Upper limits on gravitational-wave signals based on loudest events

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Abstract. Searches for gravitational-wave bursts have often focused on the loudest event(s) in searching for detections and in determining upper limits on astrophysical populations. Typical upper limits have been reported on event rates and event amplitudes which can then be translated into constraints on astrophysical populations. We describe the mathematical construction of such upper limits.

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1. Introduction

Data analysis pipelines designed to detect gravitational-wave bursts yield event candidates. While the current generation of gravitational-wave detectors and the corresponding data analysis pipelines are being commissioned, it is generally agreed that even a very significant event would require substantial scrutiny before being accepted as a detection of gravitational waves. Nevertheless, data from existing detectors can be searched for gravitational-wave bursts; the results of the searches can be used to set upper limits on the population of burst sources [1, 2, 3, 4, 5]. Furthermore, the choice of reporting an upper limit does not preclude the detection of gravitational waves, if any are present with sizable amplitude and/or rate.

For upper limits on the rate of burst events in the Universe, a traditional approach uses the number of event candidates that arise from the pipeline as the observable. Using this number, and information about the expected number of noise events arising from the pipeline, an upper limit on the rate of gravitational wave bursts can be derived. The dependence of the final limit on the discrete number of observed event candidates can be a disadvantage in the cases where the signals are weak and rare; small changes in the analysis pipeline cuts can cause discrete changes in the reported rate limit. Several published upper limits [1, 2, 3, 4] have used instead the amplitude of the largest, or loudest, event candidate to derive the reported upper limit. The loudest event method uses a continuous parameter, related to the significance of the event candidate, to determine the upper limit; the upper limit then depends continuously on the observable. A further advantage of the loudest event method is that the final
threshold level need not be determined in advance of the search. In searches for weak and rare events, this means that the loudest event method often provides the best rate limit.

In this paper, we describe the methodology used to obtain upper limits by focusing on the most significant event candidate. We derive formulae for event rate upper limits, both Bayesian and frequentist, based on the loudest event candidate including the case when a background rate can be reliably measured. The frequentist version of the loudest-event rate upper limit has been obtained previously by Cousins [6] and related methods have been explored in Ref. [7]. When the probability of a background event with amplitude greater than the loudest event is assumed to be zero, we show that the rate limit obtained is conservative in the sense that the bound is not violated in the presence of a background. We also show that setting an event rate limit based on the loudest event candidate will often out-perform rate limits derived by counting events above a pre-set amplitude threshold if the threshold choice is not optimal (in the case that all candidate events are due to detector noise). We also describe how the loudest event can be used to set an upper limit on the amplitude of the gravitational-wave strain observed from a particular class of waveforms.

2. Upper limits on event rate

Typical searches for gravitational-wave bursts involve intricate pipelines that process detector data and produce a list of candidate events. One of the parameters associated with each candidate event is an estimate of the significance of the event candidate, e.g., its amplitude or confidence. In a traditional event-counting approach to rate estimation, a threshold on this significance parameter would be used to distinguish between the events that are presumed to be of astrophysical origin and those that are presumed to be due to detector noise.

When setting an interpreted upper limit on the rate, the detection efficiency of the search is also required. The detection efficiency \( \epsilon(x) \) is a function of the amplitude \( x \) and represents the fraction of events from a (hypothetical) astrophysical population that would produce event candidates with amplitude greater than \( x \) after processing through the analysis pipeline. In some cases, a population of signals with fixed intrinsic strain-amplitude \( h \) can be constructed and the efficiency of detection for such a signal \( \epsilon(x, h) \) can be determined for a range of intrinsic signal strengths. This allows one to create a rate versus strain plot often adopted as the interpreted result for situations where a spatial distribution of sources is not known [5].

The detection efficiency is readily evaluated by Monte Carlo simulation in which simulated signals are added to (usually real) detector noise and analyzed through the pipeline. In an event-counting analysis, this efficiency is evaluated at a threshold level \( x^* \) and allows one to convert an uninterpreted bound on event rate (which is an estimate of the rate of events that exceed the threshold without regard to the origin of the events) into an interpreted bound on the event rate, which requires knowledge of the fraction of a hypothetical population that the detection pipeline could have detected.

The detection efficiency has a slightly different role in the loudest event method. The detection efficiency \( \epsilon_{\max} = \epsilon(x_{\max}) \), evaluated at the largest observed amplitude

\[ x_{\max} \]

Recall that the amplitudes are a description of the output of the analysis pipeline, which is subject to detector noise as well as the astrophysical signal, though it might be related to some intrinsic amplitude of the gravitational waves associated with the event.
$x_{\text{max}}$ is the probability that an astrophysical event would produce an amplitude greater than $x_{\text{max}}$ in the data analysis pipeline. If $\epsilon_{\text{max}}$ is small, then a large rate of events would be needed to produce one of the rare event candidates with significance greater than $x_{\text{max}}$. If $\epsilon_{\text{max}}$ is close to unity, it is very likely that an event would produce a more significant event candidate unless the rate of events is sufficiently small. This will be made mathematically precise below. It should be noted that the efficiency is essential to the loudest event statistic—there is no way to achieve an uninterpreted bound since without a hypothetical population there is no way to know how rare the loudest event is.

