WIDE ANGLE REDSHIFT DISTORTIONS REVISITED

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ABSTRACT

We explore linear redshift distortions in wide angle surveys from the point of view of symmetries. We show that the redshift space two-point correlation function can be expanded into tripolar spherical harmonics of zero total angular momentum \( S_l(x) \), and express the anisotropy of the redshift space correlation function. Moreover, only a handful of \( B_l(x) \) are non-zero: the resulting formulae reveal a hidden simplicity comparable to distant observer limit. The \( B_l(x) \) depend on spherical Bessel moments of the power spectrum and \( f = \Omega^{0.6}/b \). In the plane parallel limit, the results of Kaiser (1987) and Hamilton (1993) are recovered. The general formalism is used to derive useful new expressions. We present a particularly simple trigonometric polynomial expansion, which is arguably the most compact expression of wide angle redshift distortions. These formulae are suitable to inversion due to the orthogonality of the basis functions. An alternative Legendre polynomial expansion was obtained as well. This can be shown to be equivalent to the results of Salav et al. (1998). The simplicity of the underlying theory will admit similar calculations for higher order statistics as well.

Subject headings: cosmic microwave background — cosmology: theory — methods: statistical

1. INTRODUCTION

It has been known for decades that the two-point correlation function, or power spectrum, measured in redshift surveys is distorted by the peculiar velocities of galaxies. The anisotropy of the correlation function was demonstrated by Davis & Peebles (1983); Peebles (1980). In a seminal work, Kaiser (1987) demonstrated that in the plane parallel limit there is a simple transformation between the redshift space and real space density contrast. This results in an anisotropic enhancement of the power spectrum by \( (1 + f \mu^2)^2 \), where \( \mu \) is the cosine of the angle between the line of sight and the wave-vector. This simple formula has become the starting point of many extensions, which have used expansion into Legendre polynomials (e.g., Hamilton, 1993; Hamilton & Cúlha, 1996), or numerical methods (Zaroubi & Hoffman, 1996). Most analyses assume a small opening angle (Cole et al., 1995), i.e. they stay essentially in the distant observer limit. Others works used a expansion with formally infinite number of coefficients (Heavens & Taylo, 1993). Numerous galaxy redshift surveys have been successfully analyzed with such methods, most notably the PSCz (Tadros et al. 1999), 2dF (Peacock et al. 2001; Hawkins et al. 2003; Tegmark et al. 2002), and SDSS (Zehavi et al. 2002). For a review of methods in the above spirit and the corresponding applications, see Hamilton (1998).

To address the needs of wide angle redshift surveys, full treatment of wide angle distortions have been given by Szalay et al. (1998), where they identify the coordinates in which the expression of the redshift space two-point correlation function is compact, and most importantly finite. They have also argued, that the power spectrum will necessarily have an infinite expansion, as it arises from the convolution of the density field with a non-compact kernel. They concluded that correlation functions are more convenient for redshift space analyses then power spectra. Their results is suitable and has been used for “forward” analyses, such as the Karhunen-Loeve method, in which the correlation function is predicted and contrasted with data. Applications to the SDSS are presented most recently by Pope et al. (2004), (see also Tegmark et al. 2003b).

In this paper we analyze the symmetries of redshift space distortions. The next section shows that zero angular momentum tripolar functions form a natural basis to expand the redshift space correlation function, and that only a surprisingly small set of expansion coefficients are will be non-zero. In section 3 we present the most important properties of the basis functions, and the connection with the Kaiser-Hamilton limit. Section 4 employs the general theory to obtain compact expressions for the redshift distortions using conveniently chosen variables. In the final section we present discussions, and conclusions.

2. REDSHIFT DISTORTIONS IN LINEAR THEORY

The theory of redshift distortions is based on the redshift to real space transformation, \( s_i = x_i - f v_i x_i \hat{x}_j \hat{x}_j \), where \( \hat{x}_i \) is a unit vector pointing to the galaxy from the origin, \( f = \Omega^{0.6}/b \), and \( v_i \) is the peculiar velocity in units that its divergence equal to the density to linear order (Scoccimarro et al. 1999). The transformation of the density \( \delta \) then can be estimated via the linear theory Jacobian \( J = 1 - f \hat{x}_i \hat{x}_j \partial_j v_i \), as

\[
\delta(x) = \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot x} (1 + f \hat{x}_i \hat{k}_i)^2 \delta(k)
\]

