"Square Root" of the Proca Equation: Spin-3/2 Field Equation

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Abstract

New equations describing particles with spin 3/2 are derived. The non-local equation with the unique mass can be considered as “square root” of the Proca equation in the same sense as the Dirac equation is related to the Klein-Gordon-Fock equation. The local equation describes spin 3/2 particles with three mass states. The equations considered involve fields with spin-3/2 and spin-1/2, i.e. multi-spin 1/2, 3/2. The projection operators extracting states with definite energy, spin, and spin projections are obtained. All independent solutions of the local equation are expressed through projection matrices. The first order relativistic wave equation in the 20-dimensional matrix form, the relativistically invariant bilinear form and the corresponding Lagrangian are given. Two parameters characterizing non-minimal electromagnetic interactions of fermions are introduced, and the quantum-mechanical Hamiltonian is found. It is proved that there is only causal propagation of waves in the approach considered.

1 Introduction

Until now, we know different field equations for higher spin particles introduced in [1], [2], [3], [4], [5], [6]. It should be mentioned that the higher spin (HS) gauge theories are intensively investigated now [7]. In such an approach the higher derivatives appear in field interactions, so that the theory is non-local. The HS gauge theories include linearized electrodynamics and gravity. The superstring field theory can be connected to some version of the HS gauge theory. Anti-de Sitter (AdS) space is a natural background for interacting HS gauge fields. We also mention the Maldacena conjecture [8] that is the correspondence between field (string) theories on $(d + 1)$-dimensional

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AdS space and (super) conformal field theories (CFT) on the boundary ($d$-dimensional) of this space. For example, super $SU(N)$ Yang-Mills theory ($N = 4$) may be formulated as IIB supergravity compactified on $AdS_5 \times S^5$.


It is well known, however, that there are physical inconsistencies for describing massive particles with higher spins $s$, $s \geq 1$ in external electromagnetic fields. So, free higher spin particles can be described without difficulties, but if one introduces interactions of particles with electromagnetic fields, some problems arise [11]. For example, in the framework of the Rarita-Schwinger approach, there is superluminal speed of particles interacting (minimally) with electromagnetic fields. This indicates on noncausal propagation of interacting particles [12]. This difficulty, however, can be eliminated by considering $N = 2$ supergravity [13].

For phenomenological applications of hadron physics, one needs a consistent model with an electromagnetic background. Massive spin-3/2 electrodynamics was studied in [14].

The purpose of this paper is to take the “square root” of the Proca equation [15] in analogous procedure performed by Dirac [16] on the Klein-Gordon-Fock equation, and to obtain the spin-3/2 field equation. We concentrate here on studding solutions (pure spin states) of free equations and discussing particle electromagnetic interactions. Possibly, presented here new equations for spin-3/2 particles can be considered as effective equations for describing hadrons.

The paper is organized as follows. In Sec. 2 we derive the “square root” of the Proca equation, which is the spin-3/2 non-local field equation describing particles with the unique mass. Then, the local spin-3/2 field equation is obtained. The projection operators extracting solutions of the field equation with positive and negative energy are constructed in Sec. 3. It was proved that there are three different mass states of fields. We consider projection operators in Sec. 4, which allow us to separate states with spins $1/2$ and $3/2$ as well as spin projections. It is shown that the field equation considered describes fields with spin $1/2$ and spin $3/2$, i.e. multi-spin $1/2$, $3/2$. In Sec. 5 we represent the second order equation in the 20-dimensional matrix form of the first order relativistic wave equation. The relativistically invariant bilin-
ear form and the corresponding Lagrangian are given there. Sec. 6 is devoted to introducing two parameters characterizing non-minimal electromagnetic interactions of fermions. We also find the quantum-mechanical Hamiltonian. Sec. 7 is devoted to discussion.

The system of units $\hbar = c = 1$ is used. Four-vectors are defined as $v_\mu^2 = v_m^2 + v_4^2 = v^2 - v_0^2$, $v^2 \equiv v_m^2 = v_1^2 + v_2^2 + v_3^2$, $v_4 = iv_0$.

