On parametric type interaction between light and atomic ensembles

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One-photon and Raman type interactions between two-level atoms and narrow-band light are considered. We give some exactly solvable models of these processes when only one-photon Fock states are involved in the evolution. Possible application of these models for generation and transformation of entangled states of the W-class, some of which demonstrate hierarchy structure, are discussed. Finally, we consider preparation of entangled chains of atomic ensembles.

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I. INTRODUCTION

Interaction between light and atoms still represents a promising way of preparation of multiparticle entangled states to be main resource of many quantum communications, in particularly, a recently introduced one-way quantum computer. In this Letter we focus on some of the informational tasks which can be realized by considering a model in which an ensemble of two-level atoms interacts as a single quantum-mechanical system with one photon light. The multiparticle problem of a such type, also known as Dicke one, in general, has no exact solutions and has been studied under various approximations, particularly, in the master equation for atoms with bath-like radiation field and also using the second-order many-body perturbation theory.

One of the main obstacle in obtaining of exact solutions in this model is a large number of degrees of freedom of the multiparticle system. However, we could expect that only a little part of them will play a role, if we consider interaction with a small number of photons. Indeed, if we use, say, one-photon Fock states as initial, hence there is a small number of excitations in the system, evolution of which now can be obtained directly by making use the time-evolution operator method. It is derived in the Appendix the solutions of a general type for evolution of the wave functions of the system N atoms and quantum field when a few photons only are involved in the parametric type interaction.

These solutions describe coherent processes of exchange of excitations between light and the atomic ensemble and look like three-photon parametric phenomena well studied in Nonlinear Optics. In this way the symmetric states, belonging to the Dicke family with a subset of well known in the quantum informational theory W entangled states, can be generated. Using these solutions we focus on preparation and transformation of the W states that can accomplish many informational tasks such as telecloning, quantum cryptography, multiparticle teleportation, dense coding and probabilistic teleportation. For these states the distillation protocol and its optical implementation have been introduced. The analytical solutions for single photons interacted with three level atoms of the lambda configuration have been found based on arguments associated with electromagnetically induced transparency (EIT) and considered as a basis for storage of light. In the case of the squeezed field storage or quantum memory for light has been proposed and demonstrated experimentally. The paper is organized as follows. First we consider two hamiltonians that describe one-photon and Raman interactions between light and atoms. Then we introduce the necessary notations and features of the symmetric and W states and write some exact solutions for the problems. Finally, we discuss generation and conversion of the states from the W-class some of which have a hierarchic organization, such as considered in ref., and it is assumed to be actual in quantum bioinformatics.

II. HAMILTONIANS

Now we consider multiphoton interactions between two-level identical atoms and narrow-band light which the frequency range and detuning from resonance are small with respect to a decay rate of the atomic levels, γ. Assuming the time evolution is not long in the sense that t ≪ γ⁻¹, atomic relaxation can be neglected and hence the resonance approximation becomes valid. An effective Hamiltonian can be used to describe a set of various phenomena, particularly multiphoton processes among of which we focus only on two. The first is a one-photon resonance interaction of a single mode a. The second is the Raman type interaction of two modes b, c whose frequencies obey the relation ω_b - ω_c = ω_0, where ω_0 is a frequency of an atomic transition. Both processes can be modelled by the following Hamiltonians

\[ H_1 = i\hbar g(aS_{10} - a^\dagger S_{01}), \]
\[ H_{11} = i\hbar f(c^\dagger bS_{10} - cb^\dagger S_{01}), \]

where the atomic operators are defined by the notation \( S_{xy} = \sum_a s_{xy}(a), s_{xy}(k) = |x\rangle_a \langle y|, x, y = 0, 1, |0\rangle_a \) and \(|1\rangle_a\) are lower and upper levels of a single atom, \( Z = a, b, c \) are annihilation operators of a photon at frequency ω_Z and g, f are real coupling constants. Because the Hamiltonians commute with the operator of permutation of particles, symmetry of the input wave
function is unchanged during evolution, and any symmetric states of the Dicke family, particularly W-states, can be generated. There is a simple physical reason for this. When \( m \) atoms (from \( N \)) absorb photons, then in the case of identical particles, all possibilities to do this result in a symmetric superposition state that consists of any states with \( m \) excited atoms.