(1)

where the Einstein convention was followed for summation of multiple indices. The effects due to the gradient of the selection function and the “rocket effect” (Kaiser 1987) were neglected. As a consequence, the redshift distorted 2-point correlation function \( \xi(x_1, x_2) \) reads

\[
\xi(x_1, x_2, x_3, x_4) = (1 + \frac{1}{2} f) ^2 \xi(x)
\]
where \( P_2 \) is the second order Legendre polynomial, \( P(k) \) is the linear power spectrum. We have chosen to characterize the correlation with the two directions from the line of sight, and introduced \( x = x_1 - x_2 \), and the corresponding unit vector. By construction, these three unit vectors are in the same plane. The above formula is in full agreement with [Szalay et al. (1998)].

A few simple observations are in order with respect to the above function i) it is rotationally invariant ii) \( \xi^s(\hat{x}_1, \hat{x}_2, -\hat{x}, x) = \xi^s(\hat{x}_1, \hat{x}_2, \hat{x}, x, x) \) since the correlation function is real, iii) \( \xi^s(\hat{x}_1, \hat{x}_2, \hat{x}, x) = \xi^s(\hat{x}_2, \hat{x}_1, \hat{x}, x) \) the invariance of the correlation function under permutation, iv) the three vectors are constrained to be in the same plane, v) the unit vector \( \hat{x} \) is constrained to be between \( \hat{x}_1 \) and \( -\hat{x}_2 \). vi) We can extend the function for \( \hat{x} \) vectors outside this range, with the definition \( \xi^s(\hat{x}_1, \hat{x}_2, \hat{x}, x) \equiv \xi^s(\hat{x}_1, -\hat{x}_2, \hat{x}, x) \). This leaves Equation [2] formally valid, since \( P_l(-\mu) = (-1)^l P_l(\mu) \), and the equation contains only even Legendre polynomials.

For a function depending on three harmonics, the spherical harmonics expansion (e.g., [Varshalovich et al. 1988]) is the most natural

\[
\xi^s(\hat{x}_1, \hat{x}_2, \hat{x}, x) = \sum_{m_1, m_2, m} \left( \begin{array}{ccc} l_1 & m_1 & \frac{1}{2} \\ l_2 & m_2 & m \end{array} \right) C_{l_1 m_1}(\hat{x}_1) C_{l_2 m_2}(\hat{x}_2) C_{l m}(\hat{x}) \tag{4}
\]

where \( \left( \begin{array}{ccc} l_1 & m_1 & \frac{1}{2} \\ l_2 & m_2 & m \end{array} \right) \) is the Wigner 3j symbol, and \( C_{l m} \) are proportional to the spherical functions.

Expanding Equation [2] and using \( P_l(\hat{x}_1 \hat{x}_2) = \sum_{m} C_{lm}(\hat{x}_1) C_{lm}(\hat{x}_2) \), \( e^{ikz} = \sum_{l} (2l + 1) i^l j_l(kx) C_{lm}(\hat{k}) C_{lm}(\hat{x}) \), and the Gaunt integral, one finds that

\[
B_{li}^{00} \equiv B^{l_1 l_2} = (2l + 1) \xi_l(x) i^l \left( \begin{array}{ccc} l_1 & l_2 & 0 \\ 0 & 0 & m \end{array} \right) F_{l_1} F_{l_2},
\]

where we have introduced \( \xi_l(x) = \int \frac{dk}{2k^2} k^2 j_l(xk) P(k) \), the moments of the power spectrum with spherical Bessel functions, and

\[
F_0 = 1 + 1/3 f, F_2 = 2/3 f, \text{ otherwise } 0.
\]

From property (iii) it follows that \( B^{l_1 l_2} = B^{l_2 l_1} \).

It is worth to write this result explicitly, since only a few terms are non-zero due to the initial expression and constraints from group theory:

\[
\begin{align*}
B_{00}^{00}(x) &= (1 + \frac{4}{3} f)^2 \xi_0(x) \\
B_{20}^{02}(x) &= \frac{4}{9 \sqrt{3}} f^2 \xi_2(x) \\
B_{22}^{02}(x) &= \frac{4 \sqrt{10}}{9 \sqrt{3}} f^2 \xi_2(x) \\
B_{22}^{22}(x) &= \frac{4 \sqrt{5}}{9 \sqrt{3}} f^2 \xi_4(x).
\end{align*}
\tag{6}
\]

These functions, not unlike the \( C_l \) for the angular power spectrum, form a natural basis for maximum likelihood estimation. They can be used as an intermediate step for estimation of cosmological parameters in the linear regime.