## 2 Spin-3/2 Field Equation

We are looking for an equation which when squared will produce the Proca equation. The Proca equation \[15\] for a free particle possessing the mass $m$ is given by

$$\partial_\nu \varphi_{\mu\nu}(x) + m^2 \varphi_\mu(x) = 0,$$

(1)

where $\partial_\nu = \partial/\partial x_\nu = (\partial/\partial x_m, \partial/\partial (it))$, and antisymmetric tensor $\varphi_{\mu\nu}(x)$ can be expressed through the four-vector field $\varphi_\mu(x)$ as follows

$$\varphi_{\mu\nu}(x) = \partial_\mu \varphi_\nu(x) - \partial_\nu \varphi_\mu(x).$$

(2)

Replacing the tensor $\varphi_{\mu\nu}(x)$ from Eq. (2) into Eq. (1), one transfers the system of first order equations (1), (2) into the second order equation

$$\left[ (\partial_\alpha^2 - m^2) \delta_{\mu\nu} - \partial_\mu \partial_\nu \right] \varphi_\nu(x) = 0.$$

(3)

We can represent Eq. (3) in the matrix form

$$N \varphi(x) = 0, \quad N = \left( \partial_\alpha^2 - m^2 \right) I_4 - (\partial \cdot \partial),$$

(4)

where the $I_4$ is the unit $4 \times 4$ matrix and the $(\partial \cdot \partial)$ is the matrix-dyad with the matrix elements $(\partial \cdot \partial)_{\mu\nu} = \partial_\mu \partial_\nu$, so that the $N$ is the matrix-differential operator.

Let us consider the operator equation

$$\left[ \gamma_\mu \partial_\mu + a (\partial \cdot \partial) + m \right] \left[ \gamma_\mu \partial_\mu - a (\partial \cdot \partial) - m \right] = N,$$

(5)

where $a$, in general, is an operator, and the Dirac matrices $\gamma_\mu$ obey the commutation relations (we use notations as in \[17\])

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 \delta_{\mu\nu}.$$

(6)
We notice that Kronecker symbols are used in Eqs. (3), (6) instead of Minkowski tensors because of our Euclidean metrics (four components of all vectors are imaginary). It is easy to check that Eq. (5) is valid for two values of the $a$:

$$a = -m \pm \sqrt{m^2 + \partial_\alpha^2}. \quad (7)$$

Thus, the operators in Eqs. (5), (7) are nonlocal and act in the momentum space as

$$\frac{1}{\partial_\alpha^2} \to -\frac{1}{p^2}, \quad \sqrt{m^2 + \partial_\alpha^2} \to \sqrt{m^2 - p^2}, \quad (8)$$

where the four-momentum squared being $p^2 = p^2 - p_0^2$. From operator equation (5), we come to nonlocal equations for the spinor-vector $\psi(x) = \{\psi_\mu(x)\}$:

$$\left[ \gamma_\mu \partial_\mu + a \frac{(\partial \cdot \partial)}{\partial_\alpha^2} + m \right] \psi(x) = 0, \quad (9)$$

or

$$\left[ \gamma_\mu \partial_\mu - a \frac{(\partial \cdot \partial)}{\partial_\alpha^2} - m \right] \psi(x) = 0. \quad (10)$$

As in the similar procedure performed by Dirac on the Klein-Gordon-Fock equation, we have two equations with the opposite signs of the mass.

For massless case, one has to put $m = 0$, $a = \sqrt{\partial_\mu^2}$ in Eqs. (9), (10). So, from Eq. (9), we find that massless particles are described by a nonlocal equation

$$\left[ \gamma_\mu \partial_\mu + \frac{(\partial \cdot \partial)}{\sqrt{\partial_\mu^2}} \right] \psi(x) = 0.$$

In further, the attention will be paid on Eq. (9). The 4-vector-spinor $\psi_\mu(x)$ without supplementary conditions possesses the 16 components and describes a field with spin-3/2 (8 components, four spin projections $\pm1/2, \pm3/2$, and two values, positive and negative, of energy) and two copies of spin-1/2 fields (four components for each spin-1/2 field). Thus, the 4-vector-spinor $\psi_\mu(x)$, in general, describes fields with multi-spin 1/2, 3/2. In [18], [19], we considered the non-Abelian gauge theory of particles with multi-spin 1/2, 3/2 described by the 4-vector-spinor $\psi_\mu(x)$ which obeys the Dirac equation. The equations (9), (10) are different from that by the second order terms involving derivatives.

It follows from Eq. (5) that if the 4-vector-spinor $\psi_\mu(x)$ satisfies Eq. (9) or Eq. (10), the $\psi_\mu(x)$ also obeys the Proca Eq. (3), which acts only on
vector subspace. But Eq. (3) involves (this follows after acting the derivative \( \partial_\mu \) on the equation) the supplementary condition \( \partial_\mu \varphi_\mu(x) = 0 \). Thus, Eqs. (9), (10) involve the supplementary condition

\[
\partial_\mu \varphi_\mu(x) = 0. \tag{11}
\]

Eq. (11) reduces the degrees of freedom from 16 to 12 because Eq. (11) is valid for each spinor indexes. So, Eq. (9) (or Eq. (10)) describes one field with spin-3/2 and one field with spin-1/2 (one copy). From the Proca equation and Eq. (11) one comes to the Klein-Gordon-Fock equation for each component of the 4-vector-spinor

\[
\left( \partial_\alpha^2 - m^2 \right) \psi_\mu(x) = 0. \tag{12}
\]

