THREE-POINT STATISTICS FROM A NEW PERSPECTIVE

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ABSTRACT

Multipole expansion of spatial three-point statistics is introduced as a tool for investigating and displaying configuration dependence. The novel parametrization renders the relation between bi-spectrum and three-point correlation function especially transparent as a set of two-dimensional Hankel transforms. It is expected on theoretical grounds, that three-point statistics can be described accurately with only a few multipoles. In particular, we show that in the weakly non-linear regime, the multipoles of the reduced bispectrum, $Q_l$, are significant only up to quadrupole. Moreover, the non-linear bias in the weakly non-linear regime only affects the monopole order of these statistics. As a consequence, a simple, novel set of estimators can be constructed to constrain galaxy bias. In addition, the quadrupole to dipole ratio is independent of the bias, thus it becomes a novel diagnostic of the underlying theoretical assumptions: weakly non-linear gravity and perturbative local bias. To illustrate the use of our approach, we present predictions based on both power law, and CDM models. We show that the presently favoured SDSS-WMAP concordance model displays strong "baryon bumps" in the $Q_l$’s. Finally, we sketch out three practical techniques estimate these novel quantities: they amount to new, and for the first time edge corrected, estimators for the bispectrum.

Subject headings: cosmic microwave background — cosmology: theory — methods: statistical

1. INTRODUCTION

Three-point statistics are on track to become main stream tools in astronomy. They have been used by several authors with considerable success to constrain the statistical bias between the distribution of galaxies and dark matter (e.g. Fry 1994; Jing & Boerner 1998; Frieman & Gaztañaga 1999; Szapudi et al. 2000; Scoccimarro et al. 2001; Verde et al. 2002), as well as to constrain primordial non-Gaussianity of the Cosmic Microwave Background (Komatsu et al. 2003).

Nevertheless, the three-point correlation function and its Fourier-transform pair, the bispectrum, are complicated objects. In their most general form, they depend on three vectors, i.e. nine variables. Even after translational and rotational symmetries are taken into account, three-point functions depend on the size and shape of a triangle, i.e. still three parameters are needed. Exploration and visualization of the full configuration space becomes a surprisingly daunting task, and most previous studies have been forced to restrict their investigation to a few hand-picked triangle shapes and sizes, which are not necessarily representative.

In the past essentially three distinct parametrizations have been proposed. Peebles (1980) used the parameters $r, u, v$, where the three sides of the triangles are $r = r_1 \leq r_2 \leq r_3$, and $u = r_2/r_1$, and $v = (r_3 - r_2)/r_1$. Alternatively, Suto & Matsubara (1994); Szapudi et al. (2001) have used logarithmic bins for the three sides of a triangle. Then shapes can be uniquely described by a pair of integers $(b_1 - b_3, b_2 - b_4)$ formed from the bin numbers $b_1 \leq b_2 \leq b_3$. Finally, several authors parametrized triangles with two sides and the angle between them, $r_1, r_2, \theta$ (e.g. Fry, 1994).

Systematic exploration of the full configuration space, and comparison of results are somewhat tedious using any of the above parametrizations. Moreover, the connection between real and transform space representations (three-point function and bispectrum) is somewhat obscure, whichever description one uses. Hoping to alleviate these shortcomings, we introduce the multipole expansion of three-point statistics, which arguably expresses the rotational symmetries in the most natural way. The next section presents the definition of the three-point multipoles and the relationship between the multipoles in real and Fourier space. Section 3 applies these results in the weakly non-linear regime, bias, and presents predictions for power law and CDM models. The final section contains summary and discussion of the results, including measurement techniques for the proposed quantities.

2. MULTIPOLe EXPANSION OF THREE-POINT STATISTICS

Since the bispectrum can be parametrized with two lengths, $k_1, k_2$, and the angle $\theta$ between them, it is natural to define the following multiple expansion

$$B(k_1, k_2, \theta) = \sum_l B_l(k_1, k_2) P_l(\cos \theta) \frac{2l + 1}{4\pi},$$

where $P_l(\cos \theta)$ is the $l$-th order Legendre polynomial. Given the two scalar lengths, $B_l$ can be obtained through integration $B_l = 2\pi \int B P_l d \cos \theta$. The multipole expansion of the real space three-point correlation function, $\xi^3$, is defined entirely analogously to $B_l$.

