Palatini approach to $1/R$ gravity and its implications to the late Universe

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By applying the Palatini approach to the $1/R$-gravity model it is possible to explain the present accelerated expansion of the Universe. Investigation of the late Universe limiting case shows that: (i) due to the curvature effects the energy-momentum tensor of the matter field is not covariantly conserved; (ii) however, it is possible to reinterpret the curvature corrections as sources of the gravitational field, by defining a modified energy-momentum tensor; (iii) with the adoption of this modified energy-momentum tensor the Einstein’s field equations are recovered with two main modifications: the first one is the weakening of the gravitational effects of matter whereas the second is the emergence of an effective varying “cosmological constant”; (iv) there is a transition in the evolution of the cosmic scale factor from a power-law scaling $a \propto t^{11/18}$ to an asymptotically exponential scaling $a \propto \exp(t)$; (v) the energy density of the matter field scales as $\rho_m \propto (1/a)^{36/11}$; (vi) the present age of the Universe and the decelerated-accelerated transition redshift are smaller than the corresponding ones in the ΛCDM model.

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I. INTRODUCTION

The interpretation of recent cosmological observations concerning the redshifts of supernovae of type Ia (see e.g. [1]) suggests that the Universe has evolved from a past decelerated phase to a present accelerated period. The decelerated period is normally described by a matter dominated Universe, and in order to simulate the transition from a decelerated to an accelerated phase from Einstein’s field equations one has to introduce another cosmological fluid which is gravitationally self-repulsive. This fluid with negative pressure is identified with some form of dark energy which dominates over the matter field in the accelerated period. At least two candidates for dark energy show up in literature, namely, the quintessence and the Chaplygin gas (see e.g. [2], [3] and the references therein, respectively). Another way to obtain the decelerated-accelerated transition without the need of dark energy is to introduce curvature corrections into Einstein’s field equations (see e.g. [4], [5], [6] and the references therein).

The present work is based on the method proposed by Vollick [7] which makes use of the Palatini approach in order to derive the field equations from a modified Hilbert-Palatini action. In the gravitational part of the action figures the sum of two contributions: one of them is the usual term proportional to the curvature scalar whereas the other – which represents a correction – is inversely proportional to it. The other part is the action for the matter.

Among other results – for the late times in the Universe when the curvature corrections become relevant – we show that the energy-momentum tensor of the matter field is not covariantly conserved due to curvature effects, but it is possible to define a modified energy-momentum tensor of the sources which is covariantly conserved. The definition of the modified energy-momentum tensor is a reinterpretation of the curvature corrections as sources of the gravitational field, and it turns out to be the sum of two constituents: a matter field whose coupling to the space-time geometry is weakened and an effective varying “cosmological constant”. By considering curvature effects up to the second order approximation we also show that the cosmic scale factor evolves from a matter dominated period with a $a \propto t^{11/18}$ to a cosmological constant dominated phase where $a \propto \exp(t)$, while the energy density of the matter field scales as $\rho_m \propto (1/a)^{36/11}$. Moreover, we compare the results of the present work with those that follow from the ΛCDM model. In particular, it is shown that due to curvature effects the present age of the Universe and the redshift in the decelerated-accelerated transition are smaller than the corresponding ones in the ΛCDM model.

II. FIELD EQUATIONS

We follow the work of Vollick [7] and write the modified Hilbert-Palatini action as

$$S = \int \frac{\mathcal{L}(R)}{2\kappa} + \mathcal{L}_M \sqrt{-g} d^4x.$$  \hspace{1cm} (1)