Eq. (12) indicates that states with spin 1/2 and 3/2 of the field \( \psi_\mu(x) \) have the same mass \( m \). Therefore, on the mass shell we can replace the operator \( \partial_\alpha^2 \) in Eqs. (7), (9), (10) by the \( m^2, \partial_\alpha^2 \to m^2 \). Thus, we arrive from Eqs. (9), (7) to

\[
(\gamma_\nu \partial_\nu + m) \psi_\mu(x) - \frac{b}{m} \partial_\mu \partial_\nu \psi_\nu(x) = 0, \tag{13}
\]

where \( b = 1 \pm \sqrt{2} \). The same procedure can be applied to Eq. (10) which is different from Eq. (9) only by signs in second and third terms. It should be noted that Eq. (9) and Eq. (13) are not equivalent each other. Indeed, the supplementary condition (11) follows from Eq. (9), and Eq. (12) is valid. This means that non-local Eq. (9) describes particles with spin 3/2 and 1/2 (one copy) with the unique mass \( m \). But this is not the case for Eq. (13). Eqs. (11), (12) are not consequences of Eq. (13). Therefore, Eq. (13) describes particles with spin 3/2 and 1/2 (two copies) with some masses. We obtain the mass spectrum of particles, which are described by Eq. (13), in the next section. Eq. (13) is different from the Rarita - Schwinger equation involving two supplementary conditions [9] which describes pure spin-3/2 field. In further, we consider the case with arbitrary parameter \( b \) in Eq. (13).

### 3 Mass Projection Operators

The matrix form of Eq. (13) in momentum space for positive energy is given by

\[
\Lambda \psi(p) = 0, \quad \Lambda = i\not{\!p} + m + \frac{b}{m}(p \cdot p), \tag{14}
\]
where $\hat{p} = \gamma_\mu p_\mu$, and $(p \cdot p)$ is the matrix-dyad. For negative energy one should make the replacement $p \rightarrow -p$.

Acting on Eq. (14) by the operator $i\hat{p} - m - b(p \cdot p)/m$, one obtains the equation defining the mass spectrum

$$
\left[p^2 + m^2 + \left(\frac{b^2}{m^2}p^2 + 2b\right)(p \cdot p)\right] \psi(p) = 0. 
$$

(15)

The mass matrix

$$
M = p^2 + m^2 + \left(\frac{b^2}{m^2}p^2 + 2b\right)(p \cdot p) 
$$

(16)

obeys the minimal equation

$$
(M - p^2 - m^2) \left[M - p^2 \left(1 + \frac{b^2}{m^2}p^2 + 2b\right) - m^2\right] = 0. 
$$

(17)

Eigenvalues of the matrix $M$ obtained from Eq. (17) are $\lambda_1 = p^2 + m^2$, $\lambda_2 = p^2 (1 + b^2 p^2 / m^2 + 2b) + m^2$. Nontrivial solutions of Eq. (15) exist if $\det M = 0$, or equivalently, $\lambda_1 = 0$ or/and $\lambda_2 = 0$. The first solution $\lambda_1 = 0$ ($p^2 = -m^2$) defines the mass $m$ of the field states. The second solution $\lambda_2 = 0$ gives the equation

$$
\left[b^2 p^4 + m^2 (2b + 1) p^2 + m^4\right] \psi(p) = 0. 
$$

(18)

Eq. (18) possesses solutions:

$$
p^2 = -m^2 \left(\frac{2b + 1 \pm \sqrt{4b + 1}}{2b^2}\right),
$$

(19)

which define, in general, two different squared masses of fermions (see [20], [19] for the case of bosons):

$$
m_1^2 = m^2 \left(\frac{2b + 1 - \sqrt{4b + 1}}{2b^2}\right), \quad m_2^2 = m^2 \left(\frac{2b + 1 + \sqrt{4b + 1}}{2b^2}\right).
$$

(20)

It follows from Eq. (20) that $b \geq -1/4$. Therefore, the value $b = 1 - \sqrt{2}$ does not satisfy this requirement. Both masses are equal, $m_1 = m_2$, at $b = -1/4$. We obtain from Eq. (20) masses of particles

$$
m_1 = m \left(\frac{1 - \sqrt{4b + 1}}{2b}\right), \quad m_2 = m \left(\frac{1 + \sqrt{4b + 1}}{2b}\right).
$$

(21)
There are also solutions with opposite signs of masses. From Eqs. (21), we find the mass formula \( m_1 + m_2 = m/b \).

