Linearized gravity on the de Sitter brane in the Einstein Gauss-Bonnet theory

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We investigate the linearized gravity on a single de Sitter brane in the anti-de Sitter (AdS) bulk in the Einstein Gauss-Bonnet (EGB) theory. We find that the Einstein gravity is recovered for a high energy brane, i.e., in the limit of the large expansion rate, $H \ell \gg 1$, where $H$ is the de Sitter expansion rate and $\ell$ is the curvature radius of the AdS bulk. We also show that, in the short distance limit $r \ll \min\{\ell, H^{-1}\}$, the Brans-Dicke gravity is obtained, whereas in the large distance limit $r \gg \max\{\ell, H^{-1}\}$, a Brans-Dicke type theory is obtained for $H \ell = O(1)$, and the Einstein gravity is recovered both for $H \ell \gg 1$ and $H \ell \ll 1$. In the limit $H \ell \to 0$, these results smoothly match to the results known for the Minkowski brane.

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I. INTRODUCTION

Recent progress in string theory suggests that our universe is not a four-dimensional spacetime in reality, but is a four-dimensional submanifold "brane" embedded in a higher dimensional spacetime called "bulk". As a simple realization of this braneworld, the model proposed by Randall and Sundrum (RS) \footnote{1} has attracted much attention because of its interesting feature that gravity is localized on the brane not because of compactification but by warping of the extra dimension. This model is a solution of the 5-dimensional Einstein equations with a negative cosmological constant, where a Minkowski brane is embedded in the 5-dimensional anti-de Sitter (AdS) bulk. The linear perturbation theory in the RS model reveals that the Einstein gravity is realized on the brane in the large distance limit. However, in the short distance limit, the gravity on the brane becomes essentially 5-dimensional, which may be interpreted as due to large contribution of the Kaluza-Klein corrections \footnote{2}. Cosmological extension of this model, inclusion of black holes and so on, have been discussed by various authors \footnote{3}.

From the stringy point of view, it is plausible that there may exist many fields and higher-order curvature corrections in addition to the bulk cosmological constant. In this paper, we consider the gravitational action with the Gauss-Bonnet term added to the usual Einstein-Hilbert term. This type correction appears as low energy corrections in the perturbative approach to string theory, and is a natural extension of the Einstein-Hilbert action from 4-dimensions to higher dimensions \footnote{4, 5, 6}. Cosmological braneworld models in the Einstein Gauss-Bonnet (EGB) theory have been discussed in \footnote{7, 8, 9, 10, 11, 12, 13}, and black holes in the EGB theory have been discussed in \footnote{14, 15, 16, 17, 18, 19, 20, 21, 22, 23}.

Recently, Deruelle and Sasaki showed that in the EGB theory, the linearized gravitational force on the Minkowski brane behaves like a 4-dimensional one even in the short distance limit \footnote{20}. Then Davis showed that the Brans-Dicke gravity \footnote{21} is realized on the Minkowski brane in the short distance limit \footnote{22}. Although, the effective gravitational theory in the nonlinear regime is unknown at all, these results imply that the experimental constraint on the maximum size (curvature radius) of the extra-dimension is drastically relaxed when compared with the RS model in which the size of the extra-dimension must be less than $\sim 0.1$ mm. Thus, the EGB theory deserves more detailed investigations from various aspects.

In this paper, as a step toward understanding cosmological implications of the EGB theory, we investigate the linear perturbation of a single de Sitter brane embedded in the AdS bulk. This paper is organized as follows. In Sec. II, we describe our set-up in the EGB theory. We consider an AdS bulk with a single de Sitter brane as the background spacetime. In Sec. III, we analyze the linear perturbation theory in the bulk and on the de Sitter brane. In Sec. IV, we discuss the effective gravity theory on the brane for various limits. In the limit $H \ell \gg 1$, where $H$ is the expansion rate of the de Sitter brane and $\ell$ is the AdS curvature radius, we find that the Einstein gravity with a cosmological constant is recovered on the de Sitter brane. We also show that the Brans-Dicke gravity is obtained in the short distance limit, whereas in the large distance limit a Brans-Dicke type theory is obtained for $H \ell = O(1)$ and the Einstein gravity both for $H \ell \gg 1$ and $H \ell \ll 1$. Furthermore, it is shown that the results for the Minkowski brane
are recovered in the limit $H^\ell \to 0$, namely, the Brans-Dicke gravity in the short distance limit and the Einstein gravity in the large distance limit [22]. In Sec. V, we briefly summarize our results. In Appendix A, we review the results for the Minkowski brane [22]. In Appendix B, we define harmonic functions on the de Sitter spacetime that correspond to the Fourier modes in the Minkowski spacetime. In Appendix C, we consider the case of two de Sitter branes and show that there exists a tachyonic bound state mode that makes the system unstable, just as in the Minkowski case discussed in [23].

II. EINSTEIN GAUSS-BONNET BRANEWORLD

We consider a braneworld in the EGB theory with a cosmological constant. As usual, we assume the mirror symmetry with respect to the brane. Then we may focus on one of the two identical copies of the spacetime $M$ with the brane as the boundary $\partial M$. The action is given by [4, 6];

$$S = \int_M d^5x \sqrt{-g} \frac{1}{2\kappa_5^2} \left[ \left(5\right)R - 2\Lambda_5 + \alpha \left(5\right)R^2 - 4 \left(5\right)R_{ab} \left(5\right)R^{ab} + \left(5\right)R_{abcd} \left(5\right)R^{abcd} \right]$$

$$+ \int_{\partial M} d^4x \sqrt{-q} \left[ -\sigma + \mathcal{L}_m + \frac{1}{\kappa_5^2} \left( K + 2\alpha \left( J - 2G_{\mu\nu}K^{\mu\nu} \right) \right) \right],$$

where $\alpha$ is the coupling constant for the Gauss-Bonnet term which has dimensions $(\text{length})^2$, $\Lambda_5$ is the negative cosmological constant, $g_{ab}$ and $q_{\mu\nu}$ are the bulk and brane metrics, respectively. $\mathcal{L}_m$ is the Lagrangian density of the matter on the brane, and $\sigma$ is the brane tension. The second term in the second line in Eq. (2.1) denotes the generalized Gibbons-Hawking term [24] which is added to the boundary action in order to obtain the well-defined boundary value problem. $K_{\mu\nu}$ is extrinsic curvature of the brane and

$$J_{\mu\nu} = -\frac{2}{3} K^{\mu\rho}K_{\rho\sigma}K_{\sigma\nu} + \frac{2}{3} KK^{\mu\rho}K_{\rho\nu} + \frac{1}{3} K^{\mu\nu} \left( K^{\rho\sigma}K_{\rho\sigma} - K^2 \right).$$

The Latin indices $\{a, b, \cdots\}$ and the Greek indices $\{\mu, \nu, \cdots\}$ are used for tensors defined in the bulk and on the brane, respectively.

Extremizing the action $S$ with respect to the bulk metric, the vacuum bulk Einstein Gauss-Bonnet equation is obtained as

$$\left(5\right)G_{ab} + \Lambda_5 g_{ab} + \alpha \left[ 2 \left(5\right)R_a^{\ cde(5)}R_{b\ cde} - 2 \left(5\right)R_{cd} \left(5\right)R_{acbd} - 2 \left(5\right)R_{ac} \left(5\right)R_{b}^{\ c} + \left(5\right)R \left(5\right)R_{ab} \right]$$

$$- \frac{1}{2} g_{ab} \left[ \left(5\right)R^2 - 4 \left(5\right)R_{cd} \left(5\right)R^{cd} + \left(5\right)R_{cd} \left(5\right)R^{def} \right] = 0.$$

The brane trajectory is determined by the junction condition which is obtained by varying the action $S$ with respect to the brane metric [22, 26];

$$B_{\mu\nu} = K_{\mu\nu} - K^{\delta\mu}_{\nu} + 4\alpha \left( \frac{3}{2} J_{\mu\nu} - \frac{1}{2} J^{\delta\mu}_{\nu} - P_{\mu\rho\sigma\delta}K^{\rho\sigma} \right) = \frac{1}{2} \kappa_5^2 T^\mu_{\nu},$$

where

$$P_{\mu\rho\sigma\delta} := R_{\mu\rho\sigma} + \left( R_{\mu\sigma}q_{\rho\nu} - R_{\rho\sigma}q_{\mu\nu} + R_{\rho\nu}q_{\mu\sigma} - R_{\mu\nu}q_{\rho\sigma} \right)$$

$$- \frac{1}{2} \frac{1}{R} \left( q_{\mu\sigma}q_{\rho\nu} - q_{\mu\nu}q_{\rho\sigma} \right),$$

and $T_{\mu\nu}$ is the energy momentum tensor of the matter on the brane, defined as

$$\delta \left( \sqrt{-q} \mathcal{L}_m \right) = -\frac{1}{2} \sqrt{-q} T_{\mu\nu} \delta q^{\mu\nu}.$$

Note that the extrinsic curvature here is the one for the vector normal to $\partial M$ pointing outward from the side of $M$.

