The Holographic Dark Energy in a Non-flat Universe

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We study the model for holographic dark energy in a spatially closed universe, generalizing the proposal in hep-th/0403127 for a flat universe. We provide independent arguments for the choice of the parameter \( c = 1 \) in the holographic dark energy model. On the one hand, \( c \) can not be less than 1, to avoid violating the second law of thermodynamics. On the other hand, observation suggests \( c \) be very close to 1, it is hard to justify a small deviation of \( c \) from 1, if \( c > 1 \).

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The time evolution of the cosmic scale factor depends on the composition of mass-energy in the universe. The cosmological observations [1,2] have offered strong evidence that the expansion of our universe is accelerating, due to dark energy with negative pressure. Dark energy (or cosmological constant) problem has been a longstanding problem in fundamental theoretical physics. It might be fair to say that theorists do not have any clue as to where this dark energy comes from, and how to compute it from the first principles. It is therefore a good approach to this problem to search for plausible phenomenological models, and let experiments select one of them.

In quantum field theory, we naively estimate the value of the cosmological constant as the zero point energy with a short distance cut-off scale, for example the Planck scale, which is exceedingly larger than the observational results. On the other hand, the cosmological constant problem is a cosmological problem, since we can ignore the cosmological constant when we explore the physics at very short distance. We only need to take the cosmological constant into account when we investigate the whole universe. Bekenstein et al. [3] proposed the bound $S \leq \pi M_p^2 L^2$ on the total entropy $S$ in a volume $L^3$ and this non-extensive scaling suggests that quantum field theory breaks down in large scale. A. Cohen et al. [4] proposed a relationship between UV and IR cut-offs to rescue the local quantum field theory from the spelling of formation of black holes, this results in a up-bound on the zero-point energy density. We may view this as the holographic dark energy, the magnitude of this dark energy is consistent with the cosmological observations. However, Hsu recently pointed out that the equation of state is not correct for describing the accelerating expansion of our universe [5]. Very recently, one of us [6] suggested that we should use the proper future event horizon of our universe to cut-off the large scale, resulting in a correct equation of state of dark energy. The model proposed in [6] is consistent with the Supernova data, which has been discussed in [7] in detail, and has been expanded to the frame of Brans-Dicke scalar-tensor theory in [8]. A recent discussion on other aspects of holography and dark energy is [9]. In this paper, we study this model in the context of a non-flat universe, extending the work [6] to a closed and or an open universe.

As usually believed, an early inflation era leads to a flat universe. This is not a necessary consequence if the number of e-foldings is not very large. It is still possible that there is a contribution to the Friedmann equation from the spatial curvature, though much smaller than other energy components according to observations. Therefore, it is not just
of academic interest to study a universe with a spatial curvature marginally allowed by the inflation model as well as observations.

According to [6], the dark energy density in a flat universe is

$$\rho_\Lambda = 3c^2 M_p^2 R_h^{-2},$$

(1)

where we may always let $c > 0$, $R_h$ is the proper size of the future event horizon,

$$R_h = a(t) \int_t^\infty \frac{dt'}{a(t')} = a \int_a^\infty \frac{da'}{H' a'^2}.$$  

(2)

Since $\Omega_\Lambda = \rho_\Lambda / \rho_{cr}$ and $\rho_{cr} = 3M_p^2 H^2$ is the critical energy density of our universe, eq.(1) tells us that

$$H R_h = \frac{c}{\sqrt{\Omega_\Lambda}}.$$  

(3)

Taking derivative with respect to $t$ in both sides of eq.(2), we obtain

$$\dot{R}_h = H R_h - 1 = \frac{c}{\sqrt{\Omega_\Lambda}} - 1.$$  

(4)

Following eq.(1), the changing rate of the holographic dark energy with time is

$$\frac{d\rho_\Lambda}{dt} = -6c^2 M_p^2 R_h^{-2} \dot{R}_h = -2H(1 - \frac{1}{c} \sqrt{\Omega_\Lambda}) \rho_\Lambda.$$  

