Extending the Veneziano-Yankielowicz Effective Theory

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Abstract

We extend the Veneziano Yankielowicz (VY) effective theory in order to account for ordinary glueball states. We propose a new form of the superpotential including a chiral superfield for the glueball degrees of freedom. When integrating it “out” we obtain the VY superpotential while the N vacua of the theory naturally emerge. This fact has a counterpart in the Dijkgraaf and Vafa geometric approach. We suggest a link of the new field with the underlying degrees of freedom which allows us to integrate it “in” the VY theory. We finally break supersymmetry by adding a gluino mass and show that the Kähler independent part of the “potential” has the same form of the ordinary Yang-Mills glueball effective potential.
1 Introduction

Supersymmetric gauge theories are much studied in the hope that one day they may be relevant to understand the physics of the real world. We already know a great deal about supersymmetric gauge theories which are closer to their non supersymmetric cousins, namely $\mathcal{N} = 1$ supersymmetric gauge theories, see [1] for a review.

Effective Lagrangians are an important tool for describing strongly interacting theories in terms of their relevant degrees of freedom. A well known effective Lagrangian which economically describes the vacuum structure of super Yang-Mills has been constructed by Veneziano and Yankielowicz (VY) [2]. The Lagrangian concisely summarizes the symmetry of the underlying theory in terms of a “minimal” number of degrees of freedom which are encoded in the superfield $S$

$$S = \frac{3}{32\pi^2 N} \text{Tr} W^2,$$

where $W_\alpha$ is the supersymmetric field strength. When interpreting $S$ as an elementary field it describes a gluinoball and its associated fermionic partner. In this paper we follow the notation introduced in [3].

Besides the gluinoballs with non zero $R$-charge also glueball states with zero $R$ charge are important degrees of freedom. These states are expected to play a relevant role when breaking supersymmetry by adding a gluino mass term. This is so since the basic degrees of freedom of the pure Yang-Mills theory are glueballs. Further support for the relevance of such glueball states in super Yang-Mills comes from lattice simulations [4]. Recently it has also been argued that certain non supersymmetric theories, named orientifold, at infinite number of colors share a number of properties of the ordinary super Yang-Mills theory [5]. With such theories one can, in principle, interpolate [5] between super Yang Mills and QCD with one dirac flavor. One can also imagine a different large $N$ limit [4, 5]. We were also able to obtain a number of relevant results by including the leading $1/N$ corrections via an effective Lagrangian approach [3]. Since glueballs are present in QCD, and for a generic orientifold theory, we expect their presence also at large $N$, i.e. in the super Yang-Mills limit. It is, hence, very natural to expect these states to be present at low energies in super Yang-Mills.

However no physical glueballs appear in the VY effective Lagrangian. In this paper we extend the VY Lagrangian to take into account the glueball
states. Some attempts have already appeared in the literature \[7, 8, 9, 10, 11, 12, 13, 14\]. Shore \[7\] as well as Kaymakcalan and Schechter \[8\] proposed to use controgradient fields to include glueball states in the theory. However due to the classical field constraints used in this approach supersymmetry was not guaranteed to hold at the effective Lagrangian level. Another approach \[9, 14\] has been to rewrite $S$ as the field strength of a real gauge superfield associated to a 3-form. In order to introduce the glueball fields here a model dependent breaking of the gauge invariance has been used.

In this paper we introduce immediately a chiral superfield $\chi$ with the quantum numbers of a glueball. The basic constraints which we will use to construct the effective superpontetial involving $S$ and $\chi$ are: i) The superpotential reproduces the anomalies of super Yang-Mills; ii) The vacuum structure is unaltered even in the presence of the glueball field. These two requirements lead to a general form of the superpotential in terms of an undetermined function of the chiral field $f(\chi)$. However we will argue in favor of a specific form for $f(\chi)$ which has a number of amusing properties. For example the $N$ vacua of the theory emerge naturally when integrating out the glueball superfield $\chi$. This intriguing relation has also a counterpart in the geometric approach to the effective Lagrangian theory proposed by Dijkgraaf and Vafa \[15\] or in the more field-theoretical approach presented in \[16\]. We have also suggested an integrating in procedure which surprisingly yields exactly the specific form of the function $f(\chi)$.

Another important check is associated to supersymmetry breaking. When adding a gluino mass to the theory \`a la Masiero and Veneziano \[17\] the same choice of the function $f(\chi)$ leads to a Kähler independent part of the “potential” which has the same functional form of the glueball effective potential for the non supersymmetric Yang-Mills theory developed and used in \[18, 19, 20, 21\].