Here, $\kappa = 8\pi G$, $\mathcal{L}_M$ is the Lagrangian density of the matter field and $\mathcal{L}(R)$ refers to a generic Lagrangian density that depends on the scalar curvature $R = g^{\mu\nu} R_{\mu\nu}$, where $R_{\mu\nu}$ is the Riemann tensor given by:

$$R^\sigma_{\mu\lambda\nu} = \partial_\nu \Gamma^\sigma_{\mu\lambda} - \partial_\lambda \Gamma^\sigma_{\mu\nu} + \Gamma^\sigma_{\mu\beta} \Gamma^\beta_{\lambda\nu} - \Gamma^\alpha_{\mu\lambda} \Gamma^\sigma_{\alpha\nu}.$$  \hspace{1cm} (2)

The Palatini approach yields the field equations

$$\mathcal{L}'(R) R_{\mu\nu} - \frac{1}{2} \mathcal{L}(R) g_{\mu\nu} = -\kappa T_{\mu\nu}.$$  \hspace{1cm} (3)

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by performing the variation of the action with respect to $g_{\mu\nu}$. Above $R_{\mu\nu} = R^\sigma_{\mu\nu\sigma}$ is the Ricci tensor, the prime in the term $\mathcal{L}'(R)$ refers to a differentiation with respect to $R$ and the energy-momentum tensor $T_{\mu\nu}$ of the matter field is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta (\mathcal{L}M \sqrt{-g})}{\delta g^{\mu\nu}}.$$  \hfill (4)

Furthermore, the variation of the action with respect to $\Gamma^\sigma_{\mu\nu}$ leads to an equation which reduces to

$$\nabla_\sigma (\mathcal{L}'(R)g^{\mu\nu} \sqrt{-g}) = 0.$$ \hfill (5)

It follows from equation (5) that the affine connection is the Christoffel symbol with respect to the metric $h_{\mu\nu} \equiv \mathcal{L}'(R)g_{\mu\nu}$:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} h^{\alpha\sigma} [\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\alpha\mu} - \partial_\alpha h_{\mu\nu}].$$ \hfill (6)

The relationship between the affine connection $\Gamma^\sigma_{\mu\nu}$ and the Riemannian connection of the metric $g_{\mu\nu}$, denoted by $\tilde{\Gamma}^\sigma_{\mu\nu}$, is given by

$$\Gamma^\sigma_{\mu\nu} = \tilde{\Gamma}^\sigma_{\mu\nu} + \frac{1}{2} \left( 2\delta^\sigma_{\mu\nu} - g^{\sigma\tau} g_{\mu\nu} \partial_\tau \mathcal{L}' \right).$$ \hfill (7)

From equation (7) it follows a relationship between the Ricci tensor $R_{\mu\nu}$ and the Riemannian Ricci tensor $\tilde{R}_{\mu\nu}$, constructed from the Riemannian connection $\tilde{\Gamma}^\sigma_{\mu\nu}$, that reads

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{3}{2(L')} \partial_\nu \mathcal{L}' \partial_\mu \mathcal{L}'$$

$$+ \frac{1}{L'} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \mathcal{L}' + \frac{1}{2L'} \tilde{\nabla}^\sigma \tilde{\nabla}_\sigma \mathcal{L}' g_{\mu\nu}.$$ \hfill (8)

Above, $\tilde{\nabla}_\mu$ denotes the covariant differentiation associated with the Riemannian connection $\tilde{\Gamma}^\sigma_{\mu\nu}$.

The field equations (3) can be rewritten in terms of the Riemannian Ricci tensor $\tilde{R}_{\mu\nu}$ and of the Riemannian scalar curvature $\tilde{R}$ of the $g_{\mu\nu}$ metric, thanks to (8), as

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{R} g_{\mu\nu} = -\kappa \tilde{T}_{\mu\nu},$$ \hfill (9)

where the modified energy-momentum tensor of the sources $\tilde{T}_{\mu\nu}$ becomes

$$\tilde{T}_{\mu\nu} = \frac{T_{\mu\nu}}{L'} - \frac{1}{\kappa} \left[ \frac{3}{2(L')^2} \partial_\mu \mathcal{L}' \partial_\nu \mathcal{L}' - \frac{1}{L'} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \mathcal{L}'$$

$$- \frac{1}{2L'} \tilde{\nabla}^\sigma \tilde{\nabla}_\sigma \mathcal{L}' g_{\mu\nu} + \frac{1}{2} \left( \frac{\mathcal{L}}{L'} - \tilde{R} \right) g_{\mu\nu} \right].$$ \hfill (10)

From the Bianchi identity one can get the conservation law for the modified energy-momentum tensor of the sources, i.e., $\nabla_\nu \tilde{T}^{\nu\mu} = 0$.