Now we consider the projection operators extracting solutions of Eq. (14) corresponding to the mass \( m \) of fields. Straightforward calculations show that the operator of Eq. (14) obeys the minimal equation for the value \( b = 1 + \sqrt{2} \):

\[
\Lambda^4 - 2(1 - \sqrt{2})m\Lambda^3 + (1 - 4\sqrt{2})m^2\Lambda^2 - 2m^3\Lambda = 0. \tag{22}
\]

One may obtain the minimal equation of the operator \( \Lambda \) for arbitrary parameter \( b \). Eq. (22) allows us to construct projection operator extracting a solution of Eq. (14) corresponding to positive energy. With the help of the method [22], we find such projection operator:

\[
\Pi = \frac{1}{2m^3} \Lambda^3 + \frac{(1 - \sqrt{2})}{m^2} \Lambda^2 - \frac{(1 - 4\sqrt{2})}{2m} \Lambda + 1
\]

\[
= \frac{m - i\hat{p}}{2m} - \frac{i\hat{p}(p \cdot p)}{2m^3} + \frac{(p \cdot p)}{2m^2}
\]

which obeys the necessary equation

\[
\Pi^2 = \Pi. \tag{24}
\]

For negative energy, corresponding to antiparticles, one has to make the replacement \( p \rightarrow -p \) in the operator \( \Lambda \). Every column of the matrix \( \Pi \) is the solution, \( \psi(p) \), of Eq. (14). The matrix \( \Pi \) has eigenvalues one and zero, so that, using the transformation \( \Pi' = S\Pi S^{-1} \) (the wave function in this representation becomes \( \psi'(p) = S\psi(p) \)), the operator \( \Pi \) may be represented in the diagonal form containing only ones and zeroes. The \( \Pi \) acting on the arbitrary non-zero 4-vector-spinor \( \chi = \{ \chi_\mu \} \) will produce the solution of Eq. (14), \( \psi(p) = \Pi \chi \).

### 4 Spin Projection Operators

The 4-vector-spinor \( \psi_\mu \) transforms as a direct product of the vector representation \((1/2, 1/2)\), and the spinor representation \((1/2, 0) \oplus (0, 1/2)\) of the Lorentz group. The generators of the representation \((1/2, 1/2)\) can be taken as follows [19]:

\[
J^{(1)}_{\mu\nu} = \varepsilon^{\mu\nu\rho} - \varepsilon^{\nu\mu\rho}, \tag{25}
\]
where $4 \times 4$-matrices $\varepsilon^{\mu,\nu}$ obey equations \[ (\varepsilon^{\mu,\nu})_{\alpha\beta} = \delta_{\mu\alpha} \delta_{\nu\beta}, \quad \varepsilon^{\mu,\sigma} \varepsilon^{\rho,\nu} = \delta_{\sigma\rho} \varepsilon^{\mu,\nu}, \] \[ (\varepsilon^{\mu,\nu})_{\alpha\beta} = \delta_{\mu\alpha} \delta_{\nu\beta}, \quad \varepsilon^{\mu,\sigma} \varepsilon^{\rho,\nu} = \delta_{\sigma\rho} \varepsilon^{\mu,\nu}, \] (26) $\mu, \nu = 1, 2, 3, 4$. With the help of Eqs. (26) it is easy to verify that the generators (25) satisfy the commutation relations:

\[ [J^{(1)}_{\mu\nu}, J^{(1)}_{\alpha\beta}] = \delta_{\nu\alpha} J^{(1)}_{\mu\beta} + \delta_{\mu\beta} J^{(1)}_{\nu\alpha} - \delta_{\nu\beta} J^{(1)}_{\mu\alpha} - \delta_{\mu\alpha} J^{(1)}_{\nu\beta}. \] (27)

It should be noted that in our metrics the antisymmetric parameters of the Lorentz group $\omega_{mn}$ ($m, n = 1, 2, 3$) are real, and $\omega_{m4}$ are imaginary.

The generators of the Lorentz group corresponding to the spinor representation $(1/2, 0) \oplus (0, 1/2)$ are \[ J^{(1/2)}_{\mu\nu} = \frac{1}{4} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu), \] (28) satisfying the algebra (27).

The generators of the Lorentz group in the 16-dimensional representation $(1/2, 1/2) \otimes [(1/2, 0) \oplus (0, 1/2)]$ are given by

\[ J_{\mu\nu} = J^{(1)}_{\mu\nu} \otimes I_4 + I_4 \otimes J^{(1/2)}_{\mu\nu}, \] (29)

where the $I_4$ is the unit matrix in 4-dimensional subspaces, $\otimes$ is the direct product, and $J_{\mu\nu}$ obeys the algebra (27) corresponding to the Lorentz group.