We provide the link with the underlying degrees of freedom. This is done by first providing a classical relation between the glueball superfield $\chi$ and the controgradient fields built out of $S$ which mimics the one employed by Shore \[7\] as well as Kaymakcalan and Schechter \[8\] and then upgrading the classical constraint to a quantum one. Due to the nature of the quantum constraint the effective theory in general preserves supersymmetry. We also briefly review the three-form approach while outlining a possible way of linking the two approaches.

In section 2 we briefly review the VY theory and set the notation. Section 3 is devoted to the extension of the VY theory and contains a number of
subsections in which we provide consistency checks for the proposed extension of the VY superpotential. Finally we conclude in section 4.

2 Reviewing the VY effective Lagrangian

It is instructive to briefly review the VY Lagrangian while introducing the notation as in [3].

The underlying Lagrangian of SU($N$) supersymmetric gluodynamics is

\[ L = \frac{1}{2 g^2} \int d^2 \theta \text{Tr} W^2 + \text{H.c.} \]

\[ = -\frac{1}{4g^2} G_{\mu \nu}^{\alpha} G^{\alpha \mu \nu} + \frac{1}{2g^2} D^{\alpha} D^{\alpha} + \frac{i}{g^2} \lambda^{\alpha} \sigma^{\mu} D_{\mu} \bar{\lambda}^{\alpha}, \quad (2) \]

where $g$ is the gauge coupling, the vacuum angle is set to zero and

\[ \text{Tr} W^2 \equiv \frac{1}{2} W^{\alpha, \alpha} W_{\alpha} = -\frac{1}{2} \lambda^{\alpha, \alpha} \lambda_{\alpha}. \quad (3) \]

The low energy effective superpotential [2] constructed in terms of the composite chiral superfield $S$,

\[ S = \frac{3}{32 \pi^2 N} \text{Tr} W^2, \quad (4) \]

is:

\[ W_{VY} = \frac{2N}{3} \int d^2 \theta \left\{ S \ln \left( \frac{S}{\Lambda^3} \right)^N - NS \right\}, \quad (5) \]

where $\Lambda$ is a renormalization group invariant scale.

The chiral superfield $S$ at the component level has the standard decomposition $S(y) = \varphi(y) + \sqrt{2} \theta \chi(y) + \theta^2 F(y)$, where $y^\mu$ is the chiral coordinate, $y^\mu = x^\mu - i\theta \sigma^\mu \theta$, and

\[ \varphi = \frac{3}{64 \pi^2 N} \left[ -\lambda^{\alpha, \alpha} \lambda_{\alpha} \right], \quad (6) \]

\[ \sqrt{2} \chi = \frac{3}{64 \pi^2 N} \left[ G^{\alpha,\beta}_{\alpha,\beta} + 2i D^{\alpha} \lambda_{\alpha} \right], \quad (7) \]

\[ F = \frac{3}{64 \pi^2 N} \left[ -\frac{1}{2} G^{\alpha \mu \nu} G_{\alpha \mu \nu} + \frac{i}{2} G^{\alpha \mu \nu} G_{\alpha \mu \nu} + \text{f.t.} \right], \quad (8) \]

\[ ^{1}\text{The Grassmann integration is defined in such a way that } \int \theta^2 d^2 \theta = 1. \]
where f.t. stands for fermion terms.

The complex field $\phi$ represents the scalar and pseudoscalar gluino-balls while $\chi$ is their fermionic partner. Although it is tempting to say that $F$ represents the scalar and the pseudoscalar glueball it is an auxiliary field. Hence these states are not represented in the VY Lagrangian.

### 3 Introducing the Glueball Superfield $\chi$

One of the hardest problems in confining theories is the identification of the relevant degrees of freedom at low energies especially when the latter are not phenomenologically known. If introduced it is even harder to find their relation with the underlying gauge theory. What is straightforward though is the quantum number identification of the states of interest.

Here, assuming that the lowest component of the chiral superfield $\chi$ contains a scalar glueball, we deduce that the chiral superfield has $R$-charge zero. Covariance under superconformal transformations leads to the relation:

$$d = \frac{3}{2} n_R,$$  \hfill (9)

between the mass dimension $d$ of a generic chiral superfield and its $R$-charge $n_R^2$. The glueball superfield $\chi$ has then zero mass dimension.