III. DE SITTER BRANE IN THE EINSTEIN GAUSS-BONNET THEORY

Let us consider a de Sitter brane in the AdS bulk in the EGB theory, and investigate the linearized gravity on the de Sitter brane.
A. de Sitter brane in the Einstein Gauss-Bonnet theory

We take the Gaussian normal coordinate with respect to the brane, and assume the bulk metric in the form
\[ ds^2 = dy^2 + b^2(y)\gamma_{\mu\nu} dx^\mu dx^\nu, \] (3.1)
where \( \gamma_{\mu\nu} \) is the metric of the 4-dimensional de Sitter spacetime with \( R(\gamma) = 12H^2 \).

The background Einstein Gauss-Bonnet equation is
\[ -3H^2 + 3b''b + 3b^2 - 12\alpha\frac{b''}{b}(b^2 - H^2) = -\Lambda_5 b^2. \] (3.2)
This has a solution,
\[ b(y) = H\ell \sinh(y/\ell), \] (3.3)
where \( \ell \) is given by
\[ \frac{1}{\ell^2} = \frac{1}{4\alpha} \left(1 \pm \sqrt{1 + \frac{4\alpha\Lambda_5}{3}}\right), \] (3.4)
This agrees with the Minkowski brane case \[20, 28\]. Without loss of generality, we choose the location of the de Sitter brane at \( b(y_0) = 1 \). (3.5)

Thus \( H \) is the expansion rate of the de Sitter brane.

B. Bulk gravitational perturbations

First we consider gravitational perturbations in the bulk. We take the RS gauge \[1, 2, 29\],
\[ h_{55} = h_{5\mu} = 0, \quad h^{\mu}_\nu = D_\nu h^{\nu}_\mu = 0, \] (3.6)
where \( D_\alpha \) denotes the covariant derivative with respect to \( \gamma_{\mu\nu} \), and the perturbed metric is given by
\[ ds^2 = dy^2 + b^2(y)\left(\gamma_{\mu\nu} + h_{\mu\nu}\right) dx^\mu dx^\nu. \] (3.7)

The \((\mu,\nu)\)-components of the linearized Einstein Gauss-Bonnet equation are given by
\[ \left(1 - \tilde{\alpha}\right)\left[\frac{1}{\sinh^4(y/\ell)} \partial_y \left(\sinh^4(y/\ell) \partial_y\right) + \frac{1}{(H\ell)^2 \sinh^2(y/\ell)} \left(\Box - 2H^2\right)\right] h_{\mu\nu} = 0, \] (3.8)
where
\[ \tilde{\alpha} = \frac{4\alpha}{\ell^2}, \] (3.9)
and \( \Box = D^\mu D_\mu \) is the d’Alembertian with respect to \( \gamma_{\mu\nu} \). Throughout this paper, we assume \( \tilde{\alpha} \neq 1 \).

Equation (3.8) is separable. Setting \( h_{\mu\nu} = \psi_p(y) Y^{(p,2)}_{\mu\nu}(x^\alpha) \), we obtain
\[ \left[\frac{1}{\sinh^4(y/\ell)} \partial_y \left(\sinh^4(y/\ell) \partial_y\right) + \frac{m^2}{\ell^2 \sinh^2(y/\ell)}\right] \psi_p(y) = 0, \] (3.10)
\[ \left[\Box - (m^2 + 2)H^2\right] Y^{(p,2)}_{\mu\nu} = 0, \] (3.11)
where \( p^2 = m^2 - 9/4 \) and \( Y^{(p,2)}_{\mu\nu} \) are the tensor-type tensor harmonics on the de Sitter spacetime which satisfy the gauge condition \[30\],
\[ Y^{(p,2)}_{\mu\nu} = D_\nu Y^{(p,2)}_{\mu\nu} = 0. \] (3.12)
The properties of these harmonics are discussed in Appendix B. The equation (3.10) is the same as that for a massless scalar field in the bulk [31]. There exists a mass gap for the eigenvalue $0 < m < 3/2$ [27]. There is a unique bound state at $m = 0$, which gives $\psi_p(y) = \text{constant}$, and it is called the zero mode. For $m > 3/2$, the mass spectrum is continuous and they are called the Kaluza-Klein modes. The general solution is

$$\psi_p(y) = \frac{1}{\sinh^{3/2}(y/\ell)} \left[ A_p P_{3/2}^{ip}(\cosh(y/\ell)) + B_p Q_{3/2}^{ip}(\cosh(y/\ell)) \right],$$

(3.13)

where $P_{3/2}^{i}(z)$ and $Q_{3/2}^{i}(z)$ are the associated Legendre functions of the 1st and 2nd kinds, respectively.

For $p^2 > 0$ ($m > 3/2$), we choose those harmonic functions $Y^{(p,2)}_{\mu\nu}$ that behave as $e^{-ip t}$ in the limit $t \to \infty$. Then, assuming that there is no incoming wave from the past infinity $y = 0$, we find we should take $B_p = 0$. In fact, the asymptotic behavior of $P_{3/2}^{i}$ for $y \to 0$ is [32]

$$1 \sinh^{3/2}(y/\ell) P_{3/2}^{i}(\cosh(y/\ell)) \to \frac{2ip}{\Gamma(1-ip)} \left( \sinh(y/\ell) \cosh(y/\ell) \right)^{-ip-3/2} \approx \frac{2ip}{\Gamma(1-ip)} \left( \frac{y}{\ell} \right)^{-3/2} e^{-ip \ln(y/\ell)},$$

(3.14)

which guarantees the no incoming wave (i.e., retarded) boundary condition. Thus the bulk metric perturbations are constructed by

$$h_{\mu\nu} = \oint_C dp \psi_p(y) Y_{\mu\nu}^{(p,2)}(x^\mu),$$

(3.15)

where the contour of integration $C$ is chosen on the complex $p$-plane such that it runs from $p = -\infty$ to $p = \infty$ and covers the bound state pole at $p = 3i/2$ below the contour [33].

C. Linearized effective gravity on the brane

We now investigate the effective gravity on the brane. The position of the brane in the coordinate system is displaced in general as

$$y = y_0 - \ell \varphi(x^\mu),$$

(3.16)

where the second term in the right-hand-side describes the brane bending [20, 30]. The induced metric on the brane is given by

$$ds^2|(4) = (\gamma_{\mu\nu} + \bar{h}_{\mu\nu}) dx^\mu dx^\nu; \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - 2 \coth(y_0/\ell) \varphi \gamma_{\mu\nu}.$$

(3.17)

The extrinsic curvature on the brane is given by

$$K^\mu_{\nu} = \frac{1}{\ell} \coth(y_0/\ell) \delta^\mu_{\nu} + \frac{1}{2} h^\mu_{\nu,\nu} + \ell \left( D^\mu D_\nu + H^2 \delta^\mu_{\nu} \right) \varphi.$$

(3.18)

We consider the junction condition [20, 33]. The background part gives the relation between the brane tension and the location of the brane,

$$\kappa_5^2 \sigma = \frac{6}{\ell} \coth(y_0/\ell) \left( 1 - \frac{\alpha}{3} + \frac{2\alpha}{3 \sinh^2(y_0/\ell)} \right),$$

(3.19)

where

$$\coth(y_0/\ell) = \sqrt{1 + (H\ell)^2}, \quad \sinh(y_0/\ell) = \frac{1}{H\ell}.$$  

(3.20)

In the limit $H\ell \ll 1$, Eq. (3.19) reduces to the Minkowski tension,

$$\kappa_5^2 \sigma \simeq \frac{6}{\ell} \left( 1 - \frac{1}{3} \alpha \right).$$

(3.21)
The perturbative part of the junction condition gives
\[
(1 + \beta) \left( D_\mu D_\nu - \Box_4 \gamma_{\mu\nu} - 3H^2 \gamma_{\mu\nu} \right) \varphi + \frac{1}{2\ell} \left( 1 - \bar{\alpha} \right) h_{\mu\nu,y} - \frac{1}{2} \bar{\alpha} \coth(y_0/\ell) \left( \Box_4 - 2H^2 \right) h_{\mu\nu} = \frac{\kappa^2_5}{2\ell} S_{\mu\nu},
\]
where
\[
\beta := \frac{\cosh^2(y_0/\ell) + 1}{\sinh^2(y_0/\ell)} \bar{\alpha} = \left( 2 \coth^2(y_0/\ell) - 1 \right) \bar{\alpha} = \left( 2(H\ell)^2 + 1 \right) \bar{\alpha}.
\]

The trace of Eq. (3.22) gives the equation to determine the brane bending as
\[
\left( \Box_4 + 4H^2 \right) \varphi = -\frac{\kappa^2_5}{6(1 + \beta)\ell} S,
\]
where \( S = S^\mu_{\mu} \). Note that the field \( \varphi \) seems to be tachyonic, with mass-squared given by \(-4H^2\). However, in the case of a de Sitter brane in the Einstein gravity, there was a similar equation for the brane bending, but it was found to be non-dynamical [30]. We shall see below that the situation is quite similar in the present case of the EGB theory.

