How does the entropy/information bound work?∗

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According to the universal entropy bound, the entropy (and hence information capacity) of a complete weakly self-gravitating physical system can be bounded exclusively in terms of its circumscribing radius and total gravitating energy. The bound’s correctness is supported by explicit statistical calculations of entropy, gedanken experiments involving the generalized second law, and Bousso’s covariant holographic bound. On the other hand, it is not always obvious in a particular example how the system avoids having too many states for given energy, and hence violating the bound. We analyze in detail several purported counterexamples of this type (involving systems made of massive particles, systems at low temperature, systems with high degeneracy of the lowest excited states, systems with degenerate ground states, or involving a particle spectrum with proliferation of nearly massless species), and exhibit in each case the mechanism behind the bound’s efficacy.

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I. INTRODUCTION

Information theory started as a theory of communication—transport of information. The developers of communication channel capacity theorems paid little attention to the akin question of information storage capacity. In essence such question boils down to a more physically sounding one: what are the limitations on the magnitude of the entropy of a system characterized by general parameters such as size, energy, mass, . . . ? In 1981 I proposed [1] that the entropy of a complete physical system in asymptotically flat D = 4 spacetime, whose total mass-energy is \( E \), and which fits inside a sphere of radius \( R \), is necessarily bounded from above:

\[
S \leq \frac{2\pi ER}{\hbar c}.
\]  

(1)

The motivation for this universal entropy bound came from gedanken experiments in which an entropy-bearing object is deposited at a black hole’s horizon with the least possible energy; a violation of the generalized second law seems to occur unless the said bound applies to the object [1]. The tenor of the argument is that \( E \) is to be interpreted as the gravitating energy of the system; this prescription disposes of any ambiguity that would arise if we attempted to redefine the zero of energy.

Unruh and Wald [2] objected to the mentioned derivation by pointing out the existence of quantum buoyancy of objects in a black hole’s vicinity. Nevertheless, since quantum buoyancy is significant only at distances from the horizon of the order of the lowered object’s size, the mentioned derivation can be suitably amended to yield bound (1) even in the face of quantum buoyancy, except that a larger numerical coefficient must be accepted [3, 4]. A variant of the original gedanken experiment [5, 6] in which the object is freely dropped into a black hole (and is thus immune to quantum buoyancy) again gives bound (1), albeit with a larger coefficient. Finally, Unruh and Wald’s fluid model of the quantum buoyancy has been shown to be a gross approximation [7]; when the the responsible radiation is treated as waves, bound (1) can be recovered even with buoyancy accounted for.

Meanwhile ’t Hooft [8] and Susskind [9] introduced the holographic entropy bound

\[
S \leq \frac{\pi c^3 R^2}{\hbar G},
\]  

(2)

where \( G \) is the Newton’s constant. Again derivations from the generalized second law [9, 10] provided the clearest route. It is now understood that whereas the holographic bound is applicable to all isolated physical systems, the universal bound is relevant only for weakly self-gravitating isolated physical systems, and for these it is a much stronger bound than the holographic one [6]. Bousso has shown how to derive the holographic bound from his covariant entropy bound [11, 12], which today is the most generally applicable entropy bound known.

A strengthened version of the universal bound can be obtained from the generalized covariant entropy bound [13, 14] conjectured by Flanagan, Marolf and Wald [15]. But one should recall that the bound has its limitations. These principally belong to the strongly gravitating system regime. In common with the holographic bound, bound (1) does not apply in wildly dynamic situations such as those found inside black holes [11], and it is not guaranteed to

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work for large pieces of the universe (which, after all, are not complete systems). Bound (1) does apply in higher dimensions [12, 16] and to entire closed Robertson-Walker universes [17].

There is no controversy today as to the validity of bound (1) for complete, weakly self-gravitating, isolated objects in ordinary asymptotically flat spacetime. However, the question of exactly how the universal entropy bound (1) manages to sidestep various proposed counterexamples to it has continued to be of interest for two decades [3, 4, 18, 19, 20, 21, 22, 23, 24, 25, 26]. The present paper, which grew from unpublished material [27], is devoted to an analysis of several such attempts. In each case it lays bare assumptions made by the authors which contravene the conditions just mentioned for the validity of the bound. A paper by Boussou [28] also deals with the same issue, in some cases in a more detailed and precise way. The two papers are complementary.

II. THE REST MASS QUANDARY

A counterexample frequently adduced against the universal entropy bound by attentive listeners at lectures can be elaborated as follows. Take a number $N \gg 1$ of nonrelativistic bosons of rest mass $\mu$ confined to a space of typical extent $R$. If the bosons can occupy $\Omega$ modes (one-particle states) in all, the number of states open to them is

$$W = \frac{(N + \Omega - 1)!}{[(\Omega - 1)!N!]}$$

and the microcanonical entropy is $S = \ln W$. Now quantum mechanics tells us that the lowest lying modes have energies $\epsilon_0 = O(\hbar^2/\mu R^2)$ and this is also the typical spacing between modes. Therefore, our $N$ bosons, if they are not in very excited states, will have a total energy $E = O(N\hbar^2/\mu R^2)$. By making $\mu$ sufficiently large we make the entropy bound $2\pi ER/\hbar c$ so small that it will not be able to bound $S$. Note that the argument is one of scaling. $S$ is unaffected by a rescaling of $\mu$.