We stress that, although the conformal metric $h_{\mu\nu}$ figures in the expression (4) for the affine connection $\Gamma^\sigma_{\mu\nu}$, we do not regard it, but $g_{\mu\nu}$, as the physical metric, since it was with respect to $g_{\mu\nu}$ that we performed the variation of the action in order to obtain the field equations, and it is $g_{\mu\nu}$ that figures in the definition of the energy-momentum tensor of the matter fields, eq. (4).

From now on we shall restrict ourselves to a homogeneous and isotropic plane Universe described by the Robertson-Walker metric $ds^2 = dt^2 - a(t)^2(dx^2 + dy^2 + dz^2)$ where $a(t)$ represents the cosmic scale factor.

We write the energy-momentum tensor of the matter field as $T^\mu_{\nu} = \text{diag}(\rho_m, -p_m, -p_m, -p_m)$ where $\rho_m$ and $p_m$ denote its energy density and pressure, respectively. Moreover, we represent the modified energy-momentum tensor of the sources as $\tilde{T}^\mu_{\nu} = \text{diag}(\rho, -p, -p, -p)$. From the expression (10) one can obtain that the energy density of the modified energy-momentum tensor of the sources reads

$$\rho = \frac{\rho_m}{L'} - \frac{1}{\kappa} \left[ \frac{3}{2(L')^2} \partial_\nu \mathcal{L}' - \frac{3}{2L'} \partial_\nu \mathcal{L}'$$

$$- \frac{3}{2L'} \frac{\dot{a}}{a} \partial_\nu \mathcal{L}' + \frac{1}{2} \left( \frac{\mathcal{L}}{L'} - \tilde{R} \right) \right],$$ \hfill (11)

whereas its pressure is given by

$$p = \frac{p_m}{L'} - \frac{1}{\kappa} \left[ \frac{1}{2(L')} \partial_\nu \mathcal{L}'$$

$$+ \frac{5}{2L'} \frac{\dot{a}}{a} \partial_\nu \mathcal{L}' - \frac{1}{2} \left( \frac{\mathcal{L}}{L'} - \tilde{R} \right) \right].$$ \hfill (12)

The acceleration and Friedmann equations follow from the field equations (9), yielding

$$\frac{\dot{a}}{a} = -\kappa (\rho + 3p), \quad \left( \frac{\dot{a}}{a} \right)^2 = \frac{\kappa}{3} \rho.$$ \hfill (13)

The evolution equation for the energy density can be obtained from the two above equations or from the conservation law for the modified energy-momentum tensor of the sources $\nabla_\nu T^{\nu\mu} = 0$ in a comoving frame, and reads

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0.$$ \hfill (14)

The expressions (13) and (14) are the usual acceleration, Friedmann and energy equations for a flat Universe whose source is a fluid with energy density $\rho$ and pressure $p$.

III. CURVATURE CORRECTIONS

Let us explore the modified Lagrangian density which has a term proportional to the scalar curvature plus a
we would have since in any measurement we could perform to check it this approximation that the Newtonian limit is verified, and (10) the Einstein equations directly follow. It is in that $2\alpha/R$ approximation we have the minus sign if $-\frac{\kappa T}{\kappa T} \gg 1$ and therefore the plus sign in equation (16) will be chosen.