Due to the $\chi$ properties just found the general effective superpotential saturating all of the relevant anomalies and containing both $S$ and $\chi$ is:

$$W [S, \chi] = \frac{2N}{3} \left\{ S \ln \left( \frac{S}{\Lambda^3} \right)^N - NS - S f(\chi) \right\},$$  \hfill (10)

with $f(\chi)$ an holomorphic function of $\chi$. The latter in components reads:

$$\chi = \varphi \chi + \sqrt{2} \theta \psi \chi + \theta^2 F \chi.$$  \hfill (11)

The determination of the function $f(\chi)$ would shed light on the super Yang-Mills infrared properties. We will provide various arguments pointing to the function:

$$f(\chi) = N \ln \left[ -e^{\chi} \frac{N}{N} \ln \chi^N \right].$$  \hfill (12)

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\[^2\]Actually a generic field transforming properly (i.e. covariantly) under the superconformal group must satisfy the constraint $2n = d = \frac{3}{2} n_R$, where $n$ is the charge of the field under the superconformal transformations. One can show that invariance under dilatations implies invariance under superconformal transformations.
This function passes a number of consistency checks: i) We recover the VY superpotential when the glueball superfield is integrated out. Besides this procedure naturally leads to the $N$ independent vacua of the theory. ii) We can now better approach non supersymmetric gluondynamics when giving a mass to the gluino. The theory leads to a potential which resembles the ordinary glueball effective potential for the Yang-Mills theory. iii) The superpotential in eq. (10) has a natural interpretation in the geometric approach to the effective Lagrangian theory proposed by Dijkgraaf and Vafa. iv) A reasonable integrating in method leads to the same function.

### 3.1 Integrating Out $\chi$

The first non trivial check comes from integrating out the field $\chi$ via its equation of motion:

$$\frac{\partial W[S,\chi]}{\partial \chi} = -\frac{2N^2}{3} \frac{S}{\chi \ln \chi^N} \left[ \ln \chi^N + N \right] = -\frac{2N^2}{3} \frac{S}{\chi \ln \chi^N} \left[ \ln(0) \chi^N + 2\pi i k + N \right] = 0 , \quad (13)$$

where we have made explicit the dependence of the logarithm on the branches. $\ln(0) \chi^N$ is by definition the $k = 0$ branch of $\ln \chi^N$, i.e. its imaginary part lies within 0 and $2\pi$. The solution is:

$$\chi = \frac{1}{e} e^{-2\pi i \chi} , \quad (14)$$

which yields:

$$W_k [S] = \frac{2N}{3} \left\{ S \ln \left( \frac{S}{\Lambda^3 e^{-2\pi i k/N}} \right)^N - NS \right\} . \quad (15)$$

This reproduces the standard VY result. Besides in this way we can also account naturally for the $N$ vacua of super Yang-Mills. Somewhat surprisingly the present superpotential is equivalent to the one proposed by Kovner and Shifman in [22]. The summation over the $k$ branches is now automatic since from the start we needed to integrate over all of the allowed field $\chi$ configurations in the path integral. After having eliminated the field $\chi$ we still have to sum over the $k$ branches of the logarithm.
3.2 Approaching the Yang-Mills Theory

An important test for the proposed generalization of the VY superpotential deals with the effects of a gluino mass term and the Yang-Mills limit.

The most straightforward approach is to add a “soft” supersymmetry breaking term to the Lagrangian. Masiero and Veneziano [17] introduced the following gluino mass term,

\[ \Delta L_m = -\frac{m}{2g^2} \lambda \lambda + \text{h.c.}. \]  

which at the effective-Lagrangian level translates as

\[ \Delta L_m = \frac{m N}{g^2 3} \frac{32\pi^2}{N^2} (\varphi + \bar{\varphi}) = \frac{4m}{3\lambda} N^2 (\varphi + \bar{\varphi}), \]  

where we introduced the ’t Hooft coupling

\[ \lambda \equiv \frac{g^2 N}{8\pi^2}. \]  

It is convenient to assume the mass parameter \( m \) to be real and positive. One can always make it real and positive by redefining the vacuum angle \( \theta \). In what follows we will adopt this convention.

The softness restriction is \( m/\lambda \ll \Lambda \). Recently, soft SUSY breaking has been reanalyzed in [23], while a model for not-soft breaking has been proposed in [13].

Note that the combination \( m/\lambda \) is renormalization-group invariant to leading order, and scales as \( N^0 \); the one which is renormalization-group invariant to all orders can be found too, see [24]. Analysis of this model indicates that the theory is “trying” to approach the non-SUSY Yang-Mills case. Namely, the spin-0 and spin-1/2 particles split from each other, and their masses each pick up a piece linear in \( m \). One of the \( N \) distinct vacua of the SUSY theory becomes the true minimum. Furthermore, the vacuum value of the gluon condensate is no longer zero.

Although the previous results were encouraging, the glueball states were yet not accounted for in the standard VY approach followed in [17]. Hence it is hard to imagine how the Yang-Mills theory may have emerged after supersymmetry breaking.