What this argument glosses over is the stipulation that bound (1) applies to a complete system. It leaves out the contribution of rest energies $N\mu c^2$ to $E$, which after all do gravitate. By keeping the assumption of a nonrelativistic system, we must interpret bound (1) as

$$S < \frac{2\pi N\mu cR}{\hbar}$$

To see if this is respected we approximate $\ln W$ with Sterling’s rule assuming not only $N \gg 1$ but also $\Omega \gg 1$. Thus

$$S = (\Omega + N) \ln(\Omega + N) - N \ln N - \Omega \ln \Omega + \cdots \quad (5)$$

$$= \Omega \ln(1 + N/\Omega) + N \ln(1 + \Omega/N) + \cdots \quad (6)$$

where the ellipsis stand for corrections of order $\ln N$ and $\ln \Omega$ which are irrelevant to what follows. Now the number of available modes $\Omega$, particularly when it is large as assumed, is bounded by the volume of phase space accessible to a nonrelativistic particle. Since a nonrelativistic particle’s momentum must be restricted to some small fraction of $\mu c$ at most, we may write

$$\Omega = (\kappa \mu cR/2\pi \hbar)^3$$

where $\kappa < 1$ is some constant. It follows that

$$\frac{S}{N\mu cR/\hbar} < \frac{S}{N\Omega^{1/3}} = \frac{1}{N^{1/3}} \frac{\ln(1 + \bar{n}) + \bar{n} \ln(1 + 1/\bar{n})}{\bar{n}^{2/3}} \quad (8)$$

The function of $\bar{n} = N/\Omega$ here has a single maximum at $\bar{n} = 0.191$ with value 1.581. It follows that

$$S < 1.581N\mu cR/N^{1/3}\hbar$$

Since $N > 1$ it is obvious that the system satisfies bound (4) as required of a nonrelativistic assembly of bosons.

It is plain that all we have shown must be true also for fermions: the fermion phase space is more restricted than the boson one (Pauli principle), so other things being equal, fermion entropy is lower than boson entropy. We have thus established the correctness of the universal entropy bound for a collection of nonrelativistic particles. There is not much more to investigate regarding the universal entropy bound, unless we turn to collections of massless particles for which the more specific bound (4) is not relevant. This we do now.
III. THE LOW TEMPERATURE QUANDARY

Deutsch [20] originated the claim, which is occasionally reinvented [15, 24, 29], that a system in a thermal state violates the entropy bound (1) if its temperature $T = 1/\beta$ is sufficiently low. This is an instructive issue. It forces one to replace the definition of entropy in Sec. II by that of entropy calculated according to the canonical ensemble.

Consider a system described by some massless quantum fields, free or interacting, confined to a cavity of radius $R$. I assume there is a unique ground state of energy $\epsilon_0$. For free fields this assumption is trivial: the zero particles state is the ground state. For interacting fields it is a restrictive assumption which I shall loosen up in Sec. VI A. I also assume there is a $g$-fold degenerate excited state at energy $\epsilon_1 = \epsilon_0 + \Delta$, and higher energy states. For sufficiently large $\beta$ one may neglect the higher energy states in the partition function $Z = \sum \exp(-\beta \epsilon_i)$, and so approximate it by $\ln Z \approx -\beta \epsilon_0 + \ln(1 + g e^{-\beta \Delta})$. The mean energy is

$$E = -\frac{\partial \ln Z}{\partial \beta} = \epsilon_0 + \frac{g\Delta}{e^{\beta\Delta} + g},$$

while the entropy takes the form

$$S = \beta E + \ln Z = \frac{g\beta\Delta}{e^{\beta\Delta} + g} + \ln(1 + ge^{-\beta\Delta}).$$

The typical claim is [20]: “measure energies from the ground state so that $\epsilon_0 = 0$; then for the low temperatures $\beta > 2\pi R/\hbar c$ one gets $S > 2\pi RE/\hbar c$ and so bound (1) is violated”.

Early realistic numerical calculations of thermal quantum fields in boxes [21] did reveal that, were the ground state energy to be ignored, bound (1) would be violated at very low temperatures, typically when $E < 10^{-9}\hbar c/R$ ($R$ enters through the “energy gap” $\Delta$). It was also clear early [1, 21] that taking any reasonable positive ground state energy to be ignored, bound (1) would be violated at very low temperatures, typically when $\beta > 2\pi R/\hbar c$ and so bound (1) is violated.

Taking the zero of energy of a system at its ground state is not automatically justified because it may mean that $E$ in the formulae is distinct from the gravitating energy. But if the particles involved are massless, what is the source of nonzero $\epsilon_0$? First of all some sort of boundaries must confine the particles in the cavity. These should have some mass since they must resist pressure of the particles. In fact, their mass must be positive on grounds of causality [3]. In addition, those boundaries will be responsible for a Casimir energy connected with the particle species in question. Although Casimir energies can occasionally be negative [19], the sum of boundary and Casimir energies will be positive [4]. But without going into details, we can state that $\epsilon_0$ must be larger than $\hbar c/R$ because the system’s Compton length must be smaller than its size $R$ in order that the very notion of size be well defined. For illustrative purposes I take $R\epsilon_0/\hbar c > 2$.