To find the effective gravitational equation on the brane, we manipulate as follows. Using the expression for the induced metric on the brane, Eq. (3.17), the perturbation of the brane Einstein tensor is given by
\[
\delta G_{\mu\nu}[\hat{h}] = -\frac{1}{2} \Box_4 h_{\mu\nu} - 2H^2 h_{\mu\nu} + 2 \coth(y_0/\ell) \left[ D_\mu D_\nu - \Box_4 \gamma_{\mu\nu} \right] \varphi
- 3H^2 \left( h_{\mu\nu} - 2 \coth(y_0/\ell) \gamma_{\mu\nu} \varphi \right)
- \frac{1}{2} \left( \Box_4 - 2H^2 \right) h_{\mu\nu} + 2 \coth(y_0/\ell) \left[ D_\mu D_\nu - \Box_4 \gamma_{\mu\nu} - 3H^2 \gamma_{\mu\nu} \right] \varphi.
\]

Using the perturbed junction condition (3.22) we can eliminate the term involving \( \varphi \) from the above equation to obtain
\[
\delta G_{\mu\nu}[\hat{h}] + 3H^2 \hat{h}_{\mu\nu} = -\frac{1 - \bar{\alpha}}{2(1 + \beta)} \left( \Box_4 - 2H^2 \right) h_{\mu\nu} - \frac{1 - \bar{\alpha}}{\ell(1 + \beta)} \coth(y_0/\ell) h_{\mu\nu,y} + \frac{\kappa^2_5}{2\ell} \coth(y_0/\ell) S_{\mu\nu}.
\]

Eliminating the term proportional to \( \left( \Box_4 - 2H^2 \right) h_{\mu\nu} \) from Eqs. (3.25) and (3.20) we obtain
\[
\delta G_{\mu\nu}[\hat{h}] + 3H^2 \hat{h}_{\mu\nu} = \frac{\kappa^2_5}{2\ell} \coth(y_0/\ell) S_{\mu\nu} - \frac{1 - \bar{\alpha}}{\bar{\alpha}} \coth(y_0/\ell) \left( D_\mu D_\nu - \gamma_{\mu\nu} \Box_4 - 3H^2 \gamma_{\mu\nu} \right) \varphi
- \frac{1 - \bar{\alpha}}{2\ell} \tanh(y_0/\ell) h_{\mu\nu,y}.
\]

Together with Eq. (3.24), this may be regarded as an effective gravitational equation on the brane. The effect of the bulk gravitational field is contained in the last term proportional to \( h_{\mu\nu,y} \).

### D. Harmonic decomposition

Using the harmonic functions defined in Appendix B, we may obtain a closed (integro-differential) system on the brane. We decompose the perturbations on the brane as
\[
S_{\mu\nu} = S_{\mu\nu}^{(0)} + S_{\mu\nu}^{(2)}; \quad S_{\mu\nu}^{(0)} = \int_{-\infty}^{\infty} dp \left( S_{p,0} Y_{\mu\nu}^{(p,0)} \right), \quad S_{\mu\nu}^{(2)} = \int_{-\infty}^{\infty} dp \left( S_{p,2} Y_{\mu\nu}^{(p,2)} \right),
\]
\[
\varphi = \int_{-\infty}^{\infty} dp \varphi(p) Y_{\mu\nu}^{(p,0)},
\]
\[
h_{\mu\nu} = \int_{-\infty}^{\infty} dp h(p) Y_{\mu\nu}^{(p,2)},
\]
where \( Y_{\mu\nu}^{(p,0)} \) are the scalar harmonics and \( Y_{\mu\nu}^{(p,0)} \) are the scalar-type tensor harmonics given in terms of \( Y^{(p,0)} \), as defined in Appendix B. Note that, because of the energy-momentum conservation, \( D^\mu S_{\mu\nu} = 0 \), there is no contribution
from the vector-type tensor harmonics which do not satisfy the divergence free condition. If a bound state exists, we have to deform the contour of integration so that the corresponding pole is covered, as mentioned at the end of subsection B.

With the above decomposition, the metric perturbation on the brane $\tilde{h}_{\mu\nu}$ given by Eq. 3.14 consists of the isotropic scalar-type part and tensor-type part. The scalar-type part is determined by Eq. 3.24, which gives

$$\varphi = \frac{\kappa_5^2}{2(1+\beta)} N_\mu S_{(p,0)} = \frac{\kappa_5^2}{2(1+\beta)} \frac{1}{\sqrt{3\sqrt{(p^2 + \frac{5}{4})}}} \sqrt{(p^2 + \frac{5}{4})} H^2 S_{(p,0)},$$

(3.29)

where $N_\mu$ is the normalization factor for the harmonics defined in Appendix B. We see that the propagator part of the $\varphi$ function does not couple to either the scalar or tensor-type matter perturbations on the brane. However, we shall argue in the next subsection that the mode that corresponds to the exponential growth of the perturbation is unphysical, namely, the tachyonic mode is absent and there is no instability associated with the brane bending due to the matter source on the brane.

Before we proceed, it is useful to note the equation,

$$\left( D_\mu D_\nu - \gamma_{\mu\nu} \Box_4 - 3H^2 \gamma_{\mu\nu} \right) \varphi = \frac{\kappa_5^2}{2(1+\beta)} \frac{1}{\ell} S_{\mu\nu}^{(0)},$$

(3.30)

which directly follows from Eq. 3.29 and the definition of the scalar-type tensor harmonics $Y_{\mu\nu}^{(p,0)}$.

There is a free propagating tachyonic mode corresponding to the homogeneous solution of Eq. 3.24, which does not couple to either the scalar or tensor-type matter perturbations on the brane. However, we shall argue in the next subsection that the mode that corresponds to the exponential growth of the perturbation is unphysical, namely, the tachyonic mode is exponentially decaying with time.

The traceless part of Eq. 3.24 gives

$$h_{(p)}(y_0) = -\frac{1}{(ip + 3/2)} \frac{\ell^2 \sinh(y_0/\ell) P^{ip}_{3/2}(z_0) (1 - \alpha) P^{ip}_{1/2}(z_0) + \alpha( -ip + 3/2)(H\ell)^2 \cosh(y_0/\ell) P^{ip}_{3/2}(z_0) \kappa_5^2}{\ell \kappa_5^2} S_{(p,2)},$$

(3.31)

where $z_0 = \cosh(y_0/\ell)$. This shows that the harmonic component of the tensor-type metric perturbations on the brane has a simple pole at $p = (3/2)i$ on the complex $p$-plane, which corresponds to the zero mode.

For convenience, we also write down the $y$-derivative of $h_{(p)}$,

$$\frac{1}{\ell} \partial_y h_{(p)}(y_0) = \frac{P^{ip}_{1/2}(z_0)}{(1 - \alpha) P^{ip}_{1/2}(z_0) + \alpha( -ip + 3/2)(H\ell)^2 \cosh(y_0/\ell) P^{ip}_{3/2}(z_0) \kappa_5^2}{\ell \kappa_5^2} S_{(p,2)}.$$

(3.32)

Then, Eqs 3.24, 3.27 and 3.32 constitute the effective gravitational equations on the brane that form a closed set of integro-differential equations.

### E. Source-free tachyonic mode

Now, we discuss the source-free tachyonic mode on the brane 27. This mode corresponds to the homogeneous solution of Eq. 3.24, so does not couple to the matter perturbations on the brane.

On the complex $p$-plane, the solution corresponds to the pole at $p = (3/2)i$. Thus, the solution is given by

$$\varphi = \varphi(3i/2) Y^{(3i/2,0)}. $$

(3.33)

For this mode, the junction condition 27 tells us that it is associated with a non-vanishing $h_{\mu\nu}$. The solution in the bulk is given by 27

$$h_{\mu\nu} = \phi(y) L_{\mu\nu} \varphi, \quad L_{\mu\nu} = D_\mu D_\nu + H^2 \gamma_{\mu\nu}. $$

(3.34)

This satisfies the transverse-traceless condition and

$$\left( \Box_4 - 4H^2 \right) h_{\mu\nu} = 0. $$

(3.35)

Thus, this mode falls within the mass gap between $m = 0$ and $3/2$, with the mass $mH = \sqrt{2}H$. 