(5)

Because of the conservation of the energy-momentum tensor, the evolution of the energy density of dark energy is governed by

$$\frac{d}{da} (a^3 \rho_\Lambda) = -3a^2 p_\Lambda,$$  

(6)

thus we obtain

$$p_\Lambda = -\frac{1}{3} \frac{d\rho_\Lambda}{d\ln a} - \rho_\Lambda.$$  

(7)

and the equation of the state of the holographic dark energy is characterized by the index

$$w_\Lambda = \frac{p_\Lambda}{\rho_\Lambda} = \frac{\rho_{cr}}{\rho_\Lambda} = \frac{3}{2} \frac{1}{\sqrt{\Omega_\Lambda}}.$$  

(8)

where we used $d\ln a = H dt$. The result (8) is a consequence of the definition of the holographic dark energy, thus is independent of other components of energy in the universe. According to this result, we see that the holographic dark energy has a generic property in a flat universe: the index of the equation of state is always $w_\Lambda \simeq -\frac{1}{3}$ when any other
type of energy dominates, and is always \( w_\Lambda = -\frac{1}{3}(1 + 2/c) \), when dark energy dominates. In the latter case, it behaves almost like the cosmological constant if \( c = 1 \).

Let us focus on the flat universe first, assuming there is only the holographic dark energy, \( \Omega_\Lambda = 1 \). The entropy of our universe is

\[
S = \frac{A}{4G} = \pi M_p^2 R_h^2. \tag{9}
\]

If we require the entropy of our universe do not decrease, we need \( \dot{R}_h \geq 0 \) which implies \( c \geq 1 \) by eq.\((\text{III})\) thus \( w_\Lambda \geq -1 \). The requirement that \( c \geq 1 \) must be imposed even when matter and radiation is present, since the universe will be gradually dominated by dark energy, so \( \dot{R}_h \) approaches \( c - 1 \) in the far future. Alternatively, if one defines, as motivated by the AdS/CFT correspondence, the central charge \( c \sim M_p^2 / H^2 \), we require \( \dot{H} \leq 0 \), and thus \( p \geq -\rho \) (since \( \dot{H} \sim -(\rho + p) \)), again we find \( c \geq 1 \).

If \( c < 1 \), we will run into another trouble. The proper size of the future horizon will shrink to zero, the IR cut-off will become shorter than the UV cut-off in a finite time in the future, the very definition of the holographic dark energy breaks down. \( c < 1 \) corresponds to holographic dark energy behaving as phantom, here we have provided a global argument against phantom, which was criticized by local arguments such as instability and production of gamma rays \( [\text{10}] \).

In the original proposal \( [\text{6}] \), \( c = 1 \) is adopted, and the motivation is the following. If the event horizon is to be regarded as a black hole horizon, the total energy from dark energy must be determined by the Schwarzschild relation, this leads to result \( c = 1 \).

Of course it is desirable to have an independent argument for \( c = 1 \). We have argued that \( c \geq 1 \). If one can argue \( c \leq 1 \), then \( c = 1 \) is the only choice. We can not think of a theoretical argument for \( c \leq 1 \) at the present, the experimental result can be taken as a support to the choice \( c = 1 \), since, if \( c > 1 \), it can not deviate from 1 too much, and there appears to be no theoretical reason for a small but non-vanishing deviation. Finally, if \( c \) is strictly greater than 1, in an empty universe when there is only dark energy, the Gibbons-Hawking entropy \( [\text{9}] \) will increase in time, and it appears rather bizarre that the empty universe generates entropy constantly. Thus, \( c = 1 \) is a good choice.