This novel choice of parametrization, while in principle equivalent to previous choices, provides a surprisingly new perspective. It facilitates predictions, measurements, and their visualizations since it expresses naturally the rotational symmetry inherent in the three-dimensional statistics. To begin with, we show that there is a simple transformation between $B_l$ and $\xi^3$.
The relationship of the three-point correlation function $\xi^3$ to the bispectrum $B$ is a Fourier transform:

$$\xi(r_1, r_2, r_3) = \int \prod_{i=1}^{3} d^3 k_i B(k_1, k_2, k_3) e^{i(k_1 x_1 + k_2 x_2 + k_3 x_3)} \delta_D(k_1 + k_2 + k_3), \quad (2)$$

$\delta_D$ is the Dirac delta function, and temporarily vector notation is used. This is a six dimensional integral and difficult to treat in practice.

If the general relation is rewritten in spherical variables in terms of the above multipole expansion, the exponential can be expanded into spherical harmonics while the addition theorem can be used to expand the Legendre polynomial. The angular dependence then can be integrated using the orthogonality of spherical functions yielding the following final result

$$\xi^3_l(r_1, r_2) = \int \frac{k_1^2}{2\pi^2} dk_2 \frac{k_2^2}{2\pi^2} dk_2 (-1)^l B_l(k_1, k_2) j_l(k_1 r_1) j_l(k_2 r_2). \quad (3)$$

The formula is analogous to that of the power spectrum, where a formally 3 dimensional integral can be rendered one dimensional by $\xi(r) = \int k^2 dk/(2\pi^2) P(k) j_0(kr)$ in the natural variables.

### 3. THEORETICAL PREDICTIONS IN THE WEAKLY NON-LINEAR REGIME

Perturbation theory predicts \cite{Fry_1984, Goroff_1986, Bouchet_1995, Hivon_1999} that the bispectrum in the weakly non-linear regime is

$$\left\{ \left( \frac{4}{3} + \frac{2}{3} \mu \right) P_0(x) + \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) P_1(x) + \frac{2}{3} (1 - \mu) P_2(x) \right\} P(k_1) P(k_2) + \text{perm.} \quad (4)$$

where $x$ is the cosine of the angle between the two wave vectors, $P_l$ are Legendre polynomials, and $P(k)$ is the linear power spectrum. The parameter $\mu$ depends on the expansion history of the universe; its value is well fitted by $3/7 \Omega^{-1/40}$ \cite{Kamionkowski_1999}. We have written the above formula in terms of Legendre polynomials, to make it explicitly clear that the first permutation depends only on terms up to quadrupole. The other two permutations, however, will contribute higher $l$ terms. Typically these higher order terms drop quickly, although an infrared divergence of the power spectrum can cause oscillation for isociles ($k_1 = k_2$) configurations only.

In the following we focus on the reduced bispectrum $Q = B/(P_1 P_2 + P_2 P_3 + P_3 P_1)$, for which the dependence on the amplitude of power spectrum is scaled out, thus facilitating interpretation. Analogously to the previous definitions, we introduce the multipole expansion of the reduced bispectrum, $Q_l(k_1, k_2)$. Real space version of this quantity $Q_l(r_1, r_2)$ can be introduced exactly the same way, however, in what follows, for $Q_l$ will denote transform space quantity. In the literature the notation $Q_3$ is often used for the reduced bispectrum: since this paper deals with three-point statistics only, the index on $Q_l$ denotes multipoles.

Figure 1. plots the multipoles of the reduced bispectrum, $(2l + 1)Q_l/4\pi$ for 3 configurations, assuming a power-law power spectrum with index $n = -1$ for $\Omega = 1$. Multipoles higher then $l \approx 2$ appear to be negligible with the possible exection of a tiny negative octupole. The right panel displays the configuration dependence; $Q_l = Q_l(k_1/k_2)$. Several values of $k_2$ are plotted, nevertheless they all coincide: the reduced bispectrum depends only on the ratio of the wave vectors for a power law initial power spectrum, as expected. Therefore, in the weakly non-linear regime, most of the information contained in the full configuration space can be displayed in three graphs, which show the dependence on the ratio up to quadrupole.