The interesting quantity now is

$$S - 2\pi RE/\hbar c = \Xi(\beta\Delta) \equiv \frac{(\beta\Delta - 2\pi R\Delta)g}{e^{\beta\Delta} + g} + \ln(1 + ge^{-\beta\Delta}) - 2\pi R\epsilon_0/\hbar c.$$  \hspace{1cm} (12)

The function $\Xi(y)$ is negative for $y = 0$ and $y \to \infty$, and has a single maximum at $y = 2\pi R\Delta$ where $\Xi = \ln(1 + ge^{-2\pi R\Delta/\hbar c}) - 2\pi R\epsilon_0/\hbar c$. I thus conclude that

$$S < 2\pi RE/\hbar c + \ln(1 + ge^{-2\pi R\Delta/\hbar c}) - 2\pi R\epsilon_0/\hbar c.$$  \hspace{1cm} (13)

For the quantity in square brackets to be nonnegative it would be necessary for $g \geq e^{2\pi R\Delta/\hbar c}[e^{2\pi R\epsilon_0/\hbar c} - 1]$, i.e., $g > 2.87 \times 10^5$. However, confined quantum field systems do not exhibit such large degeneracy. For example, for a free scalar or electromagnetic field in a cubic cavity (which by virtue of high symmetry should exhibit much degeneracy), $g$ is just a few (there are a few lowest lying degenerate modes, and the lowest excitation has one quantum in one of these modes) [30]. And a scalar field with a quartic self potential also exhibits little degeneracy in its first excited levels [31]. We thus see why the square brackets in Eq. (13) are negative, so that bound (1) is obeyed for our low temperature system of massless quanta.

IV. THE HIGH DEGENERACY QUANDARY

The argument in Sec. III depends on the supposition that the degeneracy factor of the lowest lying excited states, $g$, cannot be large. Although this is true in many situations, it is not a law of nature. It is possible to contrive systems
with large $\gamma$. Further elucidation of this issue is possible by considering in some depth a purported counterexample to the entropy bound by Page [24].

Page considers a sphere of radius $R$ partitioned into $n$ concentric shells; the partitions and the inner and outer boundaries are regarded as infinitely conducting. He points out that the lowest ($\ell = 1$) three magnetic-type electromagnetic modes in the shell of median radius $r$ have frequency $\omega \approx 1/r$. Since there are $3n$ such modes (three for each shell), Page imagines populating now one, then another and so on with a single photon of energy $\sim \hbar c/r$ for the appropriate $r$. These one-photon states allow him to form a density matrix which, for equally weighted states, gives entropy $\ln(3n)$ and mean energy $\sim 2\hbar c/R$ (since $R/2$ is the median radius of the shells if they are uniformly thick). Page concludes that bound (1) is violated because the entropy grows with $n$ while the mean energy does not.

Such an argument is wrong because it misses out part of the energy. The $3n$ modes owe their existence to the infinitely conducting partitions that confine them, each to its own shell. To be highly conducting, the envisaged partitions must contain a certain number of charge carriers. As we shall see, regardless of the carriers’ nature, their aggregated masses turn out to contribute enough to the system’s total energy $E$ to make it as large as required by the entropy bound (1). Ignoring the masses of the charge carriers goes against the condition that the bound applies to a complete system: the carriers are an essential component, so their gravitating energy has to be included in $E$. The situation must be contrasted with that in which the electromagnetic field is confined to an empty parallelepiped. Detailed state counting [30] has shown that the entropy bound is satisfied even in the case of Page’s rather contrived onion-like system, the energy of the supporting structures cannot be ignored; its inclusion in the bound is justified by the above remarks.

I assume all partitions to have equal thickness $d$. One mechanism that can block the waves from crossing a partition is a high plasma frequency $\omega_p$ of the charge carriers in the partitions (which I do not assume to be electrons necessarily). We know [32] that in a plasma model of a conductor with collisionless charge carriers, the electromagnetic wave vector for frequency $\omega$ is $k = \omega c^{-1}(1 - \omega_p^2/\omega^2)^{1/2}$, so that if $\omega < \omega_p$, the fields do not propagate. Nevertheless they do penetrate a distance $\delta = \omega^{-1}(\omega_p^2/\omega^2 - 1)^{-1/2} > \omega_p^{-1}$ into the plasma before their amplitudes become insignificant. In order to prevent these evanescent waves from bridging a partition, one must thus require $\delta < d$, i.e., $\omega_p d > c$. But

$$\omega_p^2 = 4\pi N e^2/m, \quad \text{(14)}$$

where $N$ is the density of charge carriers of charge $e$ and mass $m$. Since $d < R/n$, all this gives us $(4\pi R^2 d)N > mc^2 n^2 d/e^2$. Now $4\pi R^2 d$ is the volume of material in the outermost partition. Properly accounting for the variation of partition area with its order $i$ in the sequence (we employ the sum $\sum_i i^2$), tells us that for $n \gg 1$ the total mass-energy in charge carriers in all the partitions is $E \approx nmc^2(4\pi R^2 d/3)N$. Substituting our previous bound on $(4\pi R^2 d)N$, I get $E > \frac{1}{3} n^3 m^2 c^4 d/e^2$.

Now as a matter of principle $e^2 < \hbar c$ (recall that in our world $e^2 \approx \hbar c/137$), because more strongly coupled electrodynamics would make structures, such as atoms and partitions, which are all held together electrically, unstable [33]. We also evidently have $R > nd$. Hence $ER/\hbar c > \frac{1}{3} n^4 (mc/\hbar)^2$. But a charge carrier’s Compton length has to be smaller than $d$, for otherwise the carriers would not be confined to the partitions; thus $mc d > h$. Hence $2\pi RE/\hbar c > 2n^4$ which is always larger than the entropy in photons $\ln(3n)$.