In \( [\text{6}] \), it was shown that for \( c = 1 \), the cosmic coincidence problem is resolved if the number of e-foldings is about 60, the minimal number required to solve the traditional horizon problem and flatness problem. The reasoning is simple: for dark energy to be the same order of the critical density at the present, it must be very small in the end of
inflation. Sufficient inflation red-shifts dark energy quickly, since \( w_\Lambda \) is about \(-1/3\) when the inflaton energy dominates. If \( c \) is not equal to 1, this result is still valid, and the resolution of the cosmic coincidence problem is preserved.

However, one can show that, if the inflaton energy never decays, then there will be no consistent solution to the Friedmann equation, since, as can be easily seen from the Friedmann equation, the holographic dark energy must eventually dominate the universe. This is possible only if the inflaton energy decays in a finite time, so that dark energy is red-shifted to be a very small fraction in the end of inflation but will dominate the universe in much later time. An alternative way to see this is the following. If the inflaton energy never decays and keeps almost as a constant, for \textit{all time} the Hubble constant is determined by the inflaton energy, so the size of the future horizon is roughly the inverse of the Hubble constant, thus dark energy can not be red-shifted away. Therefore, a consistent solution requires the existence of reheating, this is a rather interesting consequence of the holographic dark energy. We take this as a support to the holographic dark energy model, since this rules out an cosmological constant in addition to the holographic dark energy—after all, all zero-point energy must be included in the holographic dark energy. This issue together with other issues will be investigated elsewhere.

Now let us study the case of a universe with a spatial curvature. As said before, a closed universe with a small positive curvature (\( \Omega_k \sim 0.01 \)) is compatible with observations \[2,11,12\]. We generalize \[8\] to the spatial closed universe (results about the case of an open universe with a negative spatial curvature can be obtained from those of a closed universe by a transformation). If we still use the definition (1), when there is only dark energy and the curvature, \( \Omega_\Lambda = 1 + \Omega_k > 1 \) and \( \rho_\Lambda \) is not a constant. As a simple example, for \( c = 1 \), \( \dot{R}_h < 0 \). We know that in a flat universe, the solution with only dark energy present is a de Sitter space for \( c = 1 \) \[8\]. We can slice the de Sitter space with a positively curved spatial section, thus we expect that in this case dark energy remains a constant. Thus, we have to modify our definition (1) for the holographic dark energy in a closed universe, one choice is

\[ \rho_\Lambda = 3c^2 M_p^2 L^{-2}, \]  

(10)

here

\[ L = ar(t), \]  

(11)

and the definition of \( r(t) \) is

\[ \int_0^{r(t)} \frac{dr}{\sqrt{1 - kr^2}} = \int_t^\infty \frac{dt}{a} = \frac{R_h}{a}, \]  

(12)
or

\[ r(t) = \frac{1}{\sqrt{k}} \sin y, \quad (13) \]

where \( y = \sqrt{k} R_h/a \). According to eq. (13), \( r(t) = R_h/a \) when \( k = 0 \) and we get back to the same definition as in a flat universe. The geometric meaning of \( L \) is clear, while \( R_h \) is the radial size of the event horizon measured in the \( r \) direction, \( L \) is the radius of the event horizon measured on the sphere of the horizon. We shall see in a moment that with this definition of the holographic dark energy, the de Sitter space will be a solution for \( c = 1 \) with only dark energy present.

We want to keep the normalization \( a_0 = 1 \), using the energy density of curvature

\[ \rho_k = 3 M_p^2 k/a^2, \quad (14) \]

we obtain \( k = \Omega_k^0 H_0^2 \).

Using definitions \( \Omega_\Lambda = \rho_\Lambda/\rho_{cr} \) and \( \rho_{cr} = 3 M_p^2 H^2 \), we get

\[ HL = \frac{c}{\sqrt{\Omega_\Lambda}}. \quad (15) \]

Using (13), we obtain

\[ \dot{L} = HL + a \dot{r}(t) = \frac{c}{\sqrt{\Omega_\Lambda}} - \sqrt{1 - kr^2(t)} = \frac{c}{\sqrt{\Omega_\Lambda}} - \cos y, \quad (16) \]

and

\[ w_\Lambda = -\frac{1}{3} \left( 1 + \frac{2}{c} \sqrt{\Omega_\Lambda} \cos y \right). \quad (17) \]