**III. Symmetric Dicke and W States**

The family of the symmetric Dicke states \( \mathbb{K} \) can describe an ensemble of \( N \) identical two-level atoms in which any \( m < N \) particles are excited. In general, they read

\[
|m; N\rangle = \sum_z P_z \frac{1}{m!} \frac{1}{N-m!} \langle 1, \ldots, 1, 0, \ldots, 0 | m, N-m \rangle,
\]

where \( P_z \) is one from \( C_N^m = N! \) distinguishable permutations of particles. In \( \mathbb{K} \), the vectors are not normalized and \( \langle m; N|m; N \rangle = C_N^m \). Using the dipole momentum operator \( S_{10} = \sum_a s_{10}(a) \) one finds that

\[
S_{10}^m |0\rangle = m! |m; N\rangle.
\]

This representation tells us, that the symmetric states can be produced in any process involving interactions of collective dipole momentum of the atomic ensemble. One of the possible examples is given by the Hamiltonians \( \mathbb{U} \) and \( \mathbb{W} \). If \( m = 1 \), then

\[
|1; N\rangle = |1, 0, 0, \ldots, 0\rangle + |0, 1, 0, \ldots, 0\rangle + |0, 0, 1, \ldots, 0\rangle + |0, 0, 0, \ldots, 1\rangle.
\]

In quantum information theory the normalized vector \( 1/\sqrt{N} |1; N\rangle = W \) is well known as a multiparticle W state. The symmetrical states \( \mathbb{K} \) are entangled over many criteria and any pairs of particles are entangled in contrast with Greenberger-Horne-Zeilinger states in which two particles are separable.

The following properties of these states will be further used below

\[
S_{01} |0; N\rangle = 0,
S_{10} |m; N\rangle = (m+1) |m+1; N\rangle,
S_{01} |m; N\rangle = (N-m+1) |m-1; N\rangle,
S_{01} S_{10} |m; N\rangle = (m+1)(N-m) |m; N\rangle,
S_{10} S_{01} |m; N\rangle = m(N-m+1) |m; N\rangle.
\]

**IV. Some Exact Solutions**

The multiparticle problems of interaction between light and atoms given by the Hamiltonians \( \mathbb{U} \) and \( \mathbb{W} \) have a set of exact solutions for particular cases of initial field states. For example, consider that atoms are illuminated by light in the one-photon Fock states \(|n\rangle, |01\rangle, |10\rangle, n = 0, 1 \). In this way, one finds a simple obvious evolution for the Hamiltonian \( \mathbb{U} \)

\[
\exp \{-i \hbar^{-1} H_{11} t \} \{ \alpha |0\rangle \otimes |0; N\rangle + \beta |1\rangle \otimes |1; N\rangle \}
= \alpha \{ c |1\rangle \otimes |0; N\rangle + s \frac{1}{\sqrt{N}} |0\rangle \otimes |1; N\rangle \}
+ \beta \{ -s \sqrt{N} \langle 0 | \otimes |0; N\rangle + c |0\rangle \otimes |1; N\rangle \},
\]

where \( c, s \) are constant, \( \alpha, \beta \) are any states with \( m \). It is worth noting that Eq. \( \mathbb{U} \) describes exchange of one excitation between light and atoms similar to a beamsplitter with a single photon as its input

\[
\alpha \langle 10 \rangle + \beta \langle 01 \rangle \rightarrow \alpha \langle c |10 \rangle + s |01 \rangle \}
+ \beta \{ -s |10 \rangle + c |01 \rangle \}.
\]