Let us consider the squared Pauli-Lubanski vector for a vector subspace

\[ \sigma^{(1/2)} = \left( \frac{1}{2m} \varepsilon_{\mu\nu\alpha\beta} p_\nu J^{(1)}_{\alpha\beta} \right)^2 = \frac{1}{m^2} \left( \frac{1}{2} J^{(1)}_{\mu\alpha} p^2 - J^{(1)}_{\mu\alpha} J^{(1)}_{\nu\alpha} p_\mu p_\nu \right), \] (30)

where $p^2 \equiv p_\mu^2$, $\varepsilon_{\mu\nu\alpha\beta}$ is antisymmetric tensor Levi-Civita, $\varepsilon_{1234} = -i$. With the help of Eqs. (25), (26), one may verify that the operator (30) obeys the minimal matrix equation

\[ \sigma^{(1/2)} \left( \sigma^{(1/2)} - 2 \right) = 0. \] (31)

The eigenvalues of the squared spin operator $\sigma^{(1/2)}$ equal to $s(s + 1)$. Eq. (31) shows that the eigenvalues of $\sigma^{(1/2)}$ are zero and two, corresponding to spins $s = 0$ and $s = 1$. This is the consequence of the fact that the representation $(1/2, 1/2)$ of the Lorentz group includes two spins, $s = 0, 1$ (see [23], [19]).
The squared Pauli-Lubanski vector for a spinor subspace $\sigma^{(1/2)^2} = \left(\varepsilon_{\mu\nu\alpha\beta} p_\nu J^{(1/2)}_{\alpha\beta} / 2m\right)^2$ becomes
\[
\sigma^{(1/2)^2} = \frac{3}{4},
\] indicating that the representation $(1/2, 0) \oplus (0, 1/2)$ involves only pure spin $1/2$.

The 16-dimensional representation $(1/2, 1/2) \otimes [(1/2, 0) \oplus (0, 1/2)]$ considered can be represented as $(1, 1/2) \oplus (1/2, 1) \oplus (1/2, 0) \oplus (0, 1/2)$ and contains multi-spin $1/2, 3/2$. Indeed, one can verify that the squared Pauli-Lubanski vector for this representation
\[
\sigma^2 = \frac{1}{m^2} \left(\frac{1}{2} J^{2}_{\alpha\beta} p^2 - J_{\mu\alpha} J_{\nu\alpha} p_\mu p_\nu\right),
\] with the generators (29) satisfies the minimal matrix equation
\[
\left(\sigma^2 - \frac{3}{4}\right) \left(\sigma^2 - \frac{15}{4}\right) = 0.
\] As a result, the equation
\[
\sigma^2 \psi(p) = s(s + 1) \psi(p)
\] corresponds to eigenvalues $s(s + 1) = 3/4$ ($s = 1/2$) and $s(s + 1) = 15/4$ ($s = 3/2$). To separate states with spin $s = 1/2$ and $3/2$, we explore the method described in [22]. Following this procedure one arrives at the projection operators
\[
S^2_{1/2} = \frac{5}{4} - \frac{1}{3} \sigma^2, \quad S^2_{3/2} = \frac{1}{3} \sigma^2 - \frac{1}{4}.
\] The operators $S^2_{1/2}, S^2_{3/2}$ acting on the 16-dimensional vector extract states with spin $1/2$ and $3/2$, correspondingly, obey equations similar to Eq. (24) and also the equations: $S^2_{1/2} + S^2_{3/2} = 1$ ($1 \equiv I_{16}$ is a unit matrix in 16-dimensional space), $S^2_{1/2}S^2_{3/2} = 0$.

The total set of commuting operators includes also operators of the spin projections on the direction of the momentum $p$. Thus, we introduce spin projection operators
\[
\sigma_p = \frac{i}{2|p|} \varepsilon_{abc} p_a J_{bc} = \sigma^{(1)}_p \otimes I_4 + I_4 \otimes \sigma^{(1/2)}_p,
\]
where \(|p| = \sqrt{p_1^2 + p_2^2 + p_3^2}\), and spin projection operators for vector and spinor subspaces are defined as

\[
\sigma^{(1)}_p = -\frac{i}{2|p|} \epsilon_{abc} p_a J^{(1)}_{bc}, \quad \sigma^{(1/2)}_p = -\frac{i}{2|p|} \epsilon_{abc} p_a J^{(1/2)}_{bc}.
\] (37)