Let us first analyze the behavior of the function $\phi(y)$. It should satisfy Eq. (3.38), which becomes

$$\left[ \frac{1}{\sinh^4(y/\ell)} \partial_y \left( \sinh^4(y/\ell) \partial_y \right) + \frac{2}{\ell^2 \sinh^2(y/\ell)} \right] \phi(y) = 0. \quad (3.36)$$

The general solution is given by

$$\phi(y) = c_1 \phi_1(y) + c_2 \phi_2(y); \quad \phi_1(y) = \coth(y/\ell), \quad \phi_2(y) = 1 + \coth^2(y/\ell),$$

where the coefficients $c_1$ and $c_2$ are related through the junction condition (3.22) as

$$1 - \frac{1}{2} H^2 c_1 - H^2 \coth(y_0/\ell) \left( \frac{1 + \tilde{\alpha} \coth^n(y_0/\ell)}{1 + \beta} \right) c_2 = 0. \quad (3.38)$$

As readily seen, this mode diverges badly as $y \to 0$. Therefore, the regularity condition at $y = 0$ will eliminate this mode. Nevertheless, since its effect on the brane seems non-trivial, it is interesting to see the physical meaning of it.

We note that $\phi_1$ is a gauge mode. This can be checked by calculating the projected Weyl tensor $E_{\mu\nu} := (\tilde{C}_{\mu\nu})^{(3)}$ which is gauge-invariant. We find that only the coefficient $c_2$ survives:

$$E_{\mu\nu}(y, x^\alpha) = \frac{c_2}{\ell^2 \sinh^4(y/\ell)} \alpha_{\mu\nu}(x^\alpha). \quad (3.39)$$

This means that the junction condition (3.38) does not fix the physical amplitude $c_2$. It just fixes the gauge amplitude $c_1$.

To understand the physical meaning of this mode, it is useful to analyze the temporal behavior the projected Weyl tensor. For simplicity, let us consider a spatially homogeneous solution for $\phi$. Choosing the spatially closed chart for the de Sitter brane, for which the scale factor is given by $a(t) = H^{-1} \cosh(\lambda t)$, we find

$$\phi = C_1 P_1^{\frac{5}{2}} \frac{\left( \tanh(\lambda t) \right)}{\cosh^{3/2}(\lambda t)} + C_2 P_2^{\frac{5}{2}} \frac{\left( \tanh(\lambda t) \right)}{\cosh^{3/2}(\lambda t)}, \quad t \to \infty \sim \tilde{C}_1 e^{\lambda t} + \tilde{C}_2 e^{-\lambda t}, \quad (3.40)$$

where $\tilde{C}_1$ and $\tilde{C}_2$ differ from $C_1$ and $C_2$, respectively, by unimportant numerical factors. We see that the solution associated with $C_1$ is the one that shows instability. If we insert this solution to Eq. (3.36), however, this unstable solution disappears. In fact, we obtain

$$E_{\mu\nu} \approx \frac{15 H^2 \tilde{C}_2 c_2}{\ell^2 \sinh^4(y/\ell) e^{4\lambda t}} \approx \frac{15(\lambda t)^2 \tilde{C}_2 c_2}{16(\lambda t)^4 \sinh^4(y/\ell) a^4(t)}. \quad (3.41)$$

We note that $E_{\mu\nu}$ on the brane decays as $1/a^4(t)$. This is exactly what one expects for the behavior of the so-called dark radiation. We also note that, although $E_{\mu\nu}$ does not vanish for spatially inhomogeneous modes, they decay as $1/a^3(t)$, giving no instability to the brane.

In the Einstein case, the dark radiation term appears if there exists a black hole in the bulk. This is also true in the EGB case. There also exists a spherically symmetric black hole solution in the EGB theory [14, 15, 16, 17, 18, 19]. The metric is given by

$$ds^2 = -f(R)dt^2 + \frac{dr^2}{f(R)} + R^2 d\Omega^2_3; \quad f(R) = 1 + \frac{R^2}{4\alpha} \left( 1 + \frac{16\mu}{3R^4} + 4 \frac{\alpha L_5}{3} \right), \quad (3.42)$$

where $\mu = \kappa_3^2 M/(2\pi^2)$ and $M$ is the mass of the black hole. For this solution, the projected Weyl tensor is given by

$$E_{\mu\nu} = \frac{\mu}{R^4} \left( 1 + \frac{4}{3} \frac{\alpha L_5}{3R^4} + \frac{16\mu}{3R^4} \right)^{-3/2} \left( 1 + \frac{4}{3} \frac{\alpha L_5}{9R^4} + \frac{16\mu}{9R^4} \right) \approx \frac{\mu}{R^4} \left( 1 + \frac{4}{3} \frac{\alpha L_5}{9R^4} \right)^{-1/2}, \quad (3.43)$$

for $R \gg (\alpha \mu)^\frac{1}{2}$. Comparing Eq. (3.32) with Eq. (3.41), with the identification $R = \ell \sinh(y/\ell) \cosh(\lambda t)$, we find

$$c_2 \tilde{C}_2 \approx \frac{16\mu}{15(\lambda t)^2} \left( 1 + \frac{4}{3} \frac{\alpha L_5}{9R^4} \right)^{-1/2}. \quad (3.44)$$

Thus the solution that decays exponentially in time corresponds to adding a small black hole in the bulk.

In the two-brane system, the mode discussed here corresponds to the radion, which describes the relative displacement of the branes [29, 30]. As the case of the Einstein gravity, the radion mode is truly tachyonic. However, for the EGB theory, there is a tachyonic bound state mode other than the radionic instability, as in the limit of the Minkowski brane [29], as discussed explicitly in Appendix C. This renders the two-brane system physically unrealistic in the EGB theory.
IV. LINEARIZED GRAVITY IN LIMITING CASES

In this section, we discuss the effective gravity on the brane in various limiting cases. We find the effective gravity reduces to 4-dimensional theories in all the limiting cases.

A. High energy brane: $H\ell \gg 1$

For a high energy brane, i.e., $H\ell \gg 1$ limit, we have $\tanh(y_0/\ell) \simeq 1/(H\ell)$ and $\beta \simeq 2(H\ell)^2$. We assume that matter perturbations on the brane are dominated by the modes $p \sim O(1)$, namely, we consider the case $H\ell \gg p$. Then, from Eq. (3.30) and Eq. (3.32), we find that the second and the third terms in the right-hand-side of Eq. (3.27) are suppressed by $1/(H\ell)^2$ relative to the first term,

$$\delta G_{\mu\nu}[\tilde{h}] + 3H^2\tilde{h}_{\mu\nu} = \frac{\kappa^2}{2\ell\alpha} \tanh(y_0/\ell) \left( S_{\mu\nu} + O\left((H\ell)^{-2}\right)\right).$$

(4.1)

Thus, we obtain Einstein gravity with the cosmological constant $3H^2$, with the gravitational constant $G_4$ given by

$$8\pi G_4 = \frac{\kappa^2}{2\ell\alpha} \tanh(y_0/\ell) \approx \frac{\kappa^2}{2(H\ell)\alpha\ell}.$$ (4.2)

The terms we have neglected give the low energy non-local corrections:

$$\left(\delta G_{\mu\nu}[\tilde{h}]\right)_{\text{corr, } H\ell} = -\frac{\kappa^2}{2\ell\alpha} \tanh(y_0/\ell) \times \int_{-\infty}^{\infty} dp \left\{ Y^{(p,2)}_{\mu\nu} S^{(p,2)} + \frac{P_{1/2}^{ip}(z_0)}{(1-\bar{\alpha})P_{1/2}^{ip}(z_0) + \bar{\alpha}(-ip + 3/2)(H\ell)^2 \cosh(y_0/\ell)P_{3/2}^{ip}(z_0)} \right\}.$$ (4.3)

B. Short and large distance limits

In order to discuss short and large distance limits, it is convenient to start from the expression (3.25) for the perturbed Einstein tensor, and Eq. (3.30) which relates the brane bending scalar $\varphi$ to the scalar part of the energy momentum tensor $S^{(0)}_{\mu\nu}$. Let us recapitulate these expressions:

$$\delta G_{\mu\nu}[\tilde{h}] + 3H^2\tilde{h}_{\mu\nu} = 2\coth(y_0/\ell) \left( D_\mu D_\nu - \gamma_{\mu\nu}\Box 4 - 3H^2\gamma_{\mu\nu}\right)\varphi - \frac{1}{2} \left( \Box 4 - 2H^2\right)\tilde{h}_{\mu\nu},$$ (4.4)

$$\left( D_\mu D_\nu - \gamma_{\mu\nu}\Box 4 - 3H^2\gamma_{\mu\nu}\right)\varphi = \frac{\kappa^2}{2(1+\beta)\ell} S^{(0)}_{\mu\nu}.$$ (4.5)