This is a result valid regardless the energy content of the universe. If we take \( c = 1 \), then \( w_\Lambda \) is bounded from below by

\[ w_{\min} = -\frac{1}{3} \left( 1 + 2 \sqrt{\Omega_\Lambda} \right). \quad (18) \]

Taking \( \Omega_\Lambda = 0.73 \) for the present time, this lower bound is \(-0.90\). As we shall see shortly, \( c \) can not be less than 1, as in the flat case, thus, \( \Omega_\Lambda \) is always bounded by this number. Future experiments will either confirm this prediction or rule out this dark energy model based on this number.

According to (10) and (14), the ratio of the energy density between curvature and holographic dark energy is

\[ \frac{\Omega_k}{\Omega_\Lambda} = \frac{\rho_k}{\rho_\Lambda} = \frac{\sin^2 y}{c^2}. \quad (19) \]
We now study the behavior of the Gibbons-Hawking entropy in a closed universe. Since the space is curved in a closed universe, the Gibbons-Hawking entropy becomes

$$S = \pi M^2_p L^2.$$  \hspace{1cm} (20)

Using equation (16) and (20),

$$\frac{dS}{dt} = \frac{2S}{L} \frac{dL}{dt} = \frac{2S}{L} \left( \frac{c}{\sqrt{\Omega_\Lambda}} - \cos y \right).$$  \hspace{1cm} (21)

If the entropy can not be decreasing, we must require \( \Omega_\Lambda \cos^2 y \geq c^2 \). Applying this condition in (17), we find that the holographic dark energy can not be a phantom-like energy with \( w_\Lambda < -1 \). When there is only dark energy and the curvature, \( \Omega_\Lambda = 1 + \Omega_k \) and we have

$$\Omega_\Lambda = \left( 1 - \frac{\sin^2 y}{c^2} \right)^{-1}.$$  \hspace{1cm} (22)

and

$$\frac{dS}{dt} = \frac{2S}{L} \left( (c^2 - \sin^2 y)^{1/2} - \cos y \right).$$  \hspace{1cm} (23)

If \( c = 1, dS/dt = 0 \) and the holographic dark energy behaves like a cosmological constant with \( w_\Lambda = -1 \), this is the desired result supporting the new definition (10). If \( c > 1, dS/dt > 0 \) and \( w_\Lambda > -1 \). If \( c < 1, dS/dt < 0 \), the entropy is decreasing with the evolution with time, and the holographic dark energy looks like phantom energy with \( w_\Lambda < -1 \). So we also require \( c \geq 1 \) in a closed universe.

To describe our universe, we take matter into account. We use the Friedmann equation to relate the curvature of the universe to the energy density and expansion rate,

$$\Omega - 1 = \Omega_k, \quad \Omega = \Omega_m + \Omega_\Lambda = \frac{\rho_m}{\rho_{cr}} + \frac{\rho_\Lambda}{\rho_{cr}} \quad \text{and} \quad \Omega_k = \frac{\rho_k}{\rho_{cr}},$$  \hspace{1cm} (24)

where \( \rho_m (\rho_\Lambda) \) is the energy density of matter (dark energy). Since \( \rho_m = \rho_m^0 a^{-3} \) and \( \rho_k = \rho_k^0 a^{-2} \), the energy density of matter and curvature become

$$\rho_m = \Omega_m^0 \rho_{cr}^0 a^{-3};$$

$$\rho_k = \Omega_k^0 \rho_{cr}^0 a^{-2},$$  \hspace{1cm} (25)

where \( \rho_{cr}^0 = 3M_p^2 H_0^2 \) and we set \( a_0 = 1, a = (1 + z)^{-1} \), here \( z \) is the cosmological red-shift. Since the energy density of matter \( \rho_m \) decreases faster than \( \rho_k \), \( \rho_k \) will become larger than \( \rho_m \) in the future, if sufficient expansion is allowed. Using (24) and (25), we obtain