In our case we have glueball states even in the supersymmetric limit. We expect, by providing a mass to the gluino field, that these states go over the
ordinary Yang-Mills glueball fields. To better illustrate this phenomenon we focus on the part of the potential of the theory which is Kähler independent. At the first order in the gluino mass, after having integrated out the gluino field, one derives:

\[
V[\varphi_\chi]_k = \frac{4 m \Lambda^3}{3 \lambda} N^2 e \left[ \varphi_\chi \ln(0) \varphi_\chi + \bar{\varphi}_\chi \ln(0) \bar{\varphi}_\chi + 2\pi i \frac{k}{N} (\varphi_\chi - \bar{\varphi}_\chi) \right] + \cdots. \tag{19}
\]

The dots indicate the Kähler dependent terms which do not contribute to the vacuum expectation value of the potential, do not affect the vev of \( \varphi_\chi \) at leading order in \( m \) and will be dropped in the following. \( \varphi_\chi \) is the dimensionless glueball field. The ground state is then obtained for:

\[
ed \varphi_\chi = e^{-2\pi i \frac{k}{N}}, \tag{20}
\]

and the potential is minimized for \( k = 0 \)

\[
\langle V \rangle = -8 \frac{m \Lambda^3}{3 \lambda} N^2 \min_k \left\{ \cos 2\pi \frac{k}{N} \right\} \quad k = 0, 1, \ldots N - 1. \tag{21}
\]

In this way the degeneracy of the \( N \) supersymmetric vacua is lifted and only the \( k = 0 \) solution is selected as the ground state of the theory.

Due to the presence of the single logarithm term this potential resembles the effective potential for ordinary pure Yang-Mills theory \cite{18, 19, 20, 21}. In order to make this similarity even more transparent, and working in the \( k = 0 \) branch, we define the field \( F_Y = \frac{m \Lambda^3}{3 \lambda} e \varphi_\chi \) which has mass dimension four. The potential becomes:

\[
V[F_Y] = \frac{N^2}{2} \left[ F_Y \ln \frac{F_Y}{\Lambda_Y^4} + \text{c.c.} \right], \tag{22}
\]

with \( \Lambda_Y^4 = 8 m \Lambda^3 e/(3\lambda) \). The parameter \( \Lambda_Y \) is not the pure Yang-Mills scale yet. We recall that one loop decoupling gives the following relation between the super Yang-Mills scale, the gluino mass and the Yang-Mills scale \( \Lambda_{YM} = m^{2/11} \Lambda^{9/11} \). All the scales become comparable, i.e. \( \Lambda_Y \sim \Lambda_{YM} \sim \Lambda \) in the limit \( m \sim \Lambda \). By computing, using eq. (22), the trace of the energy momentum tensor we obtain:

\[
\theta_{Y \mu}^\nu = 4 V - 4 \left[ F_Y \frac{\partial V}{\partial F_Y} + \text{c.c.} \right] = -2 N^2 [F_Y + \bar{F}_Y]. \tag{23}
\]
If we now imagine the gluino already decoupled at a scale $m \sim \Lambda$ we can identify this expression with the one associated with the trace of the energy momentum tensor of the underlying pure Yang-Mills theory which is

$$\theta_{YM}\mu = \frac{-11 N}{3} \frac{1}{32\pi^2} G_{\mu\nu} G_{\mu\nu} . \tag{24}$$

One then finds that $Re(F_Y) \propto G_{\mu\nu}^a G^{a\mu\nu}$ while it is natural to expect that $F_Y \propto G_{\mu\nu}^a G^{\mu\nu} - i G_{\mu\nu}^a \tilde{G}^{\mu\nu}$. This also supports the toy model approach used in [13].

We note that when supersymmetry remains intact the lowest component of the superfield $\chi$ cannot be simply $G_{\mu\nu}^a G^{a\mu\nu} - i G_{\mu\nu}^a \tilde{G}^{a\mu\nu}$. Another amusing property is that the imaginary field of $F_Y$ is unstable in the previous potential (see [13] for a modern discussion). This corresponds to a negative mass square for the pseudoscalar glueball. Such a property has been a key point for solving the $U(1)_A$ problem at the effective Lagrangian level when quarks were added to the theory.

### 3.3 Integrating In $\chi$

After the interlude on supersymmetry breaking and the approach to the Yang-Mills theory we now apply the integrating in method for $S$ and $\chi$. In order to do so we first observe that the super Yang-Mills superpotential evaluated on the vacuum of the theory is:

$$W_{vac}[\Lambda,k] = -\frac{2}{3} N^2 \Lambda^3 e^{-2\pi i \frac{k}{N}} = -\frac{2}{3} N^2 \mu^3 e^{2\pi i \frac{k}{N}} . \tag{25}$$

It is immediate to see the parameter $k$ as a shift of the super Yang-Mills complexified coupling constant $\tau$:

$$\tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2} . \tag{26}$$

$S$ can be integrated in as explained in some detail in [1]. In this case the source for $S$ is $\ln \Lambda^{3N}$. This logarithm term is generated at the one loop level in super Yang-Mills. We will return to the integrating in procedure for $S$ at the end of this paragraph.