The only alternative mechanism for keeping electromagnetic waves from penetrating into a conductor is the skin effect [32]. The skin depth for electromagnetic waves of frequency $\omega$ is $\delta_s \approx \epsilon(2\pi \sigma \omega)^{-1/2}$, where $\sigma$ is the conductivity. In the simple Drude model [32], $\sigma = N e^2 (m/\tau - im\omega)^{-1}$, where $\tau$ is a charge carrier’s slowing-down timescale due to collisions, and $\epsilon = \sqrt{-1}$. The $\sigma$ in the expression for $\delta_s$ refers to an Ohmic (real) conductivity rather than to an inductive (imaginary) one. Thus one must demand that $\omega \ll 1/\tau$. This is no real restriction since one is interested in photons with the lowest possible energy. But then

$$\delta_s \gg (2\pi N e^2/mc^2)^{-1/2}. \quad \text{(15)}$$

As before one must require $\delta_s < d < R/n$. This gives $(4\pi R^2 d)N \gg 2mc^2n^2 d/e^2$ which is just a stronger version of the lower bound on $N$ we got before. Repeating the previous discussion verbatim shows that $2\pi RE \gg 4n^4$, which bounds the photon’s entropy $\ln(3n)$ comfortably.

Of course, the charge carriers (and the lattice through which they move) also contribute to the entropy. However, they constitute a nonrelativistic system, and for such the results of Sec. II assure us that the universal entropy bound is obeyed, with much room to spare. What we have thus just shown is that adding the entropy of the photons will not change the situation. Although in Page’s onion structure the photons by themselves may violate bound (1), this bound is satisfied by the complete system, photons $+$ charge carriers.
V. THE LOW EXCITATIONS QUANDARY

In the framework of the microcanonical ensemble there is a potential challenge to bound (1) if the energy gap between the ground state and the first excitations is very small. We illustrate this with Page’s example [25] of the electromagnetic field confined to a coaxial cable of length \( L \) which is coiled up so as to fit within a sphere of radius \( R \), with \( R \ll L \), before being connected end to end to form a loop.

Page’s entirely qualitative reasoning proceeds by analogy with a rectilinear coaxial cable with periodic boundary conditions. A rectilinear infinitely long coaxial cable has some electromagnetic modes which propagate along its axis with arbitrarily low frequency. Page notes that for the coiled-up cable, each right moving mode is accompanied by a degenerate (in frequency) left moving mode (basically this follows from time reversal invariance of Maxwell’s equations). He then argues that if the cable’s outer radius \( r_2 \) is small on scale \( R \), the structure of the electromagnetic modes is little affected by the cable’s curvature. This leads him to estimate the lowest frequency \( \omega_1 \) as similar to that of the rectilinear coaxial cable with periodic boundary conditions with period \( L \): \( \omega_1 \approx 2\pi c L^{-1} \). Page notes that there are three electromagnetic states with energies \( \Delta \equiv E - E_{\text{vac}} \leq 2\pi c L^{-1} \); the vacuum, and a single photon in the right- or in the left-moving mode of frequency \( \omega_1 \). Therefore, up to energy \( \Delta \) above the vacuum, there is entropy \( S = \ln 3 \approx 1 \). Since \( 2\pi R \Delta / \hbar c \approx 4\pi^2 (R/L) \), which could be very small compared to unity, Page points to this example as a violation of the entropy bound.

By now we are experienced enough to see where the error lies. The interesting question is rather whether the electromagnetic field plus coaxial cable (complete system) complies with bound (1). Now the inner conductor of the cable—let its radius be \( r_1 \)—is an essential part of the system, for without it the lowest propagating frequency would be \( \omega_1 \sim c r_2^{-1} \), where \( r_2 \) is the outer radius of the cable, and thus very large on scale \( c L^{-1} \). To play its role, the inner conductor must keep the electromagnetic fields out of it against the two mechanisms which permit such fields to penetrate into a conductor. I now elaborate on this.

As mentioned in Sec. IV, a low frequency electromagnetic wave penetrates into a conductor a distance \( \delta > \omega_p^{-1} \) with the plasma frequency \( \omega_p \) given by Eq. (14) (again \( \mathcal{N} \) here is the density of charge carriers of charge \( e \) and mass \( m \)). In order to prevent these evanescent waves from bridging the inner conductor, one must require \( r_1 > \delta \) so \( r_1 > \omega_p \) (I assume a solid inner conductor). Then by Eq. (14) \( \mathcal{N} r_1^2 > m c^2 (4\pi e^2)^{-1} \). Now the volume of material in the inner conductor is \( \pi r_1^2 L \), so that its mass-energy is at least \( \pi r_1^2 L N m c^2 \); thus the total mass-energy \( E \) of cable plus field is constrained by \( E > m c^2 \pi L (4\pi e^2)^{-1} \). However, for the charge carriers to be localized within the conductor, their Compton lengths must be smaller than \( r_1 \), so that \( m c r_1 \gg \hbar \). Hence \( E \gg \hbar c^2 L (4\pi^2 r_1^2)^{-1} \), or

\[
2\pi c L \delta > (\pi c / 2e^2) (L / r_1) (R / r_1).
\]

Since \( e^2 < \hbar c \) (see Sec. IV), and since \( L \gg R > r_1 \) by the conditions of the problem, \( 2\pi c L \delta / \hbar c \gg 1 \), and thus bound (1) comfortably bounds the photons’ entropy \( S = \ln 3 \).