Using Eqs. (6), (26), one may verify that minimal matrix equations

\[
\sigma^{(1)}_p (\sigma^{(1)}_p - 1) (\sigma^{(1)}_p + 1) = 0, \quad (\sigma^{(1/2)}_p - \frac{1}{2}) (\sigma^{(1/2)}_p + \frac{1}{2}) = 0,
\] (38)

\[
(\sigma^2_p - \frac{1}{4}) (\sigma^2_p - \frac{9}{4}) = 0,
\] (39)

are valid. With the aid of method [22], we construct from Eq. (39) the projection operators

\[
P_{\pm 1/2} = \pm \frac{1}{2} \left( \sigma_p \pm \frac{1}{2} \right) \left( \sigma_p^2 - \frac{9}{4} \right), \quad P_{\pm 3/2} = \pm \frac{1}{6} \left( \sigma_p \pm \frac{3}{2} \right) \left( \sigma_p^2 - \frac{1}{4} \right)
\] (40)

which extract spin projections \(\pm 1/2, \pm 3/2\). These operators obey the required properties (24) and

\[
P_{+1/2} + P_{-1/2} + P_{+3/2} + P_{-3/2} = 1, \quad P_{\pm 1/2} P_{\pm 3/2} = 0.
\] (41)

The projection operators extracting states with pure spin, spin projections and positive energy are given by

\[
\Delta^{(1/2)}_{\pm 1/2} = \Pi S^2_{1/2} P_{\pm 1/2}, \quad \Delta^{(3/2)}_{\pm 1/2} = \Pi S^2_{3/2} P_{\pm 1/2}, \quad \Delta^{(3/2)}_{\pm 3/2} = \Pi S^2_{3/2} P_{\pm 3/2}.
\] (42)

Thus, the projection matrix \(\Delta^{(1/2)}_{\pm 1/2}\) extracts the states with spin \(1/2\) and spin projections \(\pm 1/2\), and \(\Delta^{(3/2)}_{\pm 1/2}, \Delta^{(3/2)}_{\pm 3/2}\) correspond to spin \(3/2\) and spin projections \(\pm 1/2\) and \(\pm 3/2\), correspondingly. The projection operators \(\Delta^{(1/2)}_{\pm 1/2}, \Delta^{(3/2)}_{\pm 1/2}, \Delta^{(3/2)}_{\pm 3/2}\) are also the density matrices for pure spin states. To obtain projection matrices for negative energy, one has to make the replacement \(p \rightarrow -p\) in the mass operator \(\Lambda\), Eqs. (14), (23). We may get impure states of fields by summation of Eqs. (42), (43) over spin projections and spins.
5 First Order Relativistic wave Equation

To formulate the second order (in derivatives) Eq. (14) in the form of the first order relativistic wave equation, we introduce the 20-dimensional function

$$\Psi(x) = \{\psi_A(x)\} = \left(\begin{array}{c}
\psi_0(x) \\
\psi_\mu(x)
\end{array}\right)$$

(44)

where $A = 0, \mu$; $\psi_0(x)$ is bispinor. With the aid of the elements of the entire algebra Eq. (26), Eq. (13) may be written in the form of one equation

$$\partial_\nu \left(\epsilon^{0,\nu} + b \epsilon^{\nu,0} + \epsilon^{\alpha,\alpha} \gamma_\nu\right)_{AB} \Psi_B(x) + m \left[\epsilon^{\mu,\mu} + \epsilon^{0,0}\right]_{AB} \Psi_B(x) = 0,$$

(45)

where $\gamma$-matrices act on spinor indexes. Introducing 20-dimensional matrices

$$\beta_\nu = \left(\epsilon^{0,\nu} + b \epsilon^{\nu,0}\right) \otimes I_4 + \epsilon^{\alpha,\alpha} \otimes \gamma_\nu,$$

$$1 \equiv I_{20} = I_5 \otimes I_4, \quad I_5 = \epsilon^{\mu,\mu} + \epsilon^{0,0},$$

(46)

where $I_{20}$ is the unit matrix in 20-dimensional space, Eq. (45) becomes the relativistic wave equation of the first order:

$$(\beta_\mu \partial_\mu + m) \Psi(x) = 0.$$  

(47)

It should be noted that the unit matrix $I_4 \equiv \epsilon^{\alpha,\alpha}$ (we imply a summation over repeating indexes) in Eq. (46) acts in the 4-dimensional vector subspace. The generators of the Lorentz group in 20-dimensional space

$$J_{\mu\nu} = J_{\mu\nu}^{(1)} \otimes I_4 + I_5 \otimes J_{\mu\nu}^{(1/2)},$$

(48)

obey the commutation relations (27) and the commutation relations

$$[\beta_\lambda, J_{\mu\nu}] = \delta_{\lambda\mu} \beta_\nu - \delta_{\lambda\nu} \beta_\mu.$$  

(49)

Eq. (49) guarantees the form-invariance of Eq. (47) under the Lorentz transformations.