Here we simply upgrade the discrete shift of the complexified coupling constant to a source for the c-number field $\chi$ and define the superpotential
linear in the source:

\[
W_{\text{linear}}[\Lambda, D, \chi] = -\frac{2N^2}{3} \Lambda^3 D \chi , \quad \text{with} \quad D = -2\pi i \frac{k}{N} .
\]  

(27)

We will provide a justification of this term in the next section when we will also discuss the relation between \( \chi \) and the underlying degrees of freedom of the theory.

The relation between \( D \) and the vev of \( \chi \) is obtained via:

\[
\frac{\partial W_{\text{vac}}[\Lambda, D]}{\partial D} = -\frac{2N^2}{3} \Lambda^3 \chi ,
\]

(28)

which yields:

\[ D = \ln_{(0)} \chi \quad \text{or equivalently} \quad \ln_{(k)} \chi^N = \ln_{(0)} \chi^N + 2\pi i k = 0 , \]

(29)

defining the branch \( k \) of \( \ln \chi^N \). The dynamical superpotential as function of \( \chi \) is then \( [I] \):

\[
W[\Lambda, \chi] = \frac{2N^2}{3} \Lambda^3 \chi \left[ 2\pi i \frac{k}{N} \right] + W_{\text{vac}}[\Lambda, D] - W_{\text{linear}}[\Lambda, D, \chi]
\]

\[
= \frac{2N^2}{3} \Lambda^3 \chi \left[ \ln_{(0)} \left( \frac{\chi}{e} \right)^N + 2\pi i k \right] = \frac{2N^2}{3} \Lambda^3 \left[ \frac{\chi}{N} \ln_{(k)} \left( \frac{\chi}{e} \right)^N \right] ,
\]

(30)

where we have substituted the relation between \( D \) and \( \chi \) given in eq. [29].

We now determine the complete superpotential by finally integrating in \( S \). We recall the aforementioned one loop relation between \( \Lambda \) and \( S \) which in our normalization reads:

\[
W_{\text{loop}}[\Lambda, S] = -\frac{2}{3} N S \left[ \ln \frac{\Lambda^{3N}}{\Lambda_0^{3N}} \right] ,
\]

(31)

where \( \Lambda_0 \) is a reference scale and \( \Lambda \) is the the renormalization invariant scale of the theory. This requires:

\[
\frac{\partial W[\Lambda, \chi]}{\partial \ln \Lambda^3} = -\frac{2}{3} N^2 S .
\]

(32)

After performing the innocuous field redefinition \( \chi \to e \chi \) this leads to

\[
S = \Lambda^3 \left[ -e \frac{\chi}{N} \ln \chi^N \right] ,
\]

(33)
where \( S \) and \( \chi \) are understood as vacuum expectation values and we have dropped the subscript \((k)\) denoting the branch of the logarithm. The complete effective superpotential as function of \( S \) and \( \chi \) is:

\[
W[\chi, \chi] = - \frac{2}{3} N S \left[ \ln \left( \frac{A^{3N}}{\Lambda^{3N}} \right) \right] + W[\Lambda(S, \chi), \chi] - W_{\text{loop}}[\Lambda(S, \chi), S]
\]

\[
= \frac{2N}{3} S \left[ \ln \left( \frac{S}{\Lambda^3} \right)^N - N - N \ln \left( -e \frac{X}{N} \ln \chi \right) \right],
\]

(34)

where for \( \Lambda(S, \chi) \) we have used the relation in eq. (33). We have derived the desired function \( f(\chi) \).

### 3.4 Relation with the fundamental degrees of freedom

The effective Lagrangian describes two independent chiral superfields, i.e. \( S \) and \( \chi \). While for \( S \) we have an interpretation in terms of the fundamental fields for \( \chi \) we still lack such an identification. At a classical level we expect all of the operators to be built out of \( \text{Tr}[W^2] \), or say \( S \).

Interestingly Shore \([7]\) as well as Kaymakcalan and Schechter (KS) \([8]\) have shown that apart from \( S \) we can construct, in terms of the underlying fields, only another independent controgradient field which transforms covariantly under the superconformal transformations, i.e.:

\[
\bar{D}^2 S^{1/3} = -\frac{4}{3} \frac{\bar{F}}{\bar{\varphi}^{2/3}} - \frac{4}{9} \frac{\bar{v}^2}{\bar{\varphi}^{5/3}} - \frac{4\sqrt{2}}{3} \theta \sigma^\mu \partial_\mu \left( \frac{\bar{v}}{\bar{\varphi}^{2/3}} \right) - 4\theta^2 \Box \varphi^{1/3}.
\]

(35)

Note that the lowest component contains the \( G_{\mu\nu}G^{\mu\nu} \) as well as \( G_{\mu\nu}\tilde{G}^{\mu\nu} \) operators when expressing the fields of \( S \) as in eq. (33).

