated to the positive indices).

\[ \text{[\text{The picture is made a bit more complicated by the existence of points of space where never appear a stable set of indices.}]} \]
However, it is important to observe that, as a consequence of the “oscillatory behavior” (induced, on the spatial dependence of the Kasner indices, by the map iteration, [12–13]), the three expanding directions, as well as the collapsing ones, change through space.

Though in this asymptotic scheme we obtain a 3-dimensional expanding Universe, nevertheless the associated 4-dimensional space-time curvature diverges as $1/t^2$ ($t \to 0$). This aspect would make unsuitable the implementation of any actual cosmology in this dynamical context. However, this divergence can be removed by, first, considering the following time coordinate transformation

$$T = \frac{1}{t^{1/X}} \quad X = \text{const.} \quad X > 0$$

and then taking the conformal factor in front of the 4-dimensional metric which restores a synchronous reference; hence the line element of our model rewrites

$$ds^2 = \frac{X^2}{T^2 + 2X} \left( dt^2 - \sum_{i=1}^{3} \frac{1}{X^2} T^{2+2X (1-p_i)} \eta_{ij} dx^i dx^j \right) - \sum_{j=4}^{10} T^{-2Xp_j} \eta_{ij} dx^i dx^j. \quad (19)$$

Of course, this expression of the metric must be referred to those regions of the space where the indices $p_i$, $p_\alpha$, and $p_\beta$ assume the three allowed negative values.

As $T \to \infty$ the 4-dimensional curvature acquires a vanishing behavior, and we get a power-law inflation characterizing the 3-dimensional Universe expansion. This latter feature is particularly interesting because a power-law expansion stretches the inhomogeneities out of the (redefined) physical 3-horizon. It is worth noting that to get a real dimensional reduction, from extra-dimensions collapse, implies certain topology conditions on the model or mechanisms which prevent the direct observation. We also infer that, in our generic picture, the isometries, underlying the gauge fields representation, have to be recognized locally.

We conclude our analysis by stressing how the idea proposed in this paper, represents an additional argument confirming the privileged character of the 11-dimensional space-time in view of the implementation of a geometrical unification theory.

In fact, as discussed in [14], the 11-dimensional space-time turns out to be the most natural choice when considering the construction of a realistic Kaluza-Klein theory. This claim is supported, in first place, by a phenomenological argument, based on the necessity of representing (in the zero mode approximation) the right massless gauge fields responsible for the fundamental particles interactions. Since in the Standard Model the bosonic component is described by the gauge group $SU(3) \otimes SU(2) \otimes U(1)$, then the symmetry group of the extra-compact internal space must contain it. Now it is possible to show that the minimum dimension of a manifold, having the $SU(3) \otimes SU(2) \otimes U(1)$ symmetry, is a 7-dimensional one, i.e. $CP^2 \times S^2 \times S^1$. We see how this phenomenological constraint indicates the 11-dimensional space-time as the lowest-dimensional admissible one for a realistic Kaluza-Klein theory.

Furthermore, it represents a remarkable fact that eleven is even the maximum number
of dimensions for which supergravity theory is completely consistent. In fact, for more than eleven dimensions, the gravitino, which is a Rarita-Schwinger spin-vector, would have more than 128 degrees of freedom and, once reduced in four dimensions, it would lead to a supergravity theory containing massless particles with spin greater than two. But there are strong arguments in favor of the idea that the coupling of such kind of particles with gravity leads to inconsistency.

On the base of all these considerations it looks really remarkable that, in correspondence to an 11-dimensional space-time, this large number of peculiar aspects takes place in favor of the implementation of a geometrical Unification Theory; it leads us to argue that they these features might constitute more than mere coincidences.

V. A. Belinski is thanked for having attracted our attention on the point of view here addressed.
A. A. Kirillov and R. Ruffini are thanked for the interesting discussions on this subject and their valuable advice.

References