It is well known, that a beamsplitter can be modelled by an effective Hamiltonian of the form

\[
H_{BS} = i \hbar g (a^\dagger b - h.c.),
\]

where two modes of field \( a \) and \( b \) have equal frequencies but differ in their wave vectors or “which path”. Indeed, the beamsplitter Hamiltonian also describes a process of frequency conversion in a strong classical pump \( \omega_a - \omega_b = \Omega \), where two modes \( a \) and \( b \) have different frequencies or (“color”) and \( \Omega \) is a frequency of the pump. The process belongs to a family of the three-photon parametric phenomena well known in Quantum Optics. We then see that Eq. \( \mathbb{U} \) looks as the beamsplitter-like solution and can describe a parametric process. From the point of view of three-photon parametric phenomena the Hamiltonian \( \mathbb{W} \) could be treated as parametric interaction involving two photons of the light modes \( c \) and \( b \) and the third "photon" represents an atomic ensemble:

\[
H_{11} = i \hbar f \langle c^\dagger b S - c b^\dagger S^\dagger \rangle,
\]

where we introduce the notation \( S = S_{10}, S^\dagger = S_{01} \). In this case one finds

\[
\exp \{-i \hbar^{-1} H_{11} t \} \{ \alpha \langle 01 \rangle + \beta \langle 10 \rangle \} \otimes |\psi\rangle_A
= \alpha \{ \cos \left[ t f \sqrt{S S^\dagger} \right] |01 \rangle
+ S \frac{1}{\sqrt{S S^\dagger}} \sin \left[ t f \sqrt{S S^\dagger} \right] |10 \rangle \} \otimes |\psi\rangle_A
+ \beta \{ -S^\dagger \frac{1}{\sqrt{S S^\dagger}} \sin \left[ t f \sqrt{S S^\dagger} \right] |01 \rangle
+ \cos \left[ t f \sqrt{S S^\dagger} \right] |10 \rangle \} \otimes |\psi\rangle_A,
\]
where $|\psi\rangle_A$ is an atomic wave function. If we choose the atomic state as $|\psi\rangle_A = |m; N\rangle$, then using (9) we have

$$
(\alpha|01\rangle + \beta|10\rangle) \otimes |m; N\rangle \rightarrow \alpha \left\{ \cos \theta_m |01\rangle \otimes |m; N\rangle + \sqrt{\frac{m+1}{N-m}} \sin \theta_m |10\rangle \otimes |m+1; N\rangle \right\}

+ \beta \left\{ -\sqrt{\frac{N-m+1}{m}} \sin \theta'_m |01\rangle \otimes |m-1; N\rangle + \cos \theta'_m |10\rangle \otimes |m; N\rangle \right\}
$$

(11)

where $\theta_m = t\sqrt{(m+1)(N-m)}$, $\theta'_m = t\sqrt{m(N-m+1)}$. It follows from the presented solution (11), that during the exchange of a single excitation between two modes and atoms, a set of atomic symmetric states is produced. Both the wave vectors given by (7) and (11) describe entangled states of light and atoms. However this multipartite entanglement has a hierarchic organization consisting itself from the atomic W-states. Note that the similar W-hierarchy of the atomic states is produced. Both the wave vectors given by (7) and (11) describe entangled states of light and atoms. However this multipartite entanglement has a hierarchic organization consisting itself from the atomic W-states. Note that the similar W-hierarchy of the atomic states is produced.

V. GENERATION AND TRANSFORMATION OF W STATES

One of the main features of the state conversions given by the solutions (7) and (11) is swapping. It results in a set of processes for the preparation of multiparticle entanglement, their transformation and storage. Possible schemes could be implemented in two ways. First is a cavity version containing an atomic ensemble which interacts with high-Q cavity modes and the required initial state is injected. In the second scheme, a trapped atomic ensemble is illuminated by light from a single photon source. It follows from (11) that there are two processes of transformation at least. The first is generation of a W state: $|0; N\rangle \rightarrow |1; N\rangle$, where one atom is excited only. There is also an opposite process of decreasing of the excited atoms: $|1; N\rangle \rightarrow |1; N\rangle$. It can be treated as a disentanglement operation. Generally, if an ensemble with $m$ excited atoms are prepared, then transformation of the form $|m; N\rangle \rightarrow |m \pm 1; N\rangle$ can be achieved.