However, even when \( r_1 > \delta \) is satisfied, the waves may bridge the inner conductor if its skin depth [Eq. (15)] approaches \( r_1, \mathcal{N} r_1^2 > m c^2 (2\pi e^2)^{-1} \). But this is just a stronger version of our earlier lower bound on \( \mathcal{N} \). Repeating the previous discussion shows again that \( 2\pi c L \delta / \hbar c \), with \( E \) the total energy of the system, again bounds the photons’ entropy \( S = \ln 3 \) with plenty of room to spare.

In putting a lower bound on \( E \), I ignored the cable’s outer conductor (positive energy) and the Casimir energy. If positive this last only makes the case for the bound stronger. What if it is negative? As a rule [19] the magnitude of the electromagnetic Casimir energy of a cavity is a small fraction \( 10^{-3} \) to \( 10^{-2} \) of \( \hbar \) times the lowest eigenfrequency, here \( \omega_1 \approx 2\pi c L^{-1} \). Because \( e^2 < \hbar c \) and \( r_1 \ll L \), the lower bound on \( E \) recorded just prior to Eq. (16) is vastly greater than the Casimir energy, which may thus be neglected.

One might think that this success of the entropy bound hinges on the rather low entropy Page associated with the system. So suppose one enlists all low-lying electromagnetic modes devoid of transversal nodes and having wavelengths along the cable’s axis of the form \( L / k \) with \( k = 1, 2, 3, \ldots \). There are \( \bar{N} = \mathcal{O}(2L / r_2) \) such doubly degenerate modes with frequency below \( c / r_2 \), the frequency of the lowest-lying transversally excited mode which itself serves as base for a separate, second series of modes. How many states can one build from these \( \bar{N} \) modes, states whose energies lie below \( \hbar c / r_2 \), the energy of the lowest state arising from the second series of modes ? There are obviously \( \bar{N} \) one-photon states, fewer than \( \bar{N}^2 / 2! \) two-photon states, fewer than \( \bar{N}^3 / 3! \) three-photon states, etc. (recall: photons are indistinguishable). Together with the vacuum’s contribution of unity, the series of upper bounds sums to \( e^\bar{N} \).

Taking the logarithm we have for the entropy \( S < \mathcal{O}(2L / r_2) < (L / r_2) (R / r_2) \). But by inequality (16) together with the conditions \( e^2 < \hbar c \) and \( r_1 \ll r_2 \), this last factor is bounded from above by \( 2\pi c L \delta / \hbar c \). The futility of trying to violate bound (1) if \( E \) includes the cable’s mass-energy is thus clear. Of course, I have ignored the entropy contributed by the charge carriers. As argued already in Sec. IV, any such entropy is bounded, with room to spare, by inequality (4) just on the basis of the carriers’ rest masses. Therefore, we can afford to focus on the photon entropy alone.

To sum up, whatever the construction of the coaxial cable, the whole system complies with bound (1) as long as \( E \) includes the cable’s mass-energy.
Page felt justified in ignoring the energy associated with the cable in view of a theorem by Schiffer and me [34, 35] to the effect that a massless free scalar field confined to a cavity of whatever topology by Dirichlet boundary conditions obeys bound (1), even if \( E \) is taken as the energy above the vacuum state. This would seem to condone the interpretation of \( E \) in bound (1) as energy above the ground state. The electromagnetic field is another story. In Ref. [34] we sketched how the theorem could be extended to the electromagnetic and other noninteracting fields, but this generalization was never formally proved. In our review, Ref. [35], we work only with the scalar case. If Page’s estimate of \( \omega_1 \) is correct, such a theorem cannot apply to the electromagnetic field in a cavity with not simply connected crosssection, like the coaxial cable.

VI. THE DEGENERATE GROUND STATE QUANDARY

Confined free fields have a unique ground state, the vacuum. But nonlinear fields can have multiple degenerate vacua. For instance, the scalar field with a double well self-potential has two classically degenerate ground states, each with the field locked everywhere at the minimum of one of the wells. If the potential minima are zero, there are two states at zero energy. Although then \( S = \log 2 > 0 \), this is not a counterexample to bound (1) since \( 2\pi RE/\hbar c \) is indefinite because \( R \) is formally infinite (the field has a constant nonzero value everywhere).

A. Nonlinear scalar field

Page [25] purports to construct a real counterexample by considering configurations in which the said scalar field vanishes at a certain boundary of radius \( R \). He correctly points out that the aforesaid classical degenerate ground states engender, by quantum tunnelling between the wells, a new ground state \( \psi_0 \) (energy \( \epsilon_0 \)) with equal amplitude at each well and a first excited state \( \psi_1 \) a very small energy \( \epsilon_1 - \epsilon_0 \) above the ground state. Since the entropy of a mixed state containing \( \psi_0 \) and \( \psi_1 \) can reach \( \ln 2 \) (ground and excited states equally probable), while \( (\epsilon_1 - \epsilon_0)R/\hbar c \) can be exponentially small for deep wells, Page was convinced that the described state violates bound (1).