We can thus define the field:

\[
\Phi = \bar{D}^2 S^{1/3} + r S^{2/3},
\]

(36)

with \( r \) an unknown coefficient. \( \chi \) can be naturally introduced as:

\[
\chi = S^{-2/3} \Phi = S^{-2/3} \bar{D}^2 S^{1/3} + r.
\]

(37)

This field has the right quantum numbers to describe the glueball state we have already added in the effective theory. Note that since \( S \) acquires a
non zero vacuum expectation value it is a well defined operation to divide by powers of $S$. This relation also tells about how the fundamental degrees of freedom are related to $\chi$. Due to the presence of $\bar{D}^2$ acting on $S^{11/3}$ the $G^{\mu\nu}G_{\mu\nu}$ and $G^{\mu\nu}\tilde{G}_{\mu\nu}$ operators are partially contained in $\chi$. Note however that the interpolating field for a glueball-type of state does not need to be constructed only out of these operators.

Now let us make some considerations on the vacuum expectation values. Using the relations (36-37) in a given super Yang-Mills vacuum we deduce:

$$\langle \Phi \rangle = r \Lambda^2 \quad \text{and} \quad \langle \chi \rangle = r .$$  \hspace{1cm} (38)

This implies that the glueball field condenses, as long as $r$ is different from 0. We stress that glueball condensation when supersymmetry is intact does not signal the emergence of the gluon condensate. The latter is always guaranteed to vanish\(^3\). An instanton computation could be able to provide a value for $r$ directly from the microscopic theory.

The presence of the constant term $r$, related to a non zero vacuum expectation value for $S$, is crucial. Indeed it allows us to consider $\chi$ as an independent quantum field with respect to $S$. We note that the classical relation (obtained by setting $r = 0$) between $\chi$ and $\bar{D}^2 S^{11/3}$ is the analogous of the relation introduced by Shore \(^7\) and KS \(^8\). However in these works because of such a classical constraint $\chi$ was, in practice, never an independent field. Since a controgradient field is involved, due to the classical constraint, non holomorphic terms for the superpotential are induced. This explains why in \(^7\)\(^8\) at the effective Lagrangian level supersymmetry was not guaranteed to remain intact. We interpret instead the relation (37) as a quantum constraint between two independent fields, $S$ and $\chi$. Now the superpotential is holomorphic in these fields and in general supersymmetry holds. Upgrading $\chi$ to an independent physical field solves at once the long standing puzzle associated to the approach used by KS and Shore. We note that the upgrade from a classical to a quantum constraint is not a new idea, see \(^1\) for a review.

\(^3\)This can be easily checked at the effective Lagrangian level. Since supersymmetry does not break we must have zero vacuum energy density. However the vacuum energy density is $\frac{1}{2}(\bar{\theta}^\mu)\theta_\mu$ and $\theta_\mu$ is the trace of the energy momentum tensor. This is proportional to $G^{\mu\nu}_\mu G^{\mu\nu}_\mu$, see \(^3\).

\(^4\)Actually the KS and Shore dimensionless field is $\chi^{-1/2}$ for $r = 0$. 

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The tree-level theory we start from is

\[ L_{\text{fund}} = -\frac{2N}{3} \int d^2 \theta (2\pi i\tau) S + \text{c.c.} , \]  

(39)

It is a standard procedure to add to this Lagrangian the one loop corrections so that it properly accounts (in the Wilsonian scheme) for the perturbative dynamics. Apart from the trivial factor \(-2N/3\) one gets:

\[ \int d^2\theta \left[ 3N \ln \left( \frac{\Lambda}{\Lambda_0} \right) \right] S + \text{c.c.} . \]  

(40)

Now we want to further modify this Lagrangian to take into account also the (non-perturbative) phenomenon of gaugino condensation. We thus introduce the field \( \chi \) and write:

\[ \int d^2\theta \left\{ \left[ 3N \ln \left( \frac{\Lambda}{\Lambda_0} \right) \right] S + C \left( S - \Lambda^3 e^{-\frac{2\pi i k}{N}} \right) \chi + \text{c.c.} \right\} . \]  

(41)

Variation with respect to \( \chi \) implements gaugino condensation. To determine the constant \( C \) we note that the coefficient of \( \text{Tr}[W^2] \), i.e. \( S \), is the full coupling constant \( \hat{\tau} \). We hence deduce:

\[ 2\pi i\tau + 3N \ln \left( \frac{\Lambda}{\Lambda_0} \right) + C r = 2\pi i\hat{\tau} , \]  

(42)

where we have substituted the on-shell value for \( \chi \) (i.e. \( r \)). Here \( \tau \) is the bare coupling constant. The coefficient \( Cr \) is clearly a shift of the complexified coupling constant and as such we must have:

\[ Cr = 2\pi i k . \]  