3. PROPERTIES OF THE \( S_{l_1 l_2 l_3} \) FUNCTIONS

The \( S_{l_1 l_2 l_3} \) functions are a subset of tripolar spherical harmonics. They form an orthogonal complete basis for expanding spherically symmetric functions depending on three unit vectors. It should be emphasized that orthogonality is true only when the unit vectors are integrated over the full sphere \( d\Omega_1 d\Omega_2 d\Omega_3 \) unrestricted. From the definition it follows that

\[
\int d\Omega_1 d\Omega_2 d\Omega_3 S_{l_1 l_2 l_3}(\hat{x}_1, \hat{x}_2, \hat{x}) S_{l_1' l_2' l_3'}(\hat{x}_1, \hat{x}_2, \hat{x}) = \delta_{l_1 l_1'} \delta_{l_2 l_2'} \delta_{l_3 l_3'} \frac{(4\pi)^{3/2}}{\sqrt{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)}}.
\tag{7}
\]

The plane parallel limit [Kaiser, 1987] can be obtained by assuming that the first two unit vectors are parallel. Using the properties of the Wigner coefficients and spherical functions, it is easy to show that

\[
S_{l_1 l_2 l_3}(\hat{x}_1, \hat{x}_1, \hat{x}) = \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{array} \right) \frac{\sqrt{2}}{\sqrt{2l_1 + 1}} P_{l_3}(\hat{x}_1 \cdot \hat{x}).
\tag{8}
\]
As a special case one can see that independently of the unit vector $\hat{x}_1$

$$S_{000}(\hat{x}_1, \hat{x}_2) = S_{000}(\hat{x}_1, \hat{x}_2) = \frac{(-1)^l}{\sqrt{2\pi}} P_l(\hat{x}_1 \cdot \hat{x}_2)$$

(9)

and all other functions with any zero index are zero.

It follows from the above properties by simple algebra that in the plane parallel limit $\hat{x}_1 \simeq \hat{x}_2$

$$\xi^s(\hat{x}_1, \hat{x}_1, \hat{x}_1, \hat{x}_1) = \sum_{\alpha, \beta} \phi_{\alpha} \phi_{\beta} f(\alpha, \beta) \xi_{\alpha} \cdot \xi_{\beta}$$

in agreement with [Kaiser 1987; Hamilton 1993; Hamilton & Culhane 1996].

4. COORDINATE SYSTEMS

Since the three unit vectors are constrained in the same plane (property (iv)), the redshift space correlation function depends only on the shape and size of a triangle. Consequently, the angular dependence can be parametrized by two angles. Since the $S_{11,23,23}$ functions are rotationally invariant, we can fix the plane of the vectors, and even rotate one of them to a fixed position in order to obtain useful expressions. We explored the following choices A) the $z$ axis is perpendicular to the plane of the vectors, and we fix $\phi = 0$ for $\hat{x}$, B) $z$ axis coincides with the third unit vector, $\hat{z} = \hat{x}_1 + \hat{x}_2$. For the latter choices, we can assume that all the vectors are in the $\phi = 0$ plane.

For choice A), $S_{11,23}(\hat{x}_1, \hat{x}_2, \hat{x}) = S_{11,23}(\pi/2, \phi_1, \pi/2, \phi_2, \pi/2, 0)$, and one obtains an expansion of the form

$$\xi_{\alpha}(\phi_1, \phi_2, x) = \sum_{n_1, n_2 = 0, 2} a_{n_1 n_2} \cos(n_1 \phi_1) \cos(n_2 \phi_2) + b_{n_1 n_2} \sin(n_1 \phi_1) \sin(n_2 \phi_2),$$

where the only non-zero coefficients are:

$$a_{00} = \left(1 + \frac{2}{3} f + \frac{2 f^2}{15}\right) \xi_0(x) - \left(\frac{1}{2} + \frac{2 f^2}{21}\right) \xi_2(x) + \frac{4 f}{140} \xi_4(x)$$

$$a_{02} = a_{20} = \left(\frac{1}{2} - \frac{3 f^2}{21}\right) \xi_2(x) + \frac{4 f^2}{7} \xi_4(x)$$

$$a_{22} = \frac{f^2}{15} \xi_0(x) - \frac{f}{27} \xi_2(x) + \frac{4 f^2}{140} \xi_4(x)$$

$$b_{22} = \frac{f^2}{27} \xi_0(x) - \frac{f}{27} \xi_2(x) - \frac{4 f^2}{140} \xi_4(x)$$