1. Short distance limit: $r \ll \min\{\ell, H^{-1}\}$

For the short distance limit $p \rightarrow \infty$, using Eq. (3.31), we find

$$-\frac{1}{2} \left( \Box 4 - 2H^2\right)\tilde{h}_{\mu\nu}$$

$$= \frac{\kappa^2}{2\ell} \int_{-\infty}^{\infty} dp Y^{(p,2)}_{\mu\nu} S^{(p,2)} \frac{(H\ell)(-ip + 3/2)P_{3/2}^{ip}(z_0)/P_{1/2}^{ip}(z_0) \cosh(y_0/\ell)\Box 4 - 2H^2\right)\tilde{h}_{\mu\nu} \rightarrow \frac{\kappa^2}{2\ell\alpha} \tanh(y_0/\ell) \int_{-\infty}^{\infty} dp Y^{(p,2)}_{\mu\nu} S^{(p,2)}.$$ (4.6)
Also, using Eq. (4.3), we manipulate as

\[
2 \coth(y_0/\ell) \left( D_\mu D_\nu - \gamma_{\mu\nu} \Box_4 - 3H^2 \gamma_{\mu\nu} \right) \varphi \\
= \frac{\kappa_5^2}{2\ell\alpha} \tanh(y_0/\ell) \int_{-\infty}^{\infty} dp \, S_{(p,0)} \mu_\nu \left( D_\mu D_\nu - \gamma_{\mu\nu} \Box_4 - 3H^2 \gamma_{\mu\nu} \right) \varphi,
\]

where we have used an identity,

\[
2 \coth(y_0/\ell) = 2 \coth(y_0/\ell) - \frac{1 + \beta}{\alpha} \tanh(y_0/\ell) + \frac{1 + \beta}{\alpha} \tanh(y_0/\ell)
\]

\[
= -\frac{1 - \bar{\alpha}}{\alpha} \tanh(y_0/\ell) + \frac{1 + \beta}{\alpha} \tanh(y_0/\ell),
\]

which follows from the definition of the parameter \( \beta \), Eq. (3.23).

Substituting Eqs. (4.6) and (4.7) in Eq. (4.4), the linearized gravity on the brane at short distances becomes

\[
\delta G_{\mu\nu} h + 3H^2 h_{\mu\nu} = \frac{\kappa_5^2}{2\ell\alpha} \tanh(y_0/\ell) S_{\mu\nu} - \frac{1 - \bar{\alpha}}{\alpha} \tanh(y_0/\ell) \left( D_\mu D_\nu - \Box_4 \gamma_{\mu\nu} - 3H^2 \gamma_{\mu\nu} \right) \varphi,
\]

with

\[
\left( \Box_4 + 4H^2 \right) \varphi = -\frac{\kappa_5^2}{6(1 + \beta)\ell} S.
\]

This is a scalar-tensor type theory.

Interestingly, the scalar field \( \varphi \) which describes the brane bending degree of freedom turns to be dynamical. As we have seen in the previous subsection, there is no intrinsically dynamical mode associated with the brane bending. Therefore, this emergence of a dynamical degree of freedom is due to an accumulative effect of the whole Kaluza-Klein modes, like a collective mode. Furthermore, because of the tachyonic mass, the system appears to be unstable. However, this is not the case. Since we have taken the limit \( p \to \infty \), all the perturbations have energy much larger than \( H \), and the tachyonic mass-squared \(-4H^2\) is completely negligible. In other words, the spacetime appears to be flat at sufficiently short distance scales.

We can rewrite Eq. (4.9) in the form,

\[
\delta G_{\mu\nu} h + \Lambda_4 h_{\mu\nu} = \frac{1}{\Phi_0} \left( D_\mu D_\nu - \Box_4 \gamma_{\mu\nu} - 3H^2 \gamma_{\mu\nu} \right) \delta \Phi + \frac{8\pi G_4}{\Phi_0} S_{\mu\nu},
\]

\[
\left( \Box_4 + 4H^2 \right) \delta \Phi = \frac{8\pi G_4}{3 + 2\omega} S,
\]

with the identifications,

\[
\frac{8\pi G_4}{\Phi_0} = \frac{\kappa_5^2}{2\ell\alpha} \tanh(y_0/\ell), \quad \frac{\delta \Phi}{\Phi_0} = -\frac{1 - \bar{\alpha}}{\alpha} \tanh(y_0/\ell) \varphi, \quad \omega = \frac{3\bar{\alpha}}{1 - \bar{\alpha}} \coth^2(y_0/\ell), \quad \Lambda_4 = 3H^2.
\]

Neglecting the tachyonic mass of \( \delta \Phi \), as justified above, this is the linearized Brans-Dicke gravity with a cosmological constant \( \Lambda_4 \). For \( H\ell \ll 1 \), we have \( \tanh(y_0/\ell) \simeq \coth(y_0/\ell) \simeq 1 \). Then

\[
\frac{8\pi G_4}{\Phi_0} \simeq \frac{\kappa_5^2}{2\alpha\ell}, \quad \frac{\delta \Phi}{\Phi_0} \simeq -\frac{1 - \bar{\alpha}}{\alpha} \varphi, \quad \omega \simeq \frac{3\bar{\alpha}}{1 - \bar{\alpha}}.
\]

This is in agreement with the Minkowski brane case investigated recently. For \( H\ell \ll 1 \), we have \( \tanh(y_0/\ell) \simeq \coth(y_0/\ell) \simeq 1 \). Then

\[
\frac{8\pi G_4}{\Phi_0} \simeq \frac{\kappa_5^2}{2\alpha\ell}, \quad \frac{\delta \Phi}{\Phi_0} \simeq -\frac{1 - \bar{\alpha}}{\alpha} \varphi, \quad \omega \simeq \frac{3\bar{\alpha}}{1 - \bar{\alpha}}.
\]

The corrections are written as

\[
\left( \delta G_{\mu\nu} h \right)_{\text{corr}, p > 1} = -\frac{\kappa_5^2}{2\ell\alpha} \tanh(y_0/\ell) \int_{-\infty}^{\infty} dp \, Y_{\mu\nu}^{(p,2)} S_{(p,2)} \left[ \frac{\left( 1 - \bar{\alpha} \right) P_{1/2}^{ip}(z_0)}{\left( 1 - \bar{\alpha} \right) P_{1/2}^{ip}(z_0) + \bar{\alpha}(H\ell)^2(-ip + 3/2) \cosh(y_0/\ell) P_{3/2}^{ip}(z_0)} \right].
\]

This is in agreement with the Minkowski brane case investigated recently.
2. Large distance limit: $r \gg \max \{ \ell, H^{-1} \}$

For the limit $p \to 0$, using Eq. (3.31), we have

$$-\frac{1}{2} \left( \Box \phi - 2H^2 \right) \delta h_{\mu\nu}$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dp \left[ Y^{(p,2)}_{\mu\nu}(p,2) \right] S^{(p,2)}(p,2) \frac{\kappa^2 \ell \sinh(y_0/\ell) H^2 (-ip + 3/2) P_{1/2}^{\alpha}(z_0)/P_{1/2}^{\alpha}(z_0)}{1 - \alpha + \alpha(H\ell)^2 \cosh(y_0/\ell) (-ip + 3/2) P_{1/2}^{\alpha}(z_0)/P_{1/2}^{\alpha}(z_0)}$$

$$\simeq \frac{3\kappa^2}{4\ell} \frac{(H\ell) P_{3/2}(z_0)/P_{1/2}(z_0)}{(1 - \alpha) + (3/2)(H\ell) \coth(y_0/\ell) \alpha P_{1/2}(z_0)/P_{1/2}(z_0)} \int_{-\infty}^{\infty} dp S^{(p,0)} Y^{(p,0)}_{\mu\nu}(p,0) \, . \quad \text{(4.15)}$$

As for the term involving $\phi$, we pull out the part that takes the same form as the above equation. Using Eq. (3.29), we find

$$2 \coth(y_0/\ell) \left( D_\mu D_\nu - \gamma_{\mu\nu} \Box \phi - 3H^2 \gamma_{\mu\nu} \right) \phi$$

$$= \frac{3\kappa^2}{4\ell} \frac{(H\ell) P_{3/2}(z_0)/P_{1/2}(z_0)}{(1 - \alpha) + (3/2)(H\ell) \coth(y_0/\ell) \alpha P_{1/2}(z_0)/P_{1/2}(z_0)} \int_{-\infty}^{\infty} dp S^{(p,0)} Y^{(p,0)}_{\mu\nu}(p,0) \phi$$

$$- \frac{(H\ell)(1 - \alpha) P_{-1/2}(z_0)}{2(1 + \beta) P_{1/2}(z_0) - (H\ell) \coth(y_0/\ell) \alpha P_{-1/2}(z_0)} \left( D_\mu D_\nu - \gamma_{\mu\nu} \Box \phi - 3H^2 \gamma_{\mu\nu} \right) \phi \, , \quad \text{(4.16)}$$

where we have used the recursion relation,

$$\frac{3}{2} P_{3/2}(z_0) = 2z_0 P_{1/2}(z_0) - \frac{1}{2} P_{-1/2}(z_0) \, . \quad \text{(4.17)}$$