The Einstein field equations are recovered in the limit $2\alpha/(\kappa T) \ll 1$ and by choosing the plus sign if $T > 0$ or the minus sign if $T < 0$. In that case $R \approx \kappa T$, implying that $2\alpha/R \ll 1$ and $L' \approx 1$, and therefore, from (9) and (10) the Einstein equations directly follow. It is in this approximation that the Newtonian limit is verified, since in any measurement we could perform to check it we would have $\kappa T \geq 2\alpha$. The opposite limit, $\kappa T \ll 2\alpha$, can only be observed in cosmological scales in which $T$ is of order $10^{-26}$ kg/m$^3$.

Here we are interested in matter fields such that $p_m \leq \rho_m/3$. Assuming that the weak energy condition (7) holds ($\rho_m \geq 0$), we will always have that $T = \rho_m - 3p_m \geq 0$ and therefore the plus sign in equation (16) will be chosen.

From now on we shall restrict ourselves to the limiting case where $\kappa T/(2\alpha) \ll 1$ so that up to the second order approximation we have

$$R \simeq \alpha \left[ 1 + \frac{\kappa T}{2\alpha} \right],$$

and the Lagrangian density and its derivative with respect to $R$ become

$$\mathcal{L} \simeq \frac{2}{3} \alpha \left[ 1 + \frac{\kappa T}{\alpha} \right], \quad \mathcal{L}' \simeq \frac{4}{3} \left[ 1 - \frac{\kappa T}{4\alpha} \right],$$

respectively.

By considering the above approximations one can write the modified field equations (14) as

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{R}_{\mu\nu} = -\frac{3}{4} \kappa T_{\mu\nu} - \Lambda g_{\mu\nu},$$

thanks to (14), (17) and (18). In the above equation we have introduced a cosmological constant term

$$\Lambda = \frac{3\alpha}{8} - \frac{R}{8} = \frac{\alpha}{4} - \frac{\kappa T}{16},$$

The right-hand side of equation (14) has a simple physical interpretation: (i) in the first term the coupling between the matter with the space-time geometry is decreased by a factor $3/4$ due to curvature effects; (ii) the second term is an effective cosmological constant $\Lambda$ which is a sum of a legitimate cosmological constant $\alpha/4$ plus a varying ”cosmological constant” $-\kappa T/16$ which decreases in module with time. We stress that the two terms of the cosmological constant are due to the curvature effects as it can be seen from the first equality in equation (21).

The Bianchi identity $\nabla_\nu (\tilde{R}_{\mu\nu} - \tilde{R} g_{\mu\nu}/2) = 0$ leads to

$$\nabla_\nu T^\nu_{\mu} = -\frac{4}{3\kappa} \partial_\mu \Lambda,$$

showing that the energy-momentum tensor of the matter field is not a covariantly conservative quantity. The right-hand side of (21) may be rewritten as $-4\partial_\mu \Lambda/(3\kappa) = \partial_\mu T/12 = \partial_\mu R/(6\kappa)$ thanks to (20), hence the non-conservation of the energy-momentum tensor of the matter field is a consequence of curvature effects.

The evolution equation for the energy density of the matter field in a comoving frame follows from equation (21), yielding

$$\dot{\rho}_m + 3\frac{\dot{a}}{a} (\rho_m + p_m) = \frac{1}{12} \dot{T} = \frac{1}{6\kappa} \dot{R}. \quad (22)$$

One can infer from the above equation that the curvature drains energy from the matter field or equivalently the matter field transfer more energy to the gravitational field due to curvature effects.