(11)

According to property (v) there is a restriction that $\phi_1 \leq \phi_2$, which, however, can be lifted by symmetry property (iii). For a fixed $x$, the two angles can span the full range of the integration, if there are no restrictions from incomplete sky coverage. Then, the above becomes (double) orthogonal expansion, where the coefficients can be obtained simply by integration, e.g. $a_{n_1 n_2} \propto \int_0^{\pi/2} d\phi_1 d\phi_2 \xi_{\alpha}(\phi_1, \phi_2) \cos(2n_1 \phi_1) \cos(2n_2 \phi_2)$. If the correlation function is measured and binned according to the above expression, the four independent coefficients can be obtained either by numerical integration, or by fit; the latter would be probably preferable for realistic surveys with incomplete sky coverage.

The above expression can be inverted easily with a computer algebra package: the variables $f, \xi_0, \xi_2, \xi_4$ can be expressed analytically as a function of the four coefficients. Unfortunately, the analytical expression is too complicated to list here (it is a solution of a fourth order polynomial in $f$), but if needed it can be easily obtained with any computer algebra package, such as Mathematica. In practical applications, however, numerical inversion is expected to be more robust.

The plane parallel limit is $\phi_1 = \phi_2$: reexpressing the trigonometric functions in Legendre polynomials indeed yields, after somewhat tedious calculation, the familiar expression of [Hamilton 1993].

In coordinate system B) $S_{11,23}(\hat{x}_1, \hat{x}_2, \hat{x}) = S_{11,23}(\theta_1, 0, \theta_2, 0, 0, 0)$, but the meaning of the two angles is the same as for coordinate system A). It can be shown with simple but tedious calculation that it reduces to the same expression as above.

Finally, coordinate system C) is identical to that of [Szalay et al. 1998]: As we show next, it produces a double Legendre expansion. For this choice we have $S_{11,23}(\hat{x}_1, \hat{x}_2, \hat{x}) = S_{11,23}(\theta, 0, \theta, \pi, \gamma, 0)$ ($\phi = \pi$ ensures that the $z$ axis is between the first two unit vectors).

Using the fact that $C_{lm}(\theta, \pi) = (-1)^m C_{lm}(\theta, 0)$, inserting the Clebsch-Gordan expansion

$$C_{l_1 m_1}(\hat{x}) = \sum_{l_2 m_2} \left( \begin{array}{ccc} l_1 & l_2 & l_3 \\ 0 & m_1 & m_2 \\ 0 & 0 & m_3 \end{array} \right) C_{l_2 m_2}(\hat{x})$$

(12)

and using property (iii), one can show that in $m_3$ and $m$ are both even. This means that in that case the $S_{11,23}$ functions have only even associated Legendre functions, which can be reexpressed into a finite set of Legendre polynomials. Since there is only a few non-zero coefficients, we show the explicit result instead of the general derivation, which is not very illuminating. Finally, the redshift space two-point correlation function can be expanded into multiples of two Legendre polynomials, $\xi_{\alpha}(\cos \theta \equiv \mu_1, \cos \gamma \equiv \mu_2, x) = \sum_{l_1, l_2 = 0, 2} c_{l_1 l_2} P_{l_1}(\mu_1) P_{l_2}(\mu_2)$, where the only non-zero coefficients are

$$c_{00} = \left(1 + \frac{2 f}{3} + \frac{29 f^2}{225}\right) \xi_0(x) - \left(\frac{4 f}{3} + \frac{44 f^2}{315}\right) \xi_2(x) + \frac{32 f^2}{135} \xi_4(x)$$
In this form the plane parallel limit is $\mu_1 = 1$, i.e. $P_l(1) = 1$. It is simple (although a bit tedious) matter to show that this again returns the right answer. Again, simple, tedious calculation shows that our expression reproduces the results of Szalay et al. (1998), if in their Eq. 15 the typographical error 4/15 $\rightarrow$ 8/15 is corrected.