Most of these results hold for CDM power spectra, with the exection that $Q_l$ will depend slightly on the magnitude of the wave vectors as well. Next we present predictions for the monopole, dipole, and quadrupole moment of the reduced bispectrum in CDM models.

![Figure 1](image-url) — Left: The multipole expansion of the reduced bispectrum in a power law $n = -1$ model is plotted for three different configurations of $k_2/k_1$: isociles $1/1$ (thin dash), $2/1$ (thick dash), $4/1$ (solid). Right: the configuration dependence of the reduced bispectrum, monopole (thin dash), dipole (thick dash), quadrupole (solid). $k_2$ can be fixed at an arbitrary value due to scale invariance of the initial power spectrum (the actual plot shows three sets of five indistinguishable curves calculated numerically for the same values of $k_2$ as for Figure 2.). The x axis shows $k_1/k_2$, the shape factor.
We define our concordance cosmological model based on WMAP (Spergel et al. 2003) and SDSS (Tegmark et al. 2003; Pope & et al. 2003) with parameters $h = 0.685, \Omega = 0.309, \Omega_b = 0.0228/h^2, n = 0.966$. Figure 2 displays results for a set of fixed $k_2 = 0.01 - 0.05$ as a function of $1 \leq k_1/k_2 \leq 10$. As expected, the configuration dependence is not a function of the ratio only, but is slowly changing with the magnitude as well. Still, virtually all information can be summarized in one plot. The left panel uses a BBKS power spectrum (Bardeen et al., 1986), with no baryonic oscillations. The configuration dependence is smooth, qualitatively similar to the case of the power law power spectrum. The right panel of uses the same parameters with the more accurate EH fit power spectrum (Eisenstein & Hu, 1998); it features prominent baryonic oscillations. Baryonic oscillations are also evident on corresponding plots of the bispectrum $B_l$ as well (not displayed).

4. BIAS

The above theory can be used to construct estimators which constrain the bias in the weakly non-linear regime. Indeed, in the weakly non-linear regime one can expand the biased field as $f(\delta) = b_0 + b_1 \delta + b_2/2\delta^2$ (e.g. Fry & Gaztanaga, 1993), where $\delta$ is the purely gravitational dark matter field. Then the biased reduced bispectrum transforms as $q = Q/b + b_2/b^2$ (Fry, 1994), where the lower case denotes the galaxy (measured), and the upper case the dark matter (theory) values. It is clear that $b_2$ can only effect the monopole term. Thus a simple estimator for the bias can be constructed as

$$b = \frac{Q_1}{q_1} = \frac{Q_2}{q_2}$$
$$b_2 = q_0 b_0^2 - Q_0 b.$$  

(5)

According to the equations, the quadrupole to dipole ratio does not depend on the bias, thus it serves as a novel, useful test of the underlying assumptions: a quasi-local perturbative, deterministic bias model and perturbation theory. Figure 4. shows the dipole to quadrupole ratio for BBKS, and EH power spectra, respectively. The range of $k$’s to be used for bias extraction can be determined from contrasting the measurements with these predictions. Note that scales where baryon oscillations are prominent are barely accessible with present redshift surveys. On smaller scales non-linear evolution is likely to modify these prediction based purely on leading order perturbation theory (Meiksin et al. 1999).

![Figure 2](image1.png)  
![Figure 3](image2.png)
5. SUMMARY AND DISCUSSIONS

We have introduced the multipole expansion of three-point statistics, which expresses the inherent rotational symmetry in the most natural way. In particular, the relationship of real and transform space become entirely transparent via Hankel transforms. As a first application, we explored the meaning of the novel quantities theoretically in the weakly non-linear regime. For the multipole expansion of the reduced bispectrum, $Q_l(k_1, k_2)$, most of the information can be compressed into three functions, monopole, dipole, and quadrupole. From these, we constructed a novel estimator for the weakly non-linear bias. Since the quadrupole to dipole ratio is independent of the bias, it serves as a novel diagnostic of the underlying theoretical assumptions. This analogous redshift distortions, where the quadrupole to monopole ratio plays a similar role.