Both solutions, given by (7) and (11), describe an swapping between light and atoms. This is a basis for quantum memory, when a state of light, particularly an unknown state, can be stored in an atomic ensemble. For example, examining (7) we find

$$
\left( \alpha |1\rangle + \beta |0\rangle \right) \otimes |0; N\rangle

\rightarrow |0\rangle \otimes \left\{ \alpha \frac{1}{\sqrt{N}} |1; N\rangle + \beta |0; N\rangle \right\}.
$$

(12)

As well it follows from (11), that storage can be realized for entangled states of light $\alpha |01\rangle + \beta |10\rangle$. The same possibility has been remarked in case of an ensemble of atoms in a coupled lambda type configuration and assuming adiabatic approximation (13).

The beam splitter-like solution (7) results in a scheme that prepares entanglement of the W class from atomic ensembles. The scheme is an analog of the version with two beam splitters (14). Let us consider a consecutive interaction between light in a single photon state and two atomic ensembles having $N_1$ and $N_2$ atoms. Then the output state of light and the first ensemble can be input for the next interaction. Using (7) we find

$$
|1\rangle \otimes |0; N_1\rangle \otimes |0; N_2\rangle \rightarrow

\left( c_1 |1\rangle \otimes |0; N_1\rangle + s_1 |0\rangle \otimes (1/\sqrt{N_1}) |1; N_1\rangle \right) \otimes |0; N_2\rangle

\rightarrow c_1 c_2 |1\rangle \otimes |0; N_1\rangle \otimes |0; N_2\rangle

+ c_1 s_2 (1/\sqrt{N_1}) |0\rangle \otimes |1; N_1\rangle \otimes |0; N_2\rangle + s_1 (1/\sqrt{N_1}) |0\rangle \otimes |1; N_1\rangle \otimes |0; N_2\rangle
$$

(13)

As a result, a three-partite entanglement of the W class is prepared. Indeed, the desired consecutive interaction can be done, for example, by the Stark effect. Let two trapped atomic samples in free space be illuminated by light and two static fields are applied to the working levels. Then, by manipulating the static fields, it is possible to manipulate the resonance interaction between light and any of the atomic ensembles.

Another version of the entanglement preparation is possible when we consider interaction between light and a set of atomic ensembles. Let ensembles be located at the points $x = a_1, a_2, \ldots, a_\nu$. We will call this configuration an atomic chain. Let the chain be illuminated by light so that its interaction is described by the Hamiltonian (9), where operators $S$ are now the sum of operators of atomic ensembles, that are located in $x = a_1, \ldots, a_\nu$ and have $N_1, \ldots, N_\nu$ atoms

$$
S = \sum_x S_{10}(x)
$$

(14)

For simplicity, we will neglect all spatial behavior of light within the chain. Let $|O\rangle$ be initial state of atoms, its ground state, then using (10), we have

$$
|01\rangle \otimes |O\rangle \rightarrow

\cos \theta_0 |01\rangle \otimes |O\rangle + (1/\sqrt{N'}) \sin \theta_0 |10\rangle \otimes |C\rangle,
$$

(15)

where $\theta_0 = t\sqrt{N'}$, $N' = N_1 + \cdots + N_\nu$, and the vector $|C\rangle$ is defined as

$$
|C\rangle = \sum_x S_{10}(x)|O\rangle = |1; N_1\rangle |0, \ldots, 0\rangle + |0; 1\rangle |N_2\rangle |0, \ldots, 0\rangle + \cdots + |0, \ldots, 0\rangle |1; N_\nu\rangle.
$$

(16)

From (16) one finds the atomic chain to be in the $W$ state consisting of $\nu$ atomic ensembles.
VI. CONCLUSIONS

In summary, we have discussed simple analytical solutions describing the evolution of the wave functions for a system of N atoms and quantum field when a few photons only are involved in the parametric type interactions. From a theoretical point of view, these solutions are interesting not only for their remarkable simplicity, but also because they are one of the simplest analytic solutions that describe a set of conversion processes of the symmetric atomic states from W-class and can be useful for modelling of generation and transformation of entangled states, in particular, storing light states on atoms and as well preparing atomic entangled chains both to be processes of fundamental importance in quantum information. We acknowledge discussions with A. Kazakov. One of us (AZ) would like to thank M. Shapiro for useful remarks.