(43)

This implies that the coefficient of \( \chi \) in (41) is \( \frac{2\pi i k}{rN} \Lambda^3 e^{-\frac{2\pi i k}{N}} \) (once the \( N \) in front of the \( S \ln \Lambda^3 \) term is factored out). A source term for \( \chi \) that depends on \( k \) (the shift of the complexified coupling constant) has now been generated. This is precisely what we expected for the glueball field. It is thus tempting to identify the \( \chi \) field just introduced with the glueball superfield. This is not yet sufficient to state that we have generated a term linear in \( 2\pi i k/N \) as a source for it. However self-consistency with the integrating in procedure requires the vev of \( \chi \) (i.e. \( r \)) to have precisely the same \( k \)-dependence of the vev of \( S \) (see (29)). These findings complete the results presented in 3.3.
Inspired by the UV identification (37) we might replace (41) with

\[ \int d^2 \theta \left\{ \left[ 3N \ln \left( \frac{\Lambda}{\Lambda_0} \right) \right] S + C \left( S - \Lambda^3 e^{-\frac{2\pi ik}{N}} \right) \left( \chi - S^{-2/3} \bar{D}^2 S \right)^{1/3} + \text{c.c.} \right\} \quad (44) \]

Indeed the only difference would be a term containing a controgradient field that, although formally written as an \( F \) term, is actually a \( D \) term.

### 3.5 Dijkgraaf–Vafa Connection

A powerful tool to study the dynamics of super Yang-Mills theories is to embed them in string theory. Within this framework, various approaches and several line of investigations have been developed, all going under the rather generic name of gauge-string correspondence. One of the most interesting result which emerged is the relation of \( \mathcal{N} = 1 \) super Yang-Mills to matrix models. This relation has been found following a long path, going through topological string theory, superstring theory and D-branes.

However, after the matrix model structure of gauge theories has been conjectured in this set-up [15], it has also been possible to recover the same results in a purely field theoretical approach [16, 27] (for a nice and detailed review see [28]).

We would like to compare our findings with these more ‘stringy’ approaches. According to which particular geometric model one analyzes, there are different ways to make such a comparison (some of them are currently under investigation). Here we see that the inclusion of the glueball degrees of freedom in \( \mathcal{N} = 1 \) super Yang-Mills, pursued in the way we described in this paper, fits naturally within the Dijkgraaf–Vafa (DV) approach or the related one by Cachazo, Douglas, Seiberg and Witten, (CDSW).

Note that, even if in these approaches one deals with \( \mathcal{N} = 1 \) super Yang Mills theory coupled to a chiral adjoint scalar supermultiplet \( \Phi \) with a superpotential at tree level, it is possible to choose a quadratic superpotential, that consists just of a mass term for \( \Phi \). Assuming a large mass, it is straightforward to integrate \( \Phi \) out. In this way, at low energy one is left with pure \( \mathcal{N} = 1 \) super Yang-Mills theory. We focus on the CDSW results (we refer the reader to [16] for notation and further details) but one should keep in mind that there is an obvious translation in more geometrical terms (à la DV).

Within this approach it is possible to write the low energy superpotential
for the field $S$ as:

$$W_{\text{eff}}[S] = 2\pi i \left( \int_A T \int_B R - \int_B T \int_A R \right),$$  \hspace{1cm} (45)$$

where $A$ and $B$ are proper compact and non-compact cycles over the complex plane, which arises as the moduli space of the adjoint field $\Phi$. The appearance of a branch cut and non-trivial cycles is the result of the quantum dynamics of the underlying theory [16]. $T$ and $R$ are suitable one forms defined over the complex plane. They have the following properties [16, 25]:

\begin{align*}
\int_A T(z)dz &= N, \quad \int_B T(z)dz = -\tau + k, \hspace{1cm} (46) \\
\int_A R(z)dz &= S, \quad \int_B R(z)dz = \frac{1}{2\pi i} \frac{\partial F}{\partial S}, \hspace{1cm} (47)
\end{align*}

where $N$ is the rank of the gauge group, $F$ is called the “prepotential”, $\tau$ is the bare complexified coupling constant and $k$ is an arbitrary integer. We deduce the effective superpotential:

$$W_{\text{eff}}[S] = N \frac{\partial F}{\partial S} + 2\pi i \tau S - 2\pi i k S$$  \hspace{1cm} (48)$$

In the case we are considering (the quadratic choice for the tree-level superpotential) it is possible to compute it explicitly [26, 15, 16]:

$$W_{\text{eff}}[S] = NS \left( \log \Lambda^3_0 S + 1 \right) + 2\pi i \tau S - 2\pi i k S ,$$  \hspace{1cm} (49)$$

where $\Lambda_0$ is an ultraviolet cut-off arising from the integration on the non-compact cycle and $\tau$ is meant to be evaluated precisely at $\Lambda_0$. In this paragraph we have suppressed the $2N/3$ normalization factor.