$$\frac{\Omega_k}{\Omega_m} = a\gamma, \quad \text{or} \quad 1 - a\gamma = \frac{1 - \Omega_\Lambda}{\Omega_m},$$  \hspace{1cm} (26)
where $\gamma = \Omega_k^0 / \Omega_m^0 < 1$, constrained by cosmological observations to be no larger than a few percent. When the curvature density catches up the matter density, $\Omega_k = \Omega_m$, so $a \gamma = 1$, $\Omega_\Lambda = 1$ or $z = -1 + \gamma$.

Now using Friedmann equation (24) and (25), we have

$$\frac{1}{aH} = \frac{1}{\sqrt{\Omega_m^0} H_0} \left( \frac{1 - \Omega_\Lambda}{a^{-1} - \gamma} \right)^{1/2},$$

(27)

and

$$\rho_\Lambda = \Omega_\Lambda \rho_{cr} = \frac{\Omega_\Lambda}{1 - \Omega_\Lambda} (\Omega_m - \Omega_k) \rho_{cr} = \frac{\Omega_\Lambda}{1 - \Omega_\Lambda} (\rho_m - \rho_k) = \rho_m \frac{\Omega_\Lambda}{1 - \Omega_m} (1 - a \gamma)a^{-3}.$$  

(28)

Combining eqs. (10), (13) and (28),

$$L = a \frac{\sqrt{k}}{\sqrt{\Omega_m^0} H_0} \sin y = a \frac{c}{\sqrt{\Omega_m^0} H_0} \left( \frac{1 - \Omega_\Lambda}{\Omega_\Lambda a^{-1} - \gamma} \right)^{1/2}.$$  

(29)

Substituting eqs. (2) and (27) into (29),

$$\frac{1}{\sqrt{k}} \sin \left[ \frac{\sqrt{k}}{\sqrt{\Omega_m^0} H_0} \int_x^\infty \frac{dx'}{a^{-1} - \gamma} \left( \frac{1 - \Omega_\Lambda'}{a^{-1} - \gamma} \right)^{1/2} \right] = \frac{c}{\sqrt{\Omega_m^0} H_0} \left( \frac{1 - \Omega_\Lambda}{\Omega_\Lambda a^{-1} - \gamma} \right)^{1/2},$$

(30)

where $x = \ln a$. Taking derivative with respect to $x$ in both sides of equation (30), we get

$$\frac{\Omega_\Lambda'}{\Omega_\Lambda^2} = \left( 1 - \Omega_\Lambda \right) \left( \frac{2}{c} \frac{1}{\sqrt{\Omega_\Lambda}} \cos y + \frac{1}{1 - a \gamma} \frac{1}{\Omega_\Lambda} \right),$$

(31)

where the prime denotes the derivative with respect to $x$.

We pause here to comment that, although the differential equation (31) is a consequence of the integral equation (30), its solution does not have to be a solution of the original integral equation, since the integral equation imposes a boundary condition at $t = \infty$ or $a = \infty$, namely, $\Omega_\Lambda$ must approach 1 at $t = \infty$. Thus, the introduction of the holographic energy always requires that the energy density will be dominated by dark energy eventually, as the existence of event horizon indicates.

Applying eqs. (17) and (31), we derive the rate of evolution of $w_\Lambda$ with the cosmological red-shift $z$

$$\frac{dw_\Lambda}{dz} = \frac{1}{3c} \sqrt{\Omega_\Lambda} \left[ \frac{1 - \Omega_\Lambda}{1 - \gamma (1 + z)^{-1}} + \frac{2}{c} \sqrt{\Omega_\Lambda} (1 - \Omega_\Lambda \cos^2 y) \right] \frac{1}{1 + z}.$$  

(32)
Since \( w_k = \rho_k / p_k = -1/3 \),

\[
\frac{\ddot{a}}{a} = -\frac{1}{6 M_p^2} (\rho_\Lambda + 3 p_\Lambda + \rho_m) = \frac{1}{6 M_p^2} \left( \frac{2}{c} \sqrt{\Omega_\Lambda \cos y - \frac{\rho_m}{\rho_\Lambda}} \right) \rho_\Lambda.
\]  

(33)

Once the expansion of our universe starts to accelerate, it will accelerate forever.