The above is formally an orthogonal expansion. Due to property (v), $\theta \leq \gamma \leq \pi - \theta$ must be satisfied for any given $x$. According to property (v), however, the range can be extended, and the orthogonality of the Legendre polynomials $P_l(\mu_1)$ is ensured.

The coefficients in the above expansion can be obtained by Gauss-Legendre integration of the correlation function. Measuring the correlation function in these coordinates would produce a method which would be the closest generalization of the original Kaiser-Hamilton method.

The above form perhaps provides the most natural connection to the plane parallel limit, therefore it can be used to quantify deviations from it. Figure 3. plots wide angle to plane parallel ratio of the two most common estimators for redshift distortions: the ratio of the redshift space and real space correlation functions, and Hamilton’s $Q(s) = \xi_2(s)/(3/s^3 \int \xi_0(y)y^2dy - \xi_0(s))$, the modified quadrupole to monopole ratio. According the Figure, a simple restriction of the opening angles at $\theta < \sim 15 - 20$ degrees would ensure the accuracy of traditional measurements assuming the plan-parallel approximation.

![Figure 3](image_url)

**Fig. 1.** The two most common estimators for redshift distortions are compared to their wide angle analogue: $\xi_0(s)/\xi(r)$ (thick line), and $Q(s) = \xi_2(s)/(3/s^3 \int \xi_0(y)y^2dy - \xi_0(s))$ (thin line). The ratio of wide angle to plane parallel prediction is plotted as a function of the half opening angle $\theta$. The three sets of lines correspond to three slopes of the correlation function 1.5, 1.75, 2.0, increasing and decreasing for the three thick and thin lines, respectively. In a realistic measurement, the final result would be a weighted average over a set of opening angles, represented by the above curves.

5. CONCLUSIONS AND DISCUSSION

We have analyzed the underlying symmetries of redshift space distortions. As a result, we presented three novel expressions for the redshift space two-point correlation functions, Eqs. 6, 11, and 13. The last of these equations turns out to be identical to Szalay et al. (1998).

First, we have shown that the two-point correlation function can be expanded into tripolar harmonics and that rotational symmetry restricts the non-zero components to those with $L = 0$. We have found by direct calculation that the quadratic nature of linear
redshift distortions restricts the expansion to five unique functions $B_{1,1}^{l_1, l_2}$ of Equation (6), each depending on the distance between the two points only. These are analogous to the $C_l^i$'s of the angular power spectrum, constitute a natural basis for maximum likelihood analysis of redshift space data. The full machinery of maximum likelihood can be adapted naturally (e.g., Vogeley & Szalay 1996, Tegmark et al. 1998; Bond et al. 2000). Details are left for subsequent research.

The rotational invariance of the basis functions, and the fact that the three unit vectors are constrained into a plane, allows us to fix a plane and an orientation within that plane. Using this freedom, we identified two convenient coordinate systems, which correspond to particular choices of variables. The resulting expressions, Eqs. (11) and (12), are especially convenient, since they respectively correspond trigonometric and Legendre polynomial expansions in the two angular variables. The first expression is possibly most compact formula for linear redshift space distortions of the two-point correlation function, the second can be shown to be identical to Szalay et al. (1998). The orthogonality in these expansions presents an opportunity for applications similar in spirit to the original Kaiser-Hamilton method, but fully correct for wide angles.

This paper deals with the theory of redshift distortions, and thus lays the groundwork for possible future applications. We have not discussed practical issues, such as incomplete sky coverage, noise. Clearly, even if these issues are important, a maximum likelihood technique to find the parameters of the expansions would be still optimal. On the other hand, incomplete sky coverage and noise will cause leakage, possible emergence of higher $l$ anisotropies. We conjecture that such difficulties for direct methods could be solved similarly to Szapudi et al. (2001), where the analogous problem for the angular power spectrum was tackled. Details of practical applications are left for subsequent research.

Our calculation could be simply generalized for the effects of the gradient of the selection function. However, Equation (13) could not be generalized due to odd associated Legendre functions entering in the expansion. These cannot be re-expressed as Legendre polynomials, thus the final results would not admit a Legendre expansion. This can be explicitly demonstrated from the final results of Szalay et al. (1998). There appear to be no analogous problems when generalizing Equation (11). We conjecture that trigonometric polynomial expansion will be still possible, with terms of odd orders appearing. The simplicity of the present theory opens up the possibility of generalizations for higher orders, such as three-point correlation functions and cumulant correlators. These calculations will be presented elsewhere.

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