In the absence of an event with sufficient significance to be claimed a detection, the loudest event method alleviates the competing demands of setting a threshold that is sufficiently above the level of detector noise while being not so large that the detection efficiency becomes too low (and thus the upper limit suffers). Thus the loudest event method is appropriate when one expects true events to be rare. However, use of the loudest event is not incompatible with the goal of making a detection. Indeed, one can simultaneously claim a detection and use the loudest event to place an upper limit on the event rate.

2.1. Rate upper limits without a background estimate

If the population of astrophysical sources produces Poisson distributed events with an intrinsic event rate $R$, then the probability that all the event candidates have an amplitude less than some value $x$ is determined by this intrinsic rate, the efficiency $\epsilon(x)$ evaluated at this value of the amplitude, and the observation time $T$ via

$$P(x|\mu) = \sum_{n=0}^{\infty} \frac{[1 - \epsilon(x)]^n \mu^n e^{-\mu}}{n!} = e^{-\mu \epsilon(x)}$$

where $\mu = RT$ is the Poisson mean. This formula can be used to determine a frequentist upper limit on the rate $R$ by determining the largest amplitude $x_{\text{max}}$ recorded by the data analysis pipeline, and then solving $1 - p = P(x_{\text{max}}|\mu_p)$ for $\mu_p = R_p T$ where $p$ is the desired confidence level. The result is

$$R_p = \frac{-\ln(1 - p)}{T \epsilon_{\text{max}}}.$$  

(2)

For example, a 90%-confidence frequentist upper limit ($p = 0.9$) is

$$R_{90\%} = \frac{2.303}{T \epsilon_{\text{max}}}.$$  

(3)

Thus, an experiment will yield a value of $R_{90\%}$ that is less than the true rate only 10% of the time.

A Bayesian upper limit is determined from the posterior probability distribution $P(\mu < \mu_p|x_{\text{max}})$ which can be computed by a straightforward application of Bayes law. The probability $p(x|\mu)$ $dx$ of the loudest event being produced with an amplitude between $x$ and $x + dx$ is first obtained by taking the derivative of $P(x|\mu)$ with respect to $x$:

$$p(x|\mu) = -\mu \epsilon'(x) e^{-\mu \epsilon(x)}$$

(4)

where $\epsilon'(x) = dx(x)/dx$. The posterior probability distribution is then given by

$$P(\mu < \mu_p|x_{\text{max}}) = \mathcal{N}^{-1} \int_0^{\mu_p} d\mu p(\mu) p(x_{\text{max}}|\mu)$$

(5)
where $N = \int_0^\infty d\mu p(\mu)p(x_{\text{max}}|\mu)$ is a normalization constant and $p(\mu)$ is a prior distribution on the event rate. An upper limit on the rate at the 100\% confidence level is obtained by solving $P(\mu < \mu_p|x_{\text{max}}) = p$ for $\mu_p$. For a uniform (improper) prior $p(\mu) = \text{constant}$, this requires solving

\begin{equation}
\label{eq:limit_uniform}
p = 1 - e^{-\mu_p x_{\text{max}}}(1 + \mu_p x_{\text{max}})
\end{equation}

for $\mu_p$. A 90\% Bayesian confidence level upper limit is then

\begin{equation}
\label{eq:90_limit}
R_{90\%} = 3.890 T x_{\text{max}}.
\end{equation}

That is, there is 90\% probability that the true rate is less than $R_{90\%}$ given the observed value of $x_{\text{max}}$ and the prior assumption of a uniform rate distribution.