Thus, the effective gravitational equation is expressed as

$$\delta G_{\mu\nu}[\tilde{h}] + 3H^2 \tilde{h}_{\mu\nu} = \frac{\kappa^2}{\ell} F_T S_{\mu\nu} - F_S \left( D_\mu D_\nu - \gamma_{\mu\nu} \Box \phi - 3H^2 \gamma_{\mu\nu} \right) \phi$$

$$\left( \Box \phi + 4H^2 \right) \phi = -\frac{\kappa^2}{\ell} S \, , \quad \text{(4.18)}$$

where we have rescaled $\phi$ to $\tilde{\phi} = 6(1 + \beta) \phi$, and $F_T$ and $F_S$ are constants that represent the tensor and scalar coupling strengths, respectively, given by

$$F_T = \frac{(H\ell) \left( 4 \cosh(y_0/\ell) P_{1/2}(z_0) - P_{-1/2}(z_0) \right)}{2 \left( 2(1 + \beta) P_{1/2}(z_0) - (H\ell)^2 \cosh(y_0/\ell) \alpha P_{-1/2}(z_0) \right)} \, ,$$

$$F_S = \frac{(H\ell)(1 - \alpha) P_{-1/2}(z_0)}{6(1 + \beta) \left( 2(1 + \beta) P_{1/2}(z_0) - (H\ell)^2 \cosh(y_0/\ell) \alpha P_{-1/2}(z_0) \right)} \, . \quad \text{(4.19)}$$

In the intermediate range of $H\ell$, i.e., when $H\ell = O(1)$, then $F_T$ and $F_S$ are comparable and we obtain a Brans-Dicke type theory given by Eq. (4.11) with the identifications,

$$\frac{8\pi G_4}{\Phi_0} = \kappa^2 \ell F_T \, , \quad \frac{\delta \phi}{\Phi_0} = -F_S \phi \, , \quad \Lambda_4 = 3H^2 \, , \quad \text{(4.20)}$$

$$\omega = \frac{F_T - 3F_S}{2F_S} = \frac{6(1 + \beta) \cosh(y_0/\ell) P_{1/2}(z_0) - 3 \left( 1 + (H\ell)^2 \alpha P_{-1/2}(z_0) \right)}{(1 - \alpha) P_{-1/2}(z_0)} \, .$$

A potential problem in this case is that the tachyonic mass of the scalar field seems to make the system unstable. However, as discussed in Sec. III D, the tachyonic pole is not excited by the matter source. Further, as discussed in Sec. III E, the source-free tachyonic mode do not cause an instability either.

For $H\ell \ll 1$, $\omega \gg 1$ and the scalar field decouples to yield

$$\delta G_{\mu\nu}[\tilde{h}] + 3H^2 \tilde{h}_{\mu\nu} = \frac{\kappa^2}{\ell} \frac{\coth(y_0/\ell)}{1 + \beta} S_{\mu\nu} \, . \quad \text{(4.21)}$$
Thus we obtain the Einstein gravity with

$$8\pi G_4 = \frac{\kappa_5^2}{\ell} \frac{\coth(y_0/\ell)}{1 + \beta}. \quad (4.22)$$

In the limit $H\ell \to 0$,

$$8\pi G_4 \simeq \frac{\kappa_5^2}{\ell} \frac{1}{1 + \bar{\alpha}}. \quad (4.23)$$

This is the result for the Minkowski brane.

In the limit $H\ell \gg 1$, $\omega \gg 1$ and we recover the 4-dimensional Einstein gravity on the brane with

$$8\pi G_4 = \frac{\kappa_5^2}{2(H\ell)\bar{\alpha} \ell}. \quad (4.24)$$

Note that this is just a special case of the high energy brane case discussed in subsection A.

Thus we conclude that despite the presence of the tachyonic mass, the system is stable and well-behaved for all ranges of $H\ell$.

V. SUMMARY AND DISCUSSION

We have investigated the linear perturbation of a de Sitter brane in an Ant-de Sitter bulk in the 5-dimensional Einstein Gauss-Bonnet (EGB) theory. We have derived the effective theory on the brane which is described by a set of integro-differential equations.

To understand the nature of this theory in more details, we have investigated the behavior of the theory in various limiting cases. In contrast to the case of a braneworld in the 5-dimensional Einstein gravity, in which both the short distance and high energy brane limits exhibit 5-dimensional behavior, we have found that the gravity on the brane is effectively 4-dimensional for all the limiting cases.

For a high energy brane, i.e., in the limit $H\ell \gg 1$, the Einstein gravity is recovered, provided that the length scale of fluctuations is of order $H^{-1}$. It is found that the low energy corrections are suppressed by the factor $O((H\ell)^{-2})$.

In the short distance limit $r \ll \min\{\ell, H^{-1}\}$, the scalar field that describes brane bending becomes dynamical, and we obtain the Brans-Dicke gravity. This is consistent with the case of the Minkowski brane. A slight complication is that this brane-bending scalar field is tachyonic, with mass-squared $-4H^2$. Therefore, if it becomes dynamical, one would naively expect the theory to become unstable. However, since the energy scale of fluctuations are much larger than $H$, the fluctuations actually do not see this tachyonic mass, hence there is no instability.

In the large distance limit $r \gg \max\{\ell, H^{-1}\}$, the Einstein gravity is obtained in both limits $H\ell \ll 1$ and $H\ell \gg 1$, while a Brans-Dicke type theory is obtained for $H\ell = O(1)$. Although the scalar field of this Brans-Dicke gravity is tachyonic with mass-squared given by $-4H^2$, we have shown that this mode is not excited by the matter source, hence does not lead to an instability of the system.

In the limit $H\ell \to 0$, the previous results for the Minkowski brane have been recovered, that is, the Brans-Dicke gravity at short distances and the Einstein gravity at large distances.

In all the cases, the effective 4-dimensional gravitational constant depends non-trivially on the values of $H\ell$ and $\bar{\alpha}$, where $\bar{\alpha}$ is the non-dimensional coupling constant for the Gauss-Bonnet term. This indicates the time variation of the gravitational constant in the course of the cosmological evolution of a brane in the EGB theory. It will be interesting to investigate in more details the cosmological implications of the braneworld in the EGB theory.

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APPENDIX A: THE RESULTS FOR THE MINKOWSKI BRANE

Here, we summarize the results for the Minkowski brane.
1. Effective equations on the brane

In the RS gauge, the perturbed metric in the bulk is written
\[ ds^2 = dy^2 + b^2(y)(\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu, \quad b(y) = e^{-|y|/\ell}, \]  \hspace{1cm} (A1)
where \( \eta_{\mu\nu} \) is the Minkowski metric. The brane locates at \( y = 0 \) in the background. The background part of the Einstein Gauss-Bonnet equation \( [2,3] \) gives the relation of the AdS radius to the bulk cosmological constant, Eq. \( [3,4] \). The perturbative part of Eq. \( [2,3] \) gives
\[ \left(1 - \bar{\alpha}\right)\left(\partial_y^2 - 4\frac{1}{\ell^2}\partial_y + e^{2y/\ell}\Box_4\right)h_{\mu\nu} = 0 . \]  \hspace{1cm} (A2)
Again, we consider the case \( \bar{\alpha} \neq 1 \). The location of the brane is perturbed to be at \( y = -\ell \varphi \). Induced metric on the brane is given by
\[ ds^2 \big|_{y = 0} = (\eta_{\mu\nu} + \tilde{h}_{\mu\nu})dx^\mu dx^\nu , \quad \tilde{h}_{\mu\nu} = h_{\mu\nu} - 2\varphi\eta_{\mu\nu} . \]  \hspace{1cm} (A3)

The solution for \( h_{\mu\nu} \) on the brane which satisfies the junction condition is given by
\[ h_{\mu\nu} \big|_{y = 0} = -\frac{\kappa_5^2}{\ell} \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} \frac{\ell^2 H_2^{(1)}(q\ell)}{(1 - \bar{\alpha})q^2 H_1^{(1)}(q\ell) + \bar{\alpha}q^2 \ell^2 H_2^{(1)}(q\ell)} \left[ S_{\mu\nu}(p) - \frac{1}{3} \left( \eta_{\mu\nu} - \frac{pp_{\mu\nu}}{p^2} \right) S(p) \right] , \]  \hspace{1cm} (A4)
where \( H_\nu^{(1)} \) is the Hankel function of the first kind and \( q^2 = -p^2 \). The equation that determines the brane bending is
\[ \Box_4 \varphi = -\frac{\kappa_5^2}{6\ell^2} \frac{1}{1 + \bar{\alpha}} . \]  \hspace{1cm} (A5)