When our universe is dominated by matter, \( \Omega_\Lambda \ll 1 \), we get \( w_\Lambda \simeq -1/3 \) and \( \rho_\Lambda \propto a^{-2} \). On the other hand, after of the time when \( \Omega_m = \Omega_k \) or \( z = -1 + \gamma, 1 - a\gamma \) will be always negative and \( \Omega_\Lambda \) becomes always larger than one. According to eq. (33), \( \Omega'_\Lambda \) is positive and \( \Omega_\Lambda \) increases during the period of \( \Omega_\Lambda < 1 \). After the point \( \Omega_\Lambda = 1 \), \( \Omega_\Lambda \) will increase for a while until reaching a turning point. Passing over the turning point, \( \Omega'_\Lambda \) will become negative, and \( \Omega_\Lambda \) will steadily decrease to 1. The condition for \( \Omega'_\Lambda < 0 \) is

\[
\frac{2}{c} \sqrt{\Omega_\Lambda \cos y} + \frac{1}{1 - a\gamma} > 0.
\]  

(34)

Solving this relation, we obtain

\[
z < z_m = -1 + \gamma \left( 1 + \frac{c}{2 \sqrt{\Omega_\Lambda \cos y}} \right)^{-1}.
\]  

(35)

It says that \( \Omega_\Lambda \) will decrease after \( z = z_m \), but \( \Omega_\Lambda \) is still larger than one. According to equation (27), \( dt/da = 1/(aH) \) is always regular and we need a finite time to reach \( z = z_m \). The density parameter of the holographic dark energy \( \Omega_\Lambda \) reaches its maximum value at \( z = z_m \). Given the observational result for \( \gamma \leq 0.1 \), for a reasonable choice of \( c \) (certainly for \( c = 1 \)), the turning point \( 1 + z < 1 \), thus lies in future. We also notice that the quantity \( \sqrt{\Omega_\Lambda \cos y} \) does not reach its maximum value when \( z = z_m \). After a straightforward calculation, we find the value of \( \sqrt{\Omega_\Lambda \cos y} \) reaches its maximum value at

\[
z = z_M = -1 + \gamma \left( 1 + \frac{c}{2 \sqrt{\Omega_\Lambda}} \frac{1 - \Omega_\Lambda}{1 - \Omega_\Lambda \cos^2 y} \right)^{-1}.
\]  

(36)

Combining eqs. (19), (24) and (26), we get

\[
\Omega_\Lambda = \left[ 1 - \left( 1 - \frac{1}{a\gamma} \frac{\sin^2 y}{c^2} \right) \right]^{-1}.
\]  

(37)

Thus we obtain

\[
\frac{c}{\sqrt{\Omega_\Lambda}} - \cos y = \left( c^2 - \sin^2 y + \frac{1}{a\gamma} \sin^2 y \right)^{1/2} - \cos y.
\]  

(38)
According to this equation, if \( c \geq 1 \), the index of the state of the holographic dark energy will be always larger than \(-1\). According to (17), the holographic dark energy behaves as the phantom-like energy with \( w_{\Lambda} < -1 \) if the the maximum value of \( \sqrt{\Omega_\Lambda} \cos y \) is larger than \( c \), for \( c < 1 \).