Using the knowledge of the one loop $\beta$-function, we rewrite the superpotential as

$$W_{\text{eff}}[S] = NS \left( \log \frac{\Lambda^3}{S} + 1 \right) - 2\pi i k S$$  \hspace{1cm} (50)$$

where $\Lambda$ is now the dynamically generated scale. We see that this superpotential, a part from an overall normalization, is identical to the one in eq. [15]. Recall that the integer $k$ in [15] appeared after having integrated
out the glueball superfield. In eq. (50) instead it comes from the integration of the meromorphic one-form $T$ along the non compact cycle. To show that the result of the integration gives, besides the complexified coupling constant, an integer number $(k)$ is a non-trivial matter and was proven only ‘on-shell’ in \[27\].

At this point the glueball superfield may emerge if one modifies the prescription in eq. (46) as follows:

$$
\int_B T(z)dz = -\tau - \frac{f(\chi)}{2\pi i},
$$

with the requirement

$$
f'(<0) = 0 \text{ and } f(<0) = -2\pi i k.
$$

These are exactly the properties of the function $f(\chi)$ we introduced in eq. (12).

Thus the extended VY effective superpotential may have a natural geometric interpretation.

### 3.6 The three form approach

Long ago Gates \[29\] has classified all of the possible $p$-form gauge superfields in four-dimensional space time. To the three form is associated a real superfield $U$ (not to be confused with the real vector gauge superfield associated to the one-form) whose field strength is

$$
\bar{D}^2 U.
$$

The gauge transformation reads:

$$
U \rightarrow U + \frac{1}{2} (D^\alpha \Gamma_\alpha + \bar{D}^{\dot{\alpha}} \bar{\Gamma}_{\dot{\alpha}}),
$$

and it involves the gauge superfield associated to the 2-form $\Gamma_\alpha$. Gates observed that the field strength in eq. (53) is equivalent to a chiral multiplet with the pseudoscalar auxiliary field replaced by a four-from field strength. Since the pseudoscalar auxiliary field of $S$ has a component proportional to $G_{\mu\nu}\tilde{G}^{\mu\nu}$ which can be rewritten as a four-form field strength one can consider \[30\] expressing $S$ as the field strength of $U$ via the relation:

$$
\bar{D}^2 U = -\frac{S}{4}.
$$
Due to the gauge invariance associated to the three form the physical degrees of freedom are the ones contained in $S$. Besides the four-form is still an auxiliary field and must be integrated out in the end. So unless the gauge symmetry is broken or additional chiral superfield are explicitly added no new degrees of freedom, except for the physical ones already present in $S$, are generated. In [9,10] and [14] different gauge breaking terms were added to the theory. It was also realized in [10] that the net effect of such a gauge breaking term is that now one has two independent chiral superfields $\chi$ and $S$. $\chi$ has the same quantum numbers we considered. We expect that a superpotential identical to the one we found can emerge using the three-form approach when allowing for a more general set of gauge symmetry breaking terms while further enforcing the consistency checks. Another possibility would be to keep gauge invariance of the three form which describes $S$ and add a new independent field $\chi$.

4 Conclusions

We have proposed an extension of the VY effective theory which takes into account ordinary glueball states. The general construction principle has been to identify first the $R$-symmetry quantum number of the chiral superfield describing the glueball-type state. We have then used superconformal covariance to determine the conformal weight of the associated chiral superfield describing the glueballs. This allowed us to construct the superpotential which still saturates the anomalies of the underlying $\mathcal{N}=1$ super Yang-Mills theory. These constraints were not sufficient to fix the superpotential written in terms of $S$ and the glueball superfield $\chi$.

We proposed, however, a specific form of the superpotential which has amusing properties and passes a number of consistency checks. For example we were able to integrate “out” and “in” the field $\chi$. We obtained in the first case the standard VY theory while in the second we deduced the extended VY effective theory. We have shown that in the present approach the $N$ vacua of the theory emerge due to the presence of the glueball superfield. This fact had a natural counterpart in the geometric approach to the effective Lagrangian theory proposed by Dijkgraaf and Vafa. However a better understanding of this relation is needed. We have also broken supersymmetry by adding a gluino mass. The effective theory has led to a Kähler independent part of the “potential” which reproduces the glueball effective potential for the non
supersymmetric pure Yang-Mills theory.

Since the superpotential is known it will be interesting, in the future, to investigate in some detail some physical consequences associated, for example, to the spectrum of the theory. Moreover the generalization of the extended VY theory to orientifold field theories is an interesting avenue to explore. We plan to study also super quantum chromodynamics. We will also try to gain further insight using string theory approaches suited to describe gauge dynamics.

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