Page identifies the energy \( E \) of bound (1) with \( \epsilon_1 - \epsilon_0 \), the energy measured above the ground state. This would be correct if the ground state referred to a spatially unrestricted configuration, because then the bottom of a potential well would be the correct zero of energy (neglecting zero point fluctuations). But since the field is required to vanish at radius \( R \), the energy \( \epsilon_0 \) of the described ground state is a function of \( R \), and it makes little sense to take it as the zero of energy. For example, by expanding the system one can do work \( (-\partial E/\partial R \neq 0) \), so that its gravitating energy changes, and cannot be taken as zero for all \( R \). The gravitating energy for the equally likely mixture of ground and excited states should be identified with \( \frac{1}{2}(\epsilon_0(R) + \epsilon_1(R)) \approx \epsilon_0(R) \). The exponential smallness of \( \epsilon_1 - \epsilon_0 \) is not very relevant for the issue of the bound’s validity as I will show.

Since there are no solitons in \( D = 3 + 1 \) spacetime [36], a finite sized field configuration [the only interesting case—see (1)] must be confined by a “wall” which cannot be ignored, as Page does, if we stick to the original form of the entropy bound. There are three parts to the energy \( E \) of the complete system (before tunnelling is taken into account): the classical energy \( \epsilon_c \) of the field configuration concentrated around one well but vanishing at radial coordinate \( r = R \), the quantum correction \( \epsilon_q \) due to the zero point fluctuations about the classical configuration, and \( \epsilon_w \), the energy of the “wall” at \( r = R \). As I show below, \( \epsilon_w \) is at least of the same order as \( \epsilon_c \), and both strongly dominate the energy \( \epsilon_1 - \epsilon_0 \).

B. Classical two-well configurations

The double well potential field theory comes from the lagrangian density

\[
\mathcal{L} = -\hbar c \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{4} \lambda (\phi^2 - \phi_m^2)^2 \right].
\]

This gives the field equation

\[
\partial_\mu \partial^\mu \phi - \lambda \phi (\phi^2 - \phi_m^2) = 0.
\]

Every spherically symmetric configuration inside a spherical box of radius \( R \) will thus satisfy (I use standard spherical coordinates; \( \mu \) denotes derivative w.r.t. to \( r \))

\[
r^{-2}(r^2 \phi')' - \lambda \phi (\phi^2 - \phi_m^2) = 0.
\]
Regularity requires that \( \phi' = 0 \) at \( r = 0 \). Page chooses \( \phi = 0 \) at \( r = R \). The classical energy of such a configuration will be

\[
\epsilon_c = \frac{\hbar c}{2} \int_0^R \left[ \phi'^2 + \frac{1}{2} \lambda (\phi^2 - \phi_m^2) \right] r^2 \, dr.
\]  
(20)

Since one is interested in the ground state, I require that \( \phi \) have its first zero at \( r = R \). Multiplying Eq. (19) by \( r^2 \phi \) and integrating over the box, allows, after integration by parts and use of the boundary conditions, to show that

\[
\int_0^R \phi'^2 r^2 \, dr = \lambda \int_0^R (\phi_m^2 - \phi^2) \phi^2 r^2 \, dr
\]

whereby

\[
\epsilon_c = \frac{\lambda \hbar c}{4} \int_0^R (\phi_m^4 - \phi^4) r^2 \, dr.
\]  
(22)

It proves convenient to adopt a new, dimensionless, coordinate \( x \equiv \sqrt{\lambda} \phi_m r \) and a dimensionless scalar \( \Phi \equiv \phi/\phi_m \). Then Eq. (19) turns into a parameter-less equation:

\[
\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{d\Phi}{dx} \right) + \Phi(1 - \Phi^2) = 0.
\]  
(23)

Using \( d\Phi/dx = 0 \) at \( x = 0 \) one may integrate the equation to get

\[
\frac{d\Phi}{dx} = -\frac{1}{x^2} \int_0^x \Phi(1 - \Phi^2) \, dx.
\]  
(24)

If the integration starts with \( \Phi(0) > 1 \), then by continuity the r.h.s. of Eq. (24) is positive for small \( x \), so that \( \Phi \) grows. There is thus no way for the r.h.s. to switch sign, so \( \Phi(x) \) is monotonically increasing and can never have a zero. If \( \Phi(0) = 1 \), it is obvious that the solution of Eq. (24) is \( \Phi(x) \equiv 1 \) which cannot satisfy the boundary condition at \( r = R \). Thus the classical ground state configuration we are after requires \( \Phi(0) < 1 \).

When \( \Phi(0) < 1 \) it can also be seen from Eq. (24) that \( \Phi \) is monotonically decreasing with \( x \). For a particular \( \Phi(0) \), \( \Phi(x) \) will reach its first zero at a particular \( x \) which I refer to as \( x_0 \). This can serve as the parameter singling out the solution in lieu of \( \Phi(0) \). One thus has a family of ground state configurations \( \Phi(x, x_0) \). Each such configuration corresponds to a box of radius \( R = x_0(\sqrt{\lambda} \phi_m)^{-1} \). In terms of the new variables one can write Eq. (22) as

\[
\epsilon_c = \frac{\hbar c x_0}{4\lambda R} \int_0^{x_0} (1 - \Phi^4) x^2 \, dx.
\]  
(25)

The dependence \( \epsilon_c \propto \lambda^{-1} \) is well known from kink solutions of theory (17) in \( D = 1 + 1 \) [37], where the role of \( \hbar/cR \) is played by the effective mass of the field. Numerical integration of Eq. (23) shows that the factor \( x_0 \int_0^{x_0} (1 - \Phi^4) x^2 \, dx \) grows monotonically from 32.47 for \( \Phi(0) = 0 \) (\( x_0 = 3.1416 \)) to 232.23 for \( \Phi(0) = 0.98 \) (\( x_0 = 5.45 \)) to infinity as \( \Phi(0) \to 1 (x_0 \to \infty) \). Since \( \epsilon_c \) is not exponentially small, the quantum tunnelling corrections that Page discussed are negligible, so one need only add to \( \epsilon_c \) the zero point fluctuations energy \( \epsilon_v \) plus the wall energy \( \epsilon_w \) to get the full energy associated with the ground state. This sum plays the role of \( E \) in the bound (1).