The energy density and pressure of the modified energy-momentum tensor of the sources $\tilde{T}_{\mu\nu}$ are given by

$$\rho = \frac{3}{4} \rho_m + \left( \frac{\alpha}{4\kappa} - \frac{1}{16} T \right) = \frac{3}{4} \rho_m + \rho_\Lambda,$$

$$p = \frac{3}{4} p_m - \left( \frac{\alpha}{4\kappa} - \frac{1}{16} T \right) = \frac{3}{4} p_m + p_\Lambda,$$

thanks to (14), (17), (23) and (24). Hence, the energy density of the modified energy-momentum tensor of the sources can be interpreted as a sum of two terms. The first one is a linear function of the energy density of the matter field where the factor $3/4$ is related to the weakening of the gravitational influence of the matter field due to the curvature effects. The second term is the energy density associated with the effective cosmological constant $\rho_\Lambda = -p_\Lambda$. We note that evolution equation for the matter field (22) is recovered if we insert the expressions (23) and (24) into the evolution equation (14) for energy density of the modified energy-momentum tensor of the sources.

From now on we shall consider a Universe filled with dust so that $p_m = 0$ and hence the trace of the energy-momentum tensor of the matter reduces to $T = \rho_m$. In this case from the evolution equation (22) for the energy density of the matter field it follows that

$$\rho_m = \rho_m^0 \left( \frac{a_0}{a} \right)^{3/\beta},$$

where $\beta = 11/12$. The parameter $\beta$ was introduced in order to compare the case where the curvature effects are
considered ($\beta = 11/12$) with the case where the curvature effects are not taken into account ($\beta = 1$).

We infer from (25) that the energy density of the matter field scales as $(a_0/a)^{36/11}$ and therefore decays more rapidly in comparison with $(a_0/a)^3$ when $a_0/a > 1$. It is worth to say that for a radiation field the trace of the energy momentum tensor vanishes, and there is no effect of the curvature on the radiation field, since from equation (23) it follows that its energy density scales as $(a_0/a)^4$.

We introduce the critical density and the density parameters usually defined by

$$\rho_c = \frac{3H^2}{\kappa}, \quad \Omega_m = \frac{\rho_m}{\rho_c}, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c},$$

where $H = \dot{a}/a$ denotes the Hubble parameter. Hence one obtains from (23) that $\Omega_{\text{eff}} + \Omega_\Lambda = 1$, where $\Omega_{\text{eff}} = 3\Omega_m/4$ is the effective density parameter of the matter field which takes into account the curvature effects. Moreover, one can introduce a density parameter for the legitimate cosmological constant $\Omega_\Lambda = \alpha/(4K\rho_c)$ so that it follows the relationship $\Omega_\Lambda = 1 - \beta \Omega_{\text{eff}}$. We note that for the $\Lambda$CDM model $\beta = 1$ and we have to consider $\Omega_{\text{eff}} \equiv \Omega_m$.

The Friedmann equation (16) can now be rewritten thanks to (29) as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Omega_\Lambda^0}{H_0^2} H_0^2 \left[1 + \frac{\beta \Omega_{\text{eff}}^0}{\Omega_\Lambda^0} \left(\frac{a_0}{a}\right)^{3/\beta}\right],$$

and the general solution of this differential equation reads

$$\frac{a(t)}{a_0} = \left[\sqrt{\frac{\beta \Omega_{\text{eff}}^0}{\Omega_\Lambda^0}} \sinh \left(\frac{3}{2\beta} \sqrt{\Omega_\Lambda^0 H_0^2 t}\right)\right]^{2/\beta}. \quad (28)$$

In the limiting case where $t \ll 1/\sqrt{\Omega_\Lambda^0 H_0}$ we get from equation (28) a matter dominated Universe which evolves according to

$$\frac{a(t)}{a_0} \approx \left(\frac{3}{2} \sqrt{\beta \Omega_{\text{eff}}^0 H_0 t}\right)^{2\beta/3}. \quad (29)$$

Hence, for a matter dominated Universe where the curvature effects are taken into account $a \propto t^{11/18}$ and it expands more slowly than a matter dominated Universe without curvature effects since in the last case $a \propto t^{2/3}$.