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APPENDIX A

The presented set of exact solutions arises from the next observation on the Hamiltonian written in the form

\[ H = i \hbar \partial \]

\[ \nu = \pi \hbar - \pi \hbar, \quad (A1) \]

where there is one requirement only for operators, that \([\pi; h] = [\pi; \hbar] = 0\). Let two wave functions \(\Phi\) and \(\Phi^\dagger\) obey the conditions:

\[ h\Phi = A\Phi^\dagger, \quad h\Phi^\dagger = 0 \]

\[ h\Phi = B\Phi, \quad h\Phi^\dagger = 0, \quad (A2) \]

where \(A\) and \(B\) are c-numbers or commuting operators, that commute with \(\pi, \pi^\dagger\): \([X; Y] = 0, Y = A, B, X = \pi, \pi^\dagger\). Then considering the initial state of the form

\[ \varphi = c\Phi + e\Phi^\dagger, \quad (A3) \]

one finds its evolution

\[
\exp \left[ \partial t \right] \left( c\Phi + e\Phi^\dagger \right) = c\left\{ \cos[t\sqrt{\pi^\dagger A}]\Phi^\dagger \right. \\
+ \pi^\dagger \frac{1}{\sqrt{\pi^\dagger A}} A \sin[t\sqrt{\pi^\dagger A}]\Phi^\dagger \right\} \\
+ e\left\{ - \pi^\dagger \frac{1}{\sqrt{\pi^\dagger B}} B \sin[t\sqrt{\pi^\dagger B}]\Phi^\dagger \\
+ \cos[t\sqrt{\pi^\dagger B}]\Phi^\dagger \right\}. \quad (A4) \]

The Hamiltonian \(A1\) describes a family of physical processes between light and atoms, including multiphoton parametric phenomena. Then for appropriate input states one can find exact solution of the form \(A4\). When \(\pi^\dagger = g, \pi = g^*, h = S_{10}a^M\) we find \(M\)-photon absorption

\[ \vartheta = gS_{10}a^M - g^*S_{10}a^M, \quad (A5) \]

for which the exact solution of the form \(A4\) arises, where \(\Phi = (2M - p) \otimes |0; N\rangle, \Phi^\dagger = |M - p) \otimes |1; N\rangle, p = 1, \ldots, M, A = A(p) = (2M - p)(2M - p - 1) \cdots (M - p)^{1/2}, B = A(p - 1).\) A particular case of \(M = 1\) reduces to the considered Hamiltonians \(\Phi_1, \Phi_2\) if \(\pi^\dagger = S_{10}, h = ga, fc^\dagger b.\) The presented arguments are true, in particular, for three-photon parametric processes well known in Nonlinear Optics

\[ \vartheta = f(a^\dagger bc - ab^\dagger c^\dagger). \quad (A6) \]

The closed solution of the problem in general case are unknown for the authors, however for particular case of input states \(\Phi = |0, 1, n\rangle\) and \(\Phi^\dagger = |1, 0, n - 1\rangle,\) for which \(A = B = \sqrt{n}\), we obtain the exact solution

\[ c|0, 1, n\rangle + c|1, 0, n - 1\rangle \rightarrow c\left\{ |0, 1, n\rangle \cos[t\sqrt{n}] \\
+ |1, 0, n - 1\rangle \sin[t\sqrt{n}] \right\} \\
+ e\left\{ - |0, 1, n\rangle \sin[t\sqrt{n}] + \\
|1, 0, n - 1\rangle \cos[t\sqrt{n}] \right\}. \quad (A7) \]