I shall not bother to calculate \( \epsilon_v \) (which should include the Casimir energy). This can be done by present techniques only for the weak coupling case \( \lambda < 1 \) [37]. It is then found in other circumstances, e.g. the \( D = 1 + 1 \) kink, that \( \epsilon_v \) is small compared to \( \epsilon_c \). The situation for large \( \lambda \) (the strong coupling regime) is unclear. However, it is appropriate to recall here that the theory (17) is trivial in that it makes true mathematical sense only in the case \( \lambda = 0 \) [38]. Theorists use it for \( \lambda \neq 0 \) to obtain insights which are probably trustworthy in the small \( \lambda \) regime, but probably not for large \( \lambda \).

I now set a lower bound on \( \epsilon_w \). A look at Eq. (25) shows that for \( \Phi(0) \ll 1 \) and so \( \Phi(x) \ll 1 \), \( \epsilon_c \) scales as \( x_0^3/R \propto R^3 \). Numerically the exponent of \( R \) here only drops a little as \( \Phi(0) \) increases; for example, it is 2.86 for \( \Phi(0) = 0.98 \). So I take it as 3 for now. On virtual work grounds (consider expanding \( R \) slightly), the \( R^3 \) dependence means the \( \phi \) field exerts a suction (negative pressure) of dimension \( \approx (3\epsilon_c/4\pi R^3) \) on the inner side of the wall. By examining the force balance on a small cap of the wall, one sees that in order for the wall to withstand the negative pressure, it must support an internal compression (force per unit length) \( \tau \approx (3\epsilon_c/8\pi R^2) \) [3]. Under this compression vibrations on the wall will propagate superluminally unless the surface energy density is at least as big as \( \tau \) (dominant
energy condition). Thus one may conclude that the wall (area $4\pi R^2$) must have (positive) energy $\epsilon_w > 3\epsilon_c/2$ which adds to $\epsilon_c$ to give $E > 5\epsilon_c/2$.

As mentioned, for $\Phi(0)$ very close to unity the exponent $n$ in $\epsilon_c \propto R^n$ falls below 3; as a consequence the coefficient in the previous inequality is somewhat lower than $5/2$. However, by then $\epsilon_c R$ is already much larger than the corresponding quantity for $\Phi(0) \ll 1$ (six times larger for $\Phi(0) = 0.98$). Using the value of $\epsilon_c$ for $\Phi(0) \ll 1$ from the preceding argument, I thus conclude that for all physically relevant $\Phi(0)$, $2\pi ER/\hbar c > 127.5\lambda^{-1}$. This is certainly not exponentially small as Page originally claimed!

True, formally it seems possible to have a violation of the bound for the 50% mixture of ground and excited states ($S = \ln 2$) whenever $\lambda > 127.5/\ln 2 = 183.95$. However, this is the strong coupling regime. For all one knows the zero point energy $\epsilon_c$ may then become important and tip the scales in favor of the entropy bound. At any rate, because the theory (17) is trivial, one is more likely to be overstepping here the bounds of its applicability than to be witnessing a violation of the entropy bound at large $\lambda$. Indeed, in $D = 1 + 1$ spacetime Guendelman and I [31] found analytically all static classical configurations for the interacting theory (17) in a box, and their energies \emph{sans} the box’s. The distribution of energy levels turns out to be such that the entropy bound (1) is sustained. But I know of no analogous result in 3 + 1 dimensions

In summary, I have shown that whenever the calculation is meaningful ($\lambda$ not large), the entropy bound (1) is satisfied in Page’s example provided $E$ includes all contributions to the energy. Page [25] does not disagree with this finding, but he cites my paper with Schiffer [34] as an excuse for including in $E$ just the excitation energy above the classical ground state. However, we ourselves restricted use of this approach to an assembly of quanta of a massless \emph{noninteracting} field; the theorem for scalar fields I mentioned in Sec. V is of no help here because it holds for noninteracting fields only. Although testing the bound by ignoring the ground state energy [30, 35] is rather straightforward, it should not make us forget that in the universal bound, $E$ must include the ground state energy.