### 2.2. Accounting for a background

If the background distribution (i.e., the distribution of event candidates that result from detector noise alone) is known, or can be estimated, then this information can be incorporated into the rate upper limits derived above. In gravitational wave searches, this information is often obtained by coincidence analyses performed on time-shifted data. Let $P_0(x)$ be the probability that all background events have an amplitude less than $x$ (as a function of $x$) for the given search and observation time. Since the loudest event could have been either from the astrophysical foreground (which is a Poisson process with mean $\mu$) or from the noise-produced background, the probability that all events lie below amplitude $x$ is just the product $P(x|\mu, B) = P_0(x)e^{-\mu(x)}$ where $\mu = RT$ and $B$ symbolically represents the inclusion of information about the background. Evaluated at the loudest event, this is

\begin{equation}
\label{eq:background_prob}
P(x_{\text{max}}|\mu, B) = P_0(x_{\text{max}})e^{-\mu x_{\text{max}}}
\end{equation}

and the rate upper limit at a frequentist confidence level $p$ is

\begin{equation}
\label{eq:background_limit}
R_p = -\frac{\ln(1 - p) - \ln P_0(x_{\text{max}})}{T x_{\text{max}}}.
\end{equation}

Note that this rate limit will always be less than the rate limit in Eq. \[\text{(3)}\] (or equal in the case that $P_0(x_{\text{max}}) = 1$). This demonstrates that the no-background assumption is conservative in the sense that the rate limit in Eq. \[\text{(2)}\] will not be violated by failing to account for the background. Note, however, that the rate in Eq. \[\text{(4)}\] can become zero, or even negative, when $P_0(x_{\text{max}}) \leq 1 - p$. This pathology is intrinsic to the meaning of a frequentist upper limit: a similar pathology occurs in counting experiments if an unusually small number of events is observed in a high background experiment. A modified frequentist statistic based on the loudest event might remove this pathology.

The Bayesian upper limit, accounting for the background, is constructed as before. The probability $p(x|\mu, B) \, dx$ that the loudest event is produced with amplitude between $x$ and $x + dx$ is obtained by taking the derivative of $P(x|\mu, B)$ with respect to $x$:

\begin{equation}
\label{eq:background_pdf}
p(x|\mu, B) = [p_0(x) - \mu' P_0(x)]e^{-\mu(x)}
\end{equation}

where $p_0(x) = dP_0(x)/dx$. For a uniform prior, the 100\% confidence upper limit is determined by solving

\begin{equation}
\label{eq:background_uniform}
p = 1 - e^{\mu_p x_{\text{max}}}(1 + \xi \mu_p x_{\text{max}})
\end{equation}

for $\mu_p$. That is, there is 90\% probability that the true rate is less than $R_{90\%}$ given the observed value of $x_{\text{max}}$ and the prior assumption of a uniform rate distribution.
for $\mu_p$ where

$$\xi = \left[1 - \frac{\epsilon(x_{\max})}{\epsilon'(x_{\max})} \frac{p_0(x_{\max})}{P_0(x_{\max})}\right]^{-1} = \left[1 - \frac{d \ln P_0}{d \ln \epsilon} \right]^{-1}$$

(12)

is the correction accounting for the background. [The second expression for $\xi$ can be obtained by expressing the background probability $P_0$ as a function of $\epsilon$ rather than $x$, which requires inverting the function $\epsilon(x)$.] Notice that $0 \leq \xi \leq 1$. When $\xi = 1$, i.e. $p_0(x_{\max}) = 0$ and $P_0(x_{\max}) = 1$, the Bayesian upper limit in the case of no background is recovered. Thus the assumption underlying the no-background limit is that there is no chance that a background event could exceed $x_{\max}$. Otherwise, the Bayesian upper limit will always be smaller than the limit when no background was assumed; i.e., the no-background limit is again a conservative limit. In the extreme case of the loudest event almost certainly being due to the background ($\xi = 0$), the rate upper limit becomes

$$R_{90\%} = \frac{2.303}{T \epsilon_{\max}}.$$

(13)

In the special case of a Poisson-distributed background with mean $\mu_0(x)$ number of events with amplitudes greater than $x$, the correction $\xi$ is

$$\xi = \left[1 - \frac{\epsilon(x_{\max})}{\epsilon'(x_{\max})} \mu'_0(x_{\max})\right]^{-1}$$

(14)

where $\mu'_0(x) = d\mu_0(x)/dx$.

2.3. Rate-versus-strain plots

In some circumstances it is not possible to construct a definitive model population, yet an interpreted bound on rate is possible by setting a rate as a function of some model parameter. For example, in Ref. [5], the detection efficiency is computed for a particular waveform as a function of the intrinsic waveform strain $h$, and this is used to construct a rate-versus-strain plot. The loudest event rate limits described so far can be immediately generalized to this case by allowing the efficiency to be a function of the parameter $h$, $\epsilon = \epsilon(x, h)$. Note that $x$ represents the signal strength (or significance) as measured by the pipeline while $h$ is a measure of the intrinsic signal strength as defined by the population. (In fact, $h$ could represent any set of parameters characterizing the population.) The frequentist rate-versus-strain plot (with background) is given simply by Eq. (9) where $\epsilon_{\max} = \epsilon_{\max}(h)$ is treated as a function of the population parameter $h$.