The perturbed 4-dimensional Einstein tensor is expressed as
\[ \delta G_{\mu\nu} [\tilde{h}] = -\frac{1}{2} \Box_4 \left( h_{\mu\nu} + 2(\partial_\mu \partial_\nu - \eta_{\mu\nu} \Box_4) \varphi \right) . \]  \hspace{1cm} (A6)

Inserting Eq. \( (A4) \) into Eq. \( (A6) \), we obtain the effective equation on the brane, which reads
\[ \delta G_{\mu\nu} [\tilde{h}] \big|_{y = 0} = -\frac{\kappa_5^2}{2\bar{\alpha} \ell} \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} \frac{\ell^2 H_2^{(1)}(q\ell)}{(1 - \bar{\alpha})q^2 H_1^{(1)}(q\ell) + \bar{\alpha}q^2 \ell^2 H_2^{(1)}(q\ell)} \left[ S_{\mu\nu}(p) - \frac{1}{3} \left( \eta_{\mu\nu} - \frac{pp_{\mu\nu}}{p^2} \right) S(p) \right] \]
\[ + 2(\partial_\mu \partial_\nu - \eta_{\mu\nu} \Box_4) \varphi . \]  \hspace{1cm} (A7)

2. Short distance limit

In the short distance limit \( q\ell \gg 1 \), Eq. \( (A7) \) becomes
\[ \delta G_{\mu\nu} [\tilde{h}] \big|_{y = 0} = -\frac{\kappa_5^2}{2\bar{\alpha} \ell} S_{\mu\nu} - \left( \frac{1 - \bar{\alpha}}{\bar{\alpha}} \right) (\partial_\mu \partial_\nu - \eta_{\mu\nu} \Box_4) \varphi . \]  \hspace{1cm} (A8)

Comparing Eqs. \( (A8) \) and \( (A9) \) with the linearized Brans-Dicke gravity,
\[ \delta G_{\mu\nu} [\tilde{h}] \big|_{y = 0} = \frac{8\pi G_4}{\Phi_0} S_{\mu\nu} + \frac{1}{\Phi_0} (\partial_\mu \partial_\nu - \eta_{\mu\nu} \Box_4) \delta \Phi, \quad \Box_4 \delta \Phi = \frac{8\pi G_4}{3 + 2\omega} S, \]  \hspace{1cm} (A9)
we find the correspondences,
\[ \frac{8\pi G_4}{\Phi_0} = \frac{\kappa_5^2}{2\bar{\alpha} \ell}, \quad \frac{\delta \Phi}{\Phi_0} = -\frac{1 - \bar{\alpha}}{\bar{\alpha}} \varphi, \quad \omega = \frac{3\bar{\alpha}}{1 - \bar{\alpha}} . \]  \hspace{1cm} (A10)

The corrections are rewritten as
\[ \left( \delta G_{\mu\nu} [\tilde{h}] \right)_{\text{corr}} = -\frac{\kappa_5^2}{2\bar{\alpha} \ell} \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} \frac{(1 - \bar{\alpha})q^2 H_1^{(1)}(q\ell)}{(1 - \bar{\alpha})q^2 H_1^{(1)}(q\ell) + \bar{\alpha}q^2 \ell^2 H_2^{(1)}(q\ell)} \left[ S_{\mu\nu} - \frac{1}{3} \left( \eta_{\mu\nu} - \frac{pp_{\mu\nu}}{p^2} \right) S \right] . \]  \hspace{1cm} (A11)
3. Large distance limit

In the large distance limit \( q \ell \ll 1 \), Eq. (A7) becomes

\[
\delta G_{\mu\nu}[\tilde{h}] = \frac{\kappa^2}{\ell} \frac{1}{1 + \tilde{\alpha} S_{\mu\nu}}.
\]

(A12)

Thus we obtain the Einstein gravity with

\[
8\pi G_4 = \frac{\kappa^2}{\ell} \frac{1}{1 + \tilde{\alpha}}.
\]

(A13)

APPENDIX B: HARMONIC FUNCTIONS ON DE SITTER GEOMETRY

In this Appendix, we consider the harmonics on the de Sitter spacetime with curvature radius \( H^{-1} \). They are obtained by the Lorentzian generalization of the tensor harmonics on an \( n \)-dimensional constant curvature Riemannian space \([36]\). We focus on the tensor-type and scalar-type harmonics.

1. Tensor-type harmonics

The tensor-type tensor harmonics satisfy

\[
\left( \square_4 - (p^2 + 17/4)H^2 \right) Y^{(p,2)}_{\mu\nu}(x) = 0,
\]

(B1)

which corresponds to the 4-dimensional massive gravitons with mass-squared \( m^2 H^2 = (p^2 + 9/4)H^2 \). They satisfy the transverse-traceless condition,

\[
Y^{(p,2)\mu}_{\mu} = Y^{(p,2)\nu}_{\nu} = 0.
\]

(B2)

In reality, the tensor harmonics have 3 more indices for the spatial eigenvalues. If we adopt the flat slicing,

\[
ds^2 = -dt^2 + H^{-2}e^{2Ht} \delta_{ij} dx^i dx^j,
\]

(B3)

we can use the standard Fourier modes \( e^{ik \cdot x} \), and the spatial indices will be continuous. In addition, we also have discrete indices \( \sigma \) that describe the polarization degrees of freedom (5 in 4-dimensions). However, for notational simplicity, we omit these indices.

We ortho-normalize the tensor harmonics as

\[
\int d^4x \sqrt{-\gamma} Y^{(p,2)}_{\mu\nu} Y^{(p',2)*}_{\mu\nu} = \delta(p - p') \delta^3(k - k') \delta_{\sigma,\sigma'}.
\]

(B4)

Although we have no explicit proof for the completeness, due to our poor knowledge, we assume that \( Y^{(p,2)}_{\mu\nu} \) for \(-\infty < p < \infty\) constitute a complete set for the space of transverse-traceless tensors.

2. Scalar-type harmonics

The scalar-type harmonics \( Y^{(p,0)}(x) \) satisfy the equation for a scalar field with mass-squared \( m^2 H^2 = (p^2 + 9/4)H^2 \),

\[
\left( \square_4 - (p^2 + 17/4)H^2 \right) Y^{(p,0)}(x) = 0.
\]

(B5)

We assume they satisfy the ortho-normality condition,

\[
\int d^4x \sqrt{-\gamma} Y^{(p,0)} Y^{(p',0)*} = \delta(p - p') \delta^3(k - k').
\]

(B6)
From $Y^{(p,0)}$, the ortho-normalized scalar-type vector harmonics are constructed as
\[
Y^{(p,0)}_{\mu} = \frac{i}{H\sqrt{p^2 + 9/4}}D_{\mu}Y^{(p,0)},
\] (B7)
which satisfy
\[
\int d^4x \sqrt{-\gamma}Y^{(p,0)}_{\mu}Y^*(p',0)_{\mu} = \delta(p - p')\delta^3(k - k').
\] (B8)

The trace-free and divergence-free scalar-type tensor harmonics are constructed, respectively, as
\[
\bar{Y}^{(p,0)}_{\mu\nu} = N_p \left[ D_{\mu}D_{\nu}Y^{(p,0)} - \frac{1}{4} \left( p^2 + \frac{9}{4} \right) \gamma_{\mu\nu}H^2Y^{(p,0)} \right]
\]
\[
Y^{(p,0)}_{\mu\nu} = N_p \left[ D_{\mu}D_{\nu}Y^{(p,0)} - \left( p^2 + \frac{21}{4} \right) \gamma_{\mu\nu}H^2Y^{(p,0)} \right]
\]
\[
= \bar{Y}^{(p,0)}_{\mu\nu} - \frac{3}{4}N_p \left( p^2 + \frac{25}{4} \right) H^2\gamma_{\mu\nu}Y^{(p,0)},
\] (B9)
where
\[
|N_p|^2 = \frac{1}{3(p^2 + 21/4)(p^2 + 25/4)H^4}.
\] (B10)

Without loss of generality, we assume that $N_p$ is real and positive. The scalar-type divergence-free tensor harmonics $Y^{(p,0)}_{\mu\nu}$ satisfy the ortho-normality condition,
\[
\int d^4x \sqrt{-\gamma}Y^{(p,0)}_{\mu\nu}Y^*(p',0)_{\mu\nu} = \delta(p - p')\delta^3(k - k').
\] (B11)

**APPENDIX C: TACHYONIC BOUND STATE IN DE SITTER TWO-BRANE SYSTEM**

In [23], Charmousis and Dufaux showed that for the Minkowski two-brane system there exists a tachyonic bound state on the negative tension brane. This fact implies that the Minkowski two-brane system is unstable under the linear perturbation. Following [23], we show that there exits a tachyonic bound state also for the de Sitter two-brane system.

1. **Possibility of a negative norm state**

We consider a de Sitter two-brane system. One of the branes located at a smaller radius of the AdS space has a negative tension. We discuss only the bulk gravitational perturbations. The matter perturbations on each brane are not taken into account.