The Hermitianizing matrix $\eta$ for a 20-dimensional field $\Psi_\mu$ is given by

$$\eta = \left(\epsilon^{m,m} - \epsilon^{4,4} - b \epsilon^{0,0}\right) \otimes \gamma_4,$$

(50)

and obeys the necessary equations [21], [22]:

$$\eta \beta_i = -\beta^i \eta, \quad \eta \beta_4 = \beta^4 \eta \quad (i = 1, 2, 3).$$  

(51)
Thus, that a relativistically invariant bilinear form is
\[ \overline{\Psi} \Psi = \Psi^+ \eta \Psi, \]
(52)
where \( \Psi^+ \) is the Hermitian-conjugate wave function. Eq. (52) allows us to consider the Lagrangian in the standard form (up to a total derivative):
\[ \mathcal{L} = -\overline{\Psi}(x) (\beta^\mu \partial_\mu + m) \Psi(x). \]
(53)
The variation of the action corresponding to the Lagrangian (53) produces the Euler-Lagrange equation (47).

In accordance with the general theory, Eq. (47) defines the masses of particles: \( m/\theta_i \), where \( \theta_i \) being the eigenvalues of the matrix \( \beta_4 \). The matrix \( \beta_4 = (\varepsilon^{04} + b \varepsilon^{40}) \otimes I_4 + \varepsilon^{\alpha \alpha} \otimes \gamma_4 \) obeys the minimal matrix equation
\[ (\beta_4^2 - 1) \left[ \beta_4^2 - (2b + 1) \beta_4^2 + b^2 \right] = 0. \]
(54)
We obtain from Eq. (54) six eigenvalues of the matrix \( \beta_4 \): \( \pm 1, \pm \theta_1, \pm \theta_2 \), where
\[ \theta_1 = -\frac{1 - \sqrt{4b + 1}}{2}, \quad \theta_2 = -\frac{1 + \sqrt{4b + 1}}{2}. \]
(55)
It is easy to prove that field masses: \( m_1 = m/\theta_1, m_2 = m/\theta_2 \) coincide with expressions in Eq. (21).

It should be noted that Eqs. (47)-(55) are valid for arbitrary parameter \( b \). It is not difficult also to obtain the spin projection operators in 20-dimensional representation space using the technique of Sec. 4.

6 Electromagnetic Interactions of Fields

As follows from Eq. (54), the inverse matrix \( \beta_4^{-1} \) exists, and is given by
\[ \beta_4^{-1} = \frac{1}{b^2} \left[ \beta_4^5 - (2b + 2) \beta_4^3 + (b + 1)^2 \beta_4 \right]. \]
(56)
Therefore, all components of the wave function \( \Psi(x) \) contain time derivatives and no subsidiary conditions in the first order wave equation (47). The minimal interaction with electromagnetic fields can be obtained by the
substitution $\partial_\mu \rightarrow D_\mu = \partial_\mu - ieA_\mu$ ($A_\mu$ is the 4-vector-potential of the electromagnetic field) in Eqs. (13), (47), (53). We also introduce non-minimal interaction in Eq. (47) as follows:

$$\left[ \beta_\mu D_\mu + \frac{i}{2} (\sigma_0 P_0 + \sigma_1 P_1) \beta_\mu F_{\mu\nu} + m \right] \Psi(x) = 0,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength of the electromagnetic field,

$$\beta_{\mu\nu} = \beta_\mu \beta_\nu - \beta_\nu \beta_\mu$$

$$= I_4 \otimes (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) + b \left( \varepsilon^{\nu,0} \otimes \gamma_\mu - \varepsilon^{\mu,0} \otimes \gamma_\nu \right)$$

$$+ \varepsilon^{0,\mu} \otimes \gamma_\nu - \varepsilon^{0,\nu} \otimes \gamma_\mu + b \left( \varepsilon^{\mu,\nu} - \varepsilon^{\nu,\mu} \right) \otimes I_4.$$

and projection operators $P_0$, $P_1$, are

$$P_0 = \varepsilon^{0,0} \otimes I_4, \quad P_1 = \varepsilon^{\mu,\mu} \otimes I_4.$$

The projection operators $P_0$, $P_1$ obey the relations: $P_0^2 = P_0$, $P_1^2 = P_1$, $P_0 + P_1 = 1$. Eq. (57) is form-invariant under the Lorentz transformations and contains additional parameters $\sigma_0$, $\sigma_1$ characterizing anomalous electromagnetic interactions of fermions.

Using Eqs. (26), (44), (46), from Eq. (57), one may obtain equations:

$$(\gamma_\nu D_\nu + i\sigma_1 \gamma_\mu \gamma_\nu F_{\mu\nu} + m) \psi_\mu(x) + b (D_\mu + i\sigma_1 \gamma_\nu F_{\nu\mu}) \psi_0(x)$$

$$+ i\sigma_1 b F_{\mu\nu} \psi_\nu(x) = 0,$$

$$(D_\mu + i\sigma_0 \gamma_\mu F_{\mu\nu}) \psi_\mu(x) + (m + i\sigma_0 \gamma_\nu F_{\nu\mu}) \psi_0(x) = 0.$$  \hfill (61)

Expressing the spinor $\psi_0(x)$ from Eq. (61) and replacing it into Eq. (60), one may obtain an equation for vector-spinor $\psi_\mu(x)$. The system of equations (60), (61) describes the non-minimal electromagnetic interaction of particles with multi-spin 3/2, 1/2. Possibly, these equations can be used for a description of interacting composite fermions.