Our repackaging of three-point statistics has yielded novel predictions. In particular, the baryonic oscillations appear to be extremely prominent in the presently favoured concordance model, suggesting that our estimators will become useful for main stream parameter estimation.

In previous work, a formally similar parametrization have been used to describe projection of primordial non-Gaussianities of the CMB (Spergel & Goldberg 1999), and, in an identical result, for projecting the weakly non-linear bispectrum on the sphere (Verde et al. 2002). These papers use an intermediate quantity nearly identical to our definition of $B_l$, thus their results are directly applicable to the projection of the multipoles on the sphere.

The measurement of $B_l$ merits special consideration. The most immediate consequence of our theory is that the $\theta$ dependence of $\xi^3$ or $B_l$ should be sampled at the roots of Legendre polynomials of order $l_{\text{max}}$, where $l_{\text{max}}$ is the desired angular resolution. Then Gauss-Legendre integration can yield the $\xi^3$ or $B_l$. Starting out in real space, one can use the algorithms of Moore et al. (2001) to measure $\xi^3$, or, in transform space, one can estimate $B$ directly (Scoccimarro 2002). Either way, and Equation 3 transforms one into the other, which ultimately amounts to a novel edge corrected bispectrum estimator. This is significant improvement over all previous techniques, where no edge correction have been used for the bispectrum.

A theoretically interesting possibility is to expand the estimator of the bispectrum into bipolar spherical harmonics $\delta_k \delta_{l_1} \delta_{l_2} = \sum_{LM} b_{l_1,l_2}^{LM}(k_1, k_2) (Y_{l_1} \otimes Y_{l_2})_L$. Since the bipolar spherical harmonics are orthogonal, and $(Y_l \otimes Y_0) \sim \delta_l$, one can show that $B_l = \frac{100}{l^2} (-1)^l \sqrt{2l + 1}$. While the bipolar transformation, especially for low $l$’s, can be calculated with fast harmonic transform, the simple minded Gaussian quadrature above is likely to be more practical. Nevertheless, this sheds light on how $B_l$ is related to the symmetries on direct product of two spheres $S^2 \otimes S^2$. This is the three-point generalization of SpICE Szapudi et al. (2001), eSpICE Szapudi & et.al. (2003) algorithms for fast, edge corrected estimations of the power spectrum.

To apply these techniques successfully to the framework of high precision cosmology, it will be necessary to extend our calculations with redshift distortions (e.g. Kaiser 1987; Scoccimarro et al. 1999) taken properly into account. However, at least one (suboptimal) estimate of the real space quantities can be obtained from the transverse power (Hamilton & Tegmark 2002). The redshift space bispectrum is parametrized by the five parameters $B(k_{1,\perp},k_{2,\perp},k_{3,\perp},k_{1,\|},k_{2,\|})$, with $\perp$ denoting transverse, and $\| \perp$ parallel quantities with respect to the line of sight in the distant observer approximation. Then the real space bispectrum can be estimated from taking $k_{1,\|} \simeq k_{2,\|} \simeq 0$.

In the highly non-linear regime it is expected that $l_{\text{max}} > 2$ will be needed for full description, since halo models predict strong features in the configuration dependence of the three-point correlation function (Takada & Jain 2003). Nevertheless, it is likely that the multipole expansion will be convenient even in this case.

Several immediate generalizations to the above ideas are in the works. The above investigations will be applied to the halo model to extend the theory for dark matter and galaxies in any regime. According to Takada & Jain (2003), the halo model integrals are 7 dimensional, and intractable without approximations. This is no longer the case: with our Equation 3 the integrals reduce to three dimensions for each multipole. Generalization of the theory is fairly straightforward for projected quantities, such as angular clustering of galaxies, CMB, as well as for vector and tensor correlations, such as CMB polarizations and weak lensing. We will be exploring practical implementations of the mentioned measurement methods, with special attention to practical caveats, such as binning/regularization and Poisson noise subtraction. Multipole expansion is quite natural for redshift distortions, and we aim to generalize our calculation for the three-point multipoles, preferably without making use of the distant observer approximation as in Szalay et al. (1998). This and the above generalizations will be presented in subsequent publications.

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