In the limiting case where $t \gg 1/\sqrt{\Omega_\Lambda^0 H_0}$ we obtain from (28) a cosmological constant dominated Universe with an accelerated expansion which evolves according to

$$\frac{a(t)}{a_0} \approx \left(\frac{\beta \Omega_{\text{eff}}^0}{4\Omega_\Lambda^0}\right)^{\beta/3} \exp \left(\sqrt{\Omega_\Lambda^0 H_0 t}\right). \quad (30)$$

From equation (25) one can obtain the relationship

$$t_0 H_0 = \frac{2\beta}{3} \frac{1}{\sqrt{1 - \beta \Omega_{\text{eff}}^0}} \arcsinh \left(\frac{1 - \beta \Omega_{\text{eff}}^0}{\beta \Omega_{\text{eff}}^0}\right), \quad (31)$$

where $t_0$ denotes the present time. If we consider that $\Omega_{\text{eff}}^0 = 0.3$ and $\beta = 11/12$ it follows from (31) that $t_0 H_0 \approx 0.905$. In the $\Lambda$CDM model we would have $\beta = 1$ and get $t_0 H_0 \approx 0.964$. Hence due to curvature effects the age of the Universe becomes smaller than the one where the curvature effects are not taken into account.

The deceleration parameter $q$ has the same expression as the one without considering the curvature effects, i.e.,

$$q = -\frac{\ddot{a}}{aH^2} = \frac{3}{2} \Omega_{\text{eff}} - 1, \quad (32)$$

whereas the density parameter as a function of the redshift $z$ reads

$$\Omega_{\text{eff}} = \frac{\Omega_{\text{eff}}^0 (1 + z)^{3/\beta}}{1 + \beta \Omega_{\text{eff}}^0 (1 + z)^{3/\beta} - \beta \Omega_{\text{eff}}^0}. \quad (33)$$

We have combined equations (22) and (33) and plotted in figure 1 the deceleration parameter $q$ as a function of the redshift $z$ for $\Omega_{\text{eff}}^0 = 0.3$. The case where the curvature effects are taken into account ($\beta = 11/12$) is represented by a straight line while the $\Lambda$CDM model ($\beta = 1$) is represented by a dashed line. From these curves we infer that the deceleration parameter increases more rapidly with the increasing of the redshift for case where the curvature effects are present, since $\Omega_{\text{eff}}$ also shows this behavior due to the relationship $\rho_m \propto (1 + z)^{3/\beta}$. Moreover, one obtains that the transition from a decelerated epoch to an accelerated one takes place at a redshift $z \approx 0.544$ when the curvature effects are taken into account and at $z \approx 0.671$ in the $\Lambda$CDM model.

From the luminosity distance, which is given by

$$d_L = (1 + z)H_0^{-1} \int_0^z \frac{dz}{\sqrt{1 + \beta \Omega_{\text{eff}}^0 (1 + z)^{3/\beta} - \beta \Omega_{\text{eff}}^0}}, \quad (34)$$

FIG. 1: Deceleration parameter $q$ vs redshift $z$. With curvature effects - straight line; $\Lambda$CDM model - dashed line.
one can build the difference between the apparent magnitude \( m \) and the absolute magnitude \( M \) of a source: 

\[
\mu_0 = m - M = 5 \log d_L + 25
\]

where \( d_L \) is given in Mpc.

In figure 2 we have plotted the difference \( \mu_0 \) as function of the redshift \( z \) – by considering \( \Omega_{0\text{eff}} = 0.3 \) and \( cH_0^{-1} = 3000/0.72 \) Mpc – for the case where the curvature effects are taken into account. The circles in this figure represent the experimental values taken from the work by Riess et al. for 185 data of supernovae of type Ia. We note that the curve fits the values of the data at an acceptable level. The curve which corresponds to the \( \Lambda \)CDM model is not sketched in this figure, since its departure from the curve plotted is not too significant concerning the dispersion of the values of the data.

As a final remark we note that additional modifications emerge when higher order approximations in \( \kappa \tau / (2\alpha) \ll 1 \) are considered, such as the dependence of the equations on higher order derivatives of the trace \( T \) and the time-variation of \( \beta \).