Obviously, one may find the quantum-mechanical Hamiltonian from Eq. (57). Indeed, we obtain

$$i\beta_4 \partial_t \Psi(x) = \left[ \beta_\mu D_\mu + m + eA_\mu \beta_4 +$$

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\[ \frac{i}{2} (\sigma_0 P_0 + \sigma_1 P_1) \beta_{\mu \nu} F_{\mu \nu} \] \[ \Psi(x). \] (62)

Then, the Hamiltonian form of the equation is

\[ i \partial_t \Psi(x) = \mathcal{H} \Psi(x), \] (63)

\[ \mathcal{H} = \beta_4^{-1} \left[ \beta_a D_a + m + eA_0 \beta_4 + \frac{i}{2} (\sigma_0 P_0 + \sigma_1 P_1) \beta_{\mu \nu} F_{\mu \nu} \right], \]

where \( \beta_4 \) is given by Eq. (56).

Now we consider the consistency problem. The noncausal propagation of particles in the Rarita-Schwinger theory is connected with the subsidiary conditions (constraints). Indeed, due to these conditions there are non-dynamical auxiliary components of the wave function which should be eliminated. But for definite field strength of the electromagnetic fields it is impossible. As a result, there are noncausal propagating modes in the Rarita-Schwinger approach. We investigate this question for Eq. (57). To clear up the number of dynamical variables of the wave function (44), one has to consider the matrix \( \beta_4 \) in Eq. (57). As there exists the inverse matrix \( \beta_4^{-1} \), all components of the wave function (44) are canonical and there are no subsidiary conditions in the first order wave equation (57). According to the method [12], one should replace the derivatives in Eq. (57) by the four-vector \( n_\mu \) to investigate the characteristic surfaces. We may find the normals to the characteristic surfaces, \( n_\mu \), by considering the equation

\[ \det (\beta_4 n_\mu) = 0. \] (64)

If there are noncausal propagating components, there should be non-trivial solution to Eq. (64) for the time-like vector \( n_\mu \). In the frame of reference where \( n_\mu = (0, 0, 0, n_4) \), Eq. (64) becomes \( \det (\beta_4 n_4) = 0 \). As there are no zero eigenvalues of the matrix \( \beta_4 \) due to Eq. (54), equation \( \det (\beta_4 n_4) = 0 \) has only the trivial solution \( n_4 = 0 \), and no superluminal speed of waves in the approach considered. This is a consequence that Eq. (57) describes particles with multi-spin 1/2, 3/2. One can see the possible applications of multi-spin approach to the construction of gauge theories based on spin degrees of freedom and hadron phenomenology in [19].
7 Conclusion

We have shown that the non-local equation (9) suggested describes the field with spin 3/2 and 1/2 (one copy) and the unique mass $m$, and is related in a direct manner with the Proca equation. This equation involves only one supplementary condition $\partial_\mu \psi_\mu = 0$, which removes one state with spin-1/2 but remains another state with spin-1/2. We show that suggested local Eq. (13) describes particles with spin 3/2 and 1/2 (two copies) and three different masses: $m$, $m_1$, $m_2$, so that $m_1 + m_2 = m/b$. The massless particles obey nonlocal Eqs. (9), (10) at $m = 0$, $a = \sqrt{\partial_\mu}$. The projection operators extracting solutions of the field equation with positive and negative energy are constructed. We obtain spin projection operators which allow us to separate states with spins 1/2 and 3/2 and spin projections. The first order relativistic wave equation in the 20-dimensional matrix form is formulated and the relativistically invariant bilinear form and the corresponding Lagrangian are given. We have introduced two parameters characterizing non-minimal electromagnetic interactions of fermions and have found the quantum-mechanical Hamiltonian. It was proved that there is only causal propagation of particles in the approach considered and the theory is consistent.

The question about the gyromagnetic ratio $g$ was unanswered here. This is important for unitarity requirement and phenomenological applications. It is known [25] that low energy unitarity requires the gyromagnetic ratio to be $g = 2$ for arbitrary spin. We leave this question for further investigations.

Some possible applications of the equations studied: the effective theory of hadron interactions and the description of massive gravitinos in the theory of supergravity. This, however, requires further investigations of the equations suggested.

References