It is interesting to explore the fate of this spatial closed universe. In the far future \( z \to -1 \) and \( a \to \infty \), the energy density of matter and curvature will be red-shifted to be exceedingly smaller than dark energy, dark energy will dominate our universe and \( \Omega_\Lambda \to 1^+ \). Using equation (19), we find

\[
\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - \frac{c^2 \Omega_k}{\Omega_\Lambda}} \to 1.
\]

Now equation (31) becomes

\[
\frac{d\Omega_\Lambda}{d\ln a} = \frac{2}{c}(1 - \Omega_\Lambda).
\]

Solving this equation, we obtain

\[
\Omega_\Lambda \sim 1 + \sigma a^{-2/c},
\]

where \( \sigma \) is a integration constant and \( \sigma a^{-2} \ll 1 \). This result is consistent with our above result \( \Omega_\Lambda \to 1^+ \), and guarantees that a solution to the differential equation (31) is also a solution to the integral equation (30). Using equation (17), we obtain

\[
w_{\Lambda} \sim -1 - \frac{1}{3} \sigma a^{-2/c},
\]

and the evolution of the energy density of dark energy is

\[
\rho_{\Lambda} \simeq \frac{\Omega_\Lambda}{\Omega_\Lambda - 1} \rho_k \simeq \frac{1}{\sigma a^{-2}} \rho^0_k a^{-2} = \sigma^{-1} \rho^0_k a^{-2(1-1/c)}.
\]

In order that the energy density of the holographic dark energy will always be finite, we require \( c \geq 1 \) which is consistent with our above discussion. Specially the energy density of the holographic dark energy will become a constant for the case with \( c = 1 \). If \( c > 1 \), the energy density of the holographic dark energy will be also red-shifted to zero asymptotically in the far future.

Setting \( k = 0 \), we find \( -\frac{1}{3}(1 + 2/c) \leq w_{\Lambda} \leq -\frac{1}{3} \) in the flat case, the same as the results in [3]. For the open universe, we obtain all the relevant results from those presented above by performing the simple transformation: \( k \to -k, \rho_k \to -\rho_k, \Omega_k \to -\Omega_k \) and \( \gamma \to -\gamma \). When \( z = -1 + \gamma \), the energy density of matter equals the energy density of curvature and
\(1 - \Omega_\Lambda = 2\Omega_m = 2\Omega_k\). In this case \(\Omega_\Lambda\) is always smaller than one. The expansion of our universe will also never decelerate.

To summarize, in this paper we generalized the holographic dark energy in [6] to a non-flat universe, and studied the evolution of this dark energy in the spatial closed universe in detail. In particular, we argued that in a flat universe as well as in a closed universe, to preserve the second law of thermodynamics, the parameter \(c\) must be no less than 1, thus it is impossible to have a phantom-like holographic dark energy.

The present approach to dark energy relates the cosmic coincidence problem to the minimal number of e-foldings [6], this resolution is quite similar to the up-bound on the number of e-foldings derived by Banks and Fischler [13] (for related discussions, see [14]), this is not surprising, since the holographic dark energy takes the energy bound with a horizon into account, while Banks-Fischler’s argument uses the entropy bound with a horizon, the latter is not as tight as the former.

In two aspects the holographic dark energy model is superior to other models of dark energy, for instance quintessence models. For one thing, our model is motivated by holography which is expected to hold in a quantum gravity theory, thus the model is almost unique, as we have argued that the only parameter \(c\) is to be set to 1, and there is no room for a cosmological constant in addition to dark energy. For another, models such as quintessence models do not really solve the cosmic coincidence problem, since we need to set a new energy scale in the end of inflation, though this scale is not as low as the current Hubble scale. In our model, we relate the present dark energy density to the minimal number of e-foldings. Of course one may question why we should choose this minimal number. One answer to this question is an application of a rather weak anthropic principle, the number of e-foldings can be arbitrary, but the choice 60 is rather generic. Another answer to this question is that after all this number is needed to solve the traditional cosmological naturalness problems, we only need one solution to all these problems. In the end, Nature may introduce only one input, the eventual size of the event horizon, all other parameters such as number of e-foldings and the CMB power spectrum are all consequences of this input.

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