The bulk component of the perturbed Einstein Gauss-Bonnet equation including the boundary branes are written in the Sturm-Liouville form as
\[
\left\{ \left( b^4 - \bar{\alpha}\ell^2 b^2 (y^2 - b^2 H^2) \right) \psi_{p,y} \right\}_{y} = -b^2 \left( 1 - \bar{\alpha}\ell^2 \frac{H'}{b} \right) \left( p^2 + \frac{9}{4} \right) H^2\psi_p.
\] (C1)

Using Eq. (C1), the boundary condition on each brane is derived. For $H = 0$ and $b(y) = e^{-|y|/\ell}$, Eq. (C1) naturally reduces to the Minkowski version, Eq. (8) in [23].

1. **On positive tension brane**

Imposing the $Z_2$ symmetry, the warp factor around the positive tension brane is expressed as
\[
b(y) = H\ell \sinh\left( \frac{y - |y - y_+|}{\ell} \right).
\] (C2)

Integrating Eq. (C1) around $y = y_+$ and using the $Z_2$-symmetry,
\[
\partial_y\psi_p(y_+ - 0) = \frac{\zeta}{\ell} \left( p^2 + \frac{9}{4} \right) \cosh(y_+ / \ell) \psi_p(y_+),
\] (C3)
where
\[ \zeta := \frac{\bar{\alpha}}{1 - \bar{\alpha}}. \] (C4)

2. **On negative tension brane**

Similarly, the \( Z_2 \) symmetry gives the warp factor around the negative tension brane as
\[ b(y) = \ell \sinh \left( \frac{|y - y_+| + y_-}{\ell} \right). \] (C5)

Integrating Eq. (C1) around \( y = y_- \) and using the \( Z_2 \)-symmetry,
\[ \partial_y \psi_p(y_- + 0) = \frac{\zeta}{\ell} \left( \frac{p^2 + 9/4}{\sinh^2(y_-/\ell)} \right) \psi_p(y_-). \] (C6)

For both branes, the boundary conditions are of a mixed (Robin) type. This renders us impossible to prove the positivity of the norm. Namely, we have
\[
\int_{y_-}^{y_+} dy \left( b^4 - \bar{\alpha} \ell^2 (b^2 v^2 - b^2 H^2) \right) (\partial_y \psi_p)^2 \\
= (\ell^2)^4 \left( p^2 + \frac{9}{4} \right) \left[ \frac{\bar{\alpha}}{2 \ell} \left( \sinh(2y_+ / \ell) \psi_p^2(y_+) - \sinh(2y_- / \ell) \psi_p^2(y_-) \right) + \frac{(1 - \bar{\alpha})}{\ell^2} \int_{y_-}^{y_+} dy \sin^2(\ell y / \ell) \psi_p^2(y) \right].
\] (C7)

Thus the norm is no longer positive definite for \( p^2 + 9/4 > 0 \).

2. **Condition for the existence of tachyonic bound state**

In order to determine whether a tachyonic bound state exists, we need to analyze the mass spectrum. The tachyonic eigenmode, if it exists, is written by
\[ \psi_q(y) = \frac{1}{\sinh^{3/2}(y/\ell)} \left[ A_q P_{3/2}^{-q}(\cosh(y/\ell)) + B_q P_{3/2}^q(\cosh(y/\ell)) \right], \] (C8)

where \( m^2 = -\mu^2, q := \sqrt{\mu^2 + 9/4}, \) and \( q^2 = -p^2 \). The \( y \)-derivative of it is
\[ \partial_y \psi_q = -\frac{1}{\ell \sinh^{5/2}(y/\ell)} \left[ \left( \frac{3}{2} - q \right) A_q P_{1/2}^{-q}(\cosh(y/\ell)) + \left( \frac{3}{2} + q \right) B_q P_{1/2}^q(\cosh(y/\ell)) \right]. \] (C9)

Using the boundary condition on each brane, Eqs. (C8) and (C9), we obtain
\[ A_q \left( \frac{3}{2} - q \right) \left( (z_+^2 - 1) P_{1/2}^{-q}(z_+) + \zeta \left( \frac{3}{2} + q \right) z_+ P_{3/2}^{-q}(z_+) \right) \\
+ B_q \left( \frac{3}{2} + q \right) \left( (z_+^2 - 1) P_{1/2}^q(z_+) + \zeta \left( \frac{3}{2} - q \right) z_+ P_{3/2}^q(z_+) \right) = 0, \]
\[ A_q \left( \frac{3}{2} - q \right) \left( (z_-^2 - 1) P_{1/2}^{-q}(z_-) + \zeta \left( \frac{3}{2} + q \right) z_- P_{3/2}^{-q}(z_-) \right) \\
+ B_q \left( \frac{3}{2} + q \right) \left( (z_-^2 - 1) P_{1/2}^q(z_-) + \zeta \left( \frac{3}{2} - q \right) z_- P_{3/2}^q(z_-) \right) = 0, \] (C10)

where \( z_{\pm} = \cosh(y_{\pm}/\ell) \).

For a non-trivial solution for \( A_q \) and \( B_q \) to exist, the determinant must vanish. Thus
\[
\left( (z_+^2 - 1) P_{1/2}^{-q}(z_+) + \zeta \left( \frac{3}{2} + q \right) z_+ P_{3/2}^{-q}(z_+) \right) \left( (z_-^2 - 1) P_{1/2}^{-q}(z_-) + \zeta \left( \frac{3}{2} - q \right) z_- P_{3/2}^{-q}(z_-) \right) \\
- \left( (z_+^2 - 1) P_{1/2}^q(z_+) + \zeta \left( \frac{3}{2} + q \right) z_+ P_{3/2}^q(z_+) \right) \left( (z_-^2 - 1) P_{1/2}^q(z_-) + \zeta \left( \frac{3}{2} - q \right) z_- P_{3/2}^q(z_-) \right) = 0. \] (C11)

The pole at \( q = 3/2 \), which corresponds to the zero mode, is divided out in deriving Eq. (C11). If there exists a solution of Eq. (C11) at \( q > 3/2 \), it implies the existence of a tachyonic bound state.
3. Existence of a tachyonic bound state

From Eq. (C11),

\[
\frac{(z^2 - 1)P_{1/2}^q(z_-) + \zeta \left( \frac{3}{2} - q \right) z_- P_{3/2}^q(z_-)}{(z^2 - 1)P_{1/2}^q(z_+) + \zeta \left( \frac{3}{2} + q \right) z_+ P_{3/2}^q(z_+)} = \frac{(z^2 - 1)P_{1/2}^q(z_-) + \zeta \left( \frac{3}{2} - q \right) z_- P_{3/2}^q(z_-)}{(z^2 - 1)P_{1/2}^q(z_+) + \zeta \left( \frac{3}{2} + q \right) z_+ P_{3/2}^q(z_+)}.
\]

(C12)

Using the definition of the Legendre functions \[52\],

\[
P_{\nu}^q(z) = \frac{1}{\Gamma(1 - \mu)} \left( \frac{z+1}{z-1} \right)^{\mu/2} F_1 \left[ -\nu, \nu + 1; 1 - \mu; \frac{1 - z}{2} \right],
\]

(C13)

we see that the left-hand-side of Eq. (C12) is generally much larger than the right-hand-side for \( q \gg 1 \) for fixed \( z_+ \) and \( z_- \). Therefore, in order for this equation to be satisfied, we must have

\[
q - \frac{3}{2} \approx \frac{(z^2 - 1)P_{1/2}^q(z_-)}{\zeta z_- P_{3/2}^q(z_-)} \rightarrow \frac{z_-^2 - 1}{\zeta z_-} \quad \text{for} \quad q \rightarrow \infty.
\]

(C14)

This is a consistent solution for \( \zeta \ll 1 \). Thus a tachyonic bound state exists in the de Sitter brane case as well.

The tachyon mass is given by

\[
\mu H = \sqrt{q^2 - 9/4} H \simeq \frac{(z^2 - 1)H \ell}{\zeta z_- \ell}.
\]

(C15)

In the low energy limit, we have \( z_+ > z_- \gg 1 \) and \( H \ell \simeq 1/z_+ \ll 1 \). Hence, the above reduces to

\[
\mu H \simeq \frac{\Omega}{\zeta \ell},
\]

(C16)

where

\[
\Omega := \frac{b(z_-)}{b(z_+)} \simeq \frac{z_-}{z_+} \sim e^{-(y_+ - y_-)/\ell}.
\]

(C17)

This agrees with the result for the Minkowski brane \[23\].

On the other hand, in the high energy limit, \( H \ell \gg 1 \), we have

\[
\mu H \simeq \frac{\Omega^2 H}{\zeta (H \ell)^2} \ll \frac{H}{\zeta}.
\]

(C18)

Thus the high background expansion rate of the brane suppresses the tachyonic mass, giving a tendency to stabilize the two-brane system.

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