2.4. Advantage of the rate limit based on the loudest event

The choice of the threshold required for an event counting approach to constructing an upper limit necessitates a balance between the competing goals of setting the threshold high enough that one is not dominated by a large number of false alarms (which would result in an artificially high upper limit) and setting the threshold low enough that the detection efficiency is not compromised (which would again result in an artificially high upper limit). In the case that there is no true signal present, the loudest event statistic represents, in some sense, tuning the threshold to the nearly optimal level: right at the loudest event. To illustrate this, consider the case in which the noise consists of a large number $N$ background events where the probability that any single
event has an amplitude less than $x$ is $e^{-x^2/2}$ so that $P_0(x) \sim e^{-x^2/2}$. (This would be the expected behavior for the problem of detecting binary inspirals in Gaussian noise.) Assume that the detection efficiency is $\epsilon \sim x^{-3}$ for large amplitude (as would be expected for gravitational wave sources distributed homogeneously in space). A Monte Carlo realization of 10000 instances of noise with each instance consisting of 10000 noise events, shows the upper limit that would be obtained for various choices of threshold via an event counting method, see Fig. 1. Also shown on this plot is the upper limit that would be obtained using a conservative loudest event method (i.e., with no background accounting) for both the Bayesian and frequentist statistics. (These are shown as horizontal lines in the plot although these statistics are not functions of a threshold.) This plot shows that the upper limit obtained by a counting method with a threshold depends rather sensitively on the threshold level, and that the loudest event methods perform nearly as well as possible unless a nearly perfect threshold level can be achieved.

3. Upper limits on event strength

Another type of gravitational wave search does not seek to bound a rate on the number of events occurring during the observational period but rather to set a limit on the strength of any single event that may be present during that observation time. A significantly strong candidate may be described as a detection independently of such an upper limit. Such upper limits have been sought in past gravitational-wave searches [1, 8]. The crucial assumption in such an approach is that a single event is present, as would be the case for a triggered search (e.g., a search for a gravitational wave burst associated with an observed supernova) or for a search for a known pulsar. The central quantity is then the probability that all triggers have amplitude less than the observed maximum $x_{\text{max}}$ given the presence of a signal with intrinsic strain $h$, i.e., $P(x < x_{\text{max}}|h)$ [this is what was previously denoted $1 - \epsilon_{\text{max}}(h)$]. This probability distribution can be estimated using injections into off-source times (or frequencies), then the $100p\%$ frequentist upper limit on the signal strain is determined by solving

$$\left(1 - p\right) = P(x < x_{\text{max}}|h_p)$$

for $h_p$. Notice that there are some problems with this limit. Suppose, for example, that $x_{\text{max}}$ is unusually small in the sense that the $P(x < x_{\text{max}}|h = 0)$ is small. Then the value of $h_p$ might be abnormally low. [This happens when the false alarm probability $1 - P(x < x_{\text{max}}|h = 0)$ is close to the required confidence, $p$, for the upper limit.] Once again, this is an artifact of the frequentist approach to the upper limit and could be avoided by a Bayesian analysis.

4. Summary

We have described several upper-limit statistics based on the loudest observed event. These statistics can be used to bound the rate of events coming from a known source population, to construct a rate-versus-strain exclusion curve, or to bound the amplitude of an event if at most one event is known to be present. The loudest event methods have advantages over traditional methods that involve counting events that exceed a threshold in that the loudest event methods: (i) alleviate the (often difficult) task of threshold choice, (ii) use a continuous parameter (the amplitude of the loudest event) rather than a discrete parameter (the number of events above threshold), and
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Figure 1. The median value and interquartile ranges of the frequentist 90% confidence upper limit on the rate of a Poisson process as a function of amplitude threshold $x^*$ based on counting the number of events and ignoring the noise background. The results are based on a Monte-Carlo simulation involving 10000 instances of 10000 noise events drawn from a probability distribution of the form $P_0(x) \sim e^{-x^2/2}$ assuming an efficiency which varies as $\epsilon \sim x^{-3}$. Also shown on the figure are the median values and interquartile ranges for the frequentist and Bayesian 90% confidence upper limits on the rate using the loudest event. These results are shown as lines for ease of comparison since there is no threshold dependence. Notice the narrowness in the choice of threshold $x^*$ that would give rise to an upper limit that is competitive with the limit provided by a loudest event statistic.

(iii) will usually out-perform the event counting method (meaning that the upper limit will be more constraining at a given confidence level).

A number of alternatives to the loudest event method for use in gravitational wave data analysis remain to be explored. For example, a “next-loudest event” statistic may be more robust against a potentially damaging single noise glitch. An interesting approach has been developed by Yellin [7], which establishes a limit based on the largest gap in the efficiency between events.

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References

[9] This result was first obtained by L. S. Finn (private communication).