C. Multiwell potential

Page [24] also confronts bound (1) with a theory like (17) but with a potential having three equivalent wells. Presumably one would like one of these centered at $\phi = 0$, with the other two flanking it symmetrically. Then Page’s conclusion that there are three exponentially close states (in energy) is untenable. This would require three classically degenerate configurations, which certainly exist in open space (field $\phi$ fixed at one of three well bottoms). However, one is here considering a finite region with $\phi = 0$ on the boundary $r = R$. One exact solution is indeed $\phi \equiv 0$, and it has zero energy (the zero point fluctuation energy correction will, however, depend on $R$). Then there are two degenerate solutions in which the field starts at $r = 0$ in one side well and then moves to the central one with $\phi \to 0$ as $r \to R$. By analogy with our earlier calculations, the common classical energy of these two configurations will be of $O(\hbar c/\lambda R)$. It cannot thus be regarded as the zero of energy; this role falls to the energy of the $\phi = 0$ configuration. When tunnelling between wells is taken into account, one has a truly unique ground state and two excited states of classical origin split slightly in energy (plus the usual gamut of quantum excitations). The entropy of an equally weighted mixture of these states is $\ln 3$. Ignoring Casimir energy, its mean energy $E$ is $\frac{\lambda}{2}(\epsilon_c + \epsilon_w)$ of an excited state, that is $O(13\hbar c/\lambda R)$, so the entropy bound is easily satisfied, at least in the weak coupling regime where the theory makes sense.

When the potential has $n = 5, 7, 9, \cdots$ equivalent wells with one centered at $\phi = 0$ and the rest disposed symmetrically about it, there will be a single zero-energy configuration ($\phi \equiv 0$), and $\frac{1}{2}(n-1)$ pairs of degenerate configurations with successively ascending $R$-dependent energies. For $n = 4, 6, 8, \cdots$ wells there is no zero-energy configuration, but there are $\frac{1}{2}n$ pairs of degenerate configurations with $R$ dependent energies. Because of the extra energy splitting appearing here already classically, I expect by analogy with the previous results, that the appropriate mean configuration energy (perhaps supplemented by wall energy), when multiplied by $2\pi R/\hbar c$, will bound the maximum entropy, $\ln n$, from above.

D. Zero mode systems

Years ago Unruh [22] proposed a counterexample to the universal entropy bound which has some resemblance to those based on degenerate ground states. He focused on a system with a zero (frequency) mode. For example, a real massless scalar field confined to a box by Neumann boundary conditions has a zero frequency mode: $\phi = q + pt$ with $q$ and $p$ real constants. Unruh noted that a vacuum state of such a scalar field is exclusively characterized by its $q$ and $p$. Noting further that one can add an arbitrary constant to $q$, he asserted that the system has an infinite number of degenerate vacua, with a common energy determined by $p^2$. This would evidently violate bound (1) by an infinite factor.
The problem with this proposal [23] is the identification of zero modes with distinct $q$ with one and the same system. This might make sense if the zero mode could be populated with quanta just as a $\omega \neq 0$ mode can. But in fact, a zero mode represents the classical part of $\phi$. It does not represent quanta: different $q$ do not stand for different “occupation numbers” in the ground state, but for different systems. The $q$ is very much like the order parameter in a superfluid which tells us the density of particles in the superfluid ground state, which in turn serves as base for all other, excited, states. Changing the order parameter defines a different superfluid (different particle density). Nobody would think of counting all superfluid systems to compute an entropy (which would of necessity be infinite). Likewise, it makes no sense to ascribe entropy to the multiplicity of $q$ values.

VII. THE PROLIFERATION OF SPECIES QUANDARY

A popular challenge to the entropy bound [2, 15, 24] imagines an hypothetical proliferation of particle species. Suppose there were to exist as many copies $\tilde{N}$ of a field e.g. the electromagnetic one, as one ordered. The entropy in a box containing a fixed energy allocated to the said fields should grow with $\tilde{N}$ because the bigger $\tilde{N}$, the more ways there are to split up the energy. Thus eventually the entropy should surpass the entropy bound. Numerical estimates show that it would take $\tilde{N} \sim 10^9$ to do the trick [21]. A similar picture seems to come from Eq. (13); the degeneracy factor $g$ should grow proportionally to $\tilde{N}$ making the factor in square brackets large so that, it would seem, one could not use the argument based on (13) to establish that $S < 2\pi R E/\hbar c$. These observations constitute the quandary of the proliferation of species.

There are several approaches to its resolution. As already remarked in Ref. [3] and reiterated later [4, 7], the above reasoning fails to take into account that each field species contributes to the Casimir energy which gets lumped in $\epsilon_0$. If these contributions are positive, then the negative term in the square bracket in Eq. (13) eventually dominates the logarithm as $\tilde{N}$ grows, and for large $\tilde{N}$ one again recovers the entropy bound (1). If they are negative (which implies a Casimir suction proportional to $\tilde{N}$ on the walls which delineate the system), then the scalar field example suggests that the wall energy, which must properly be included in $\epsilon_0$, should suffice to make the overall $\epsilon_0$ positive [4]. This would go a long way towards making the entropy bound safe.

Bousso [28] has pointed out that when some species contribute positive Casimir energies and some contribute negatively, a near cancellation of the Casimir energy could take place making the above saving strategy irrelevant. He suggests that radiative corrections to the interactions responsible for confining the particles of interest in the cavity make a contribution to $\epsilon_0$ proportional to $\tilde{N}$ on the walls which suffices to make the entropy bound work.

There is an alternative view [7, 39]: the seeming clash between entropy bound and an exceedingly large number of species merely tells us that physics is consistent only in a world with a limited number of particle species, such as ours. Indeed, as Brustein, Eichler, Foffa and Oaknin have argued [40], a very large number of species will make the vacuum of quantum field theory unstable against collapse into a “black hole slush”, unless we are willing to accept a rather modest ultraviolet cutoff for the theory. The unlimited proliferation of species may not even be physically consistent, and cannot thus constitute an argument against the entropy bound.

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