Non-commutative dynamics and roton-like spectra in bosonic and fermionic condensates

Paolo Castorina,1 Giuseppe Riccobene,2 and Dario Zappalà3
1Dept. of Physics, University of Catania and INFN, Sezione di Catania, Via S. Sofia 64, I-95123, Catania, Italy
2 Scuola Superiore di Catania, via S.Paolo 73, I-95123, Catania, Italy
3INFN, Sezione di Catania, and Dept. of Physics, University of Catania, via S. Sofia 64, I-95123, Catania, Italy

(Dated: May 12, 2004)

The relation between symmetry breaking in non-commutative cut-off field theories and transitions to inhomogeneous phases in condensed matter is discussed. The non-commutative dynamics can be regarded as an effective description of the mechanisms which lead to inhomogeneous phase transitions and their relation to the roton-like excitation spectrum. The typical infrared-ultraviolet mixing in non-commutative theories contains the peculiar ingredients to describe the interplay between short and long distance particle interactions which is responsible for the non-uniform background and the roton spectrum both in bosonic and fermionic condensates.

PACS numbers: 11.10.Nx 11.30.Qc

The relation between symmetry breaking in quantum field theory and phase transitions in condensed matter is well known. Spontaneous symmetry breaking in $\lambda\phi^4$ with a constant vacuum expectation of the field corresponds to a Bose -Einstein condensation (BEC) and there is a strong analogy between the cut-off field theory and the hard sphere Bose gas description of superfluidity. Chiral symmetry breaking is the field theoretical version of superconductivity and the ground state is described by a condensation of fermion-antifermion pairs corresponding to the Cooper pairs in the Bardeen-Cooper-Schrieffer (BCS) state. The previous correspondences are usually restricted to the cases of constant order parameters, that is of transitions to uniform phases, because this guarantees the translational and rotational invariance of the field theory.

On the other hand, in condensed matter, due to particle interactions or to external experimental setups (for example a trapped condensate), one considers transitions from homogeneous to inhomogeneous phases with non-constant order parameters. For a bosonic system, according to the ground state which is an oscillating function, is associated to a roton-like behavior of the excitation spectrum. For the fermionic systems it is possible to build superconducting states, like the Larkin-Ovchinnikov-Fulde-Ferrell (LOFF) state, with energy lower than the BCS state, where the Cooper pairs have a non-zero total momentum and then the fermionic condensate is not uniform.

In the application of non-commutative dynamics to the lowest Landau level, to quantum Hall effect and to fluid dynamics has been discussed. In it has been proposed that one can continue to follow the previous analogies with condensed matter also for transition to inhomogeneous phases if the quantum field theory is generalized to non-commutative coordinates. In particular, the non-commutative generalization of $\lambda\phi^4$ shows that the spontaneous symmetry breaking occurs for a non-uniform stripe phase and in the non-commutative Gross-Neveu (GN) model there is an inhomogeneous chiral symmetry breaking corresponding to spin density waves.

In this letter we consider the symmetry breaking in cut-off non-commutative field theories as an effective approach to understand the general dynamical mechanisms which lead to inhomogeneous phase transitions and the correlated roton-like excitation spectrum. Indeed the non-commutative dynamics indicates that also for condensed matter systems:

a) there is a strong link between a roton-like excitation spectrum and a transition to an inhomogeneous background, which confirms;

b) for bosonic systems the non-uniform behavior and the correlated roton spectrum is due to non-local particle interactions (also induced by external setups), i.e., to the interplay between short and long distance effects;

c) for fermionic systems there are: a transition to inhomogeneous states with a pairing with total momentum $P \neq 0$; a roton dip in the excitation spectrum; effects due to momenta non-equal to the Fermi momentum (this is mainly due to the infrared/ultraviolet (IR/UV) connection (for a review see [10]));

d) the observed phase transitions turn out to be first order.

We shall first discuss the relation between the non-commutative scalar field theory and the roton excitation in BEC and then we consider fermionic systems.

On general grounds, in a bosonic condensate the roton spectrum is due to a non-local interatomic potential $V(\vec{r} - \vec{r}')$, with a momentum dependent Fourier transform. Indeed, the roton spectrum has been re-
cently obtained by considering a BEC with a ground state wave function which takes explicitly into account the van der Waals interaction [12] or by a dipole-dipole induced atomic interaction in trapped condensates [13, 14, 15, 16] or by considering a BEC close to the solid phase [17]. Since the BEC with the local (pseudo)potential $\delta(r-r')$ is analogous to the spontaneous symmetry breaking in $\lambda \phi^4$ theory, one can assume that some relevant physical effects due to the non-local repulsive interaction can be described by generalizing the self-interacting field theory in such a way to introduce an effective non-local coupling.

A simple approach is to consider the non-commutative $\lambda \phi^4$ theory with action

$$S(\phi) = \int d^4x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right)$$

(1)

where the star (Moyal) product is defined by $(i, j = 1, \ldots, 4)$

$$\phi^4(x) = \phi(x) \ast \phi(x) \ast \phi(x) \ast \phi(x) = \exp \left[ \frac{i}{2} \sum_{i<j} \theta_{ij} \partial_{x_i} \phi \partial_{x_j} \phi \right] \left( \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \right) \bigg|_{x_i=x}(2)$$

The “deformation” of the self-interaction term by the Moyal product gives a momentum dependent repulsive effect which is responsible, as we shall see below, for the roton spectrum and for the phase transition to an inhomogeneous background.

In [19] the spontaneous symmetry breaking for the theory in Eqs. (1) and (2) has been analyzed with the following results:

1) The transition occurs to a stripe phase where the order parameter is $\phi(\vec{x}) = A \cos(\vec{Q} \cdot \vec{x})$;

2) $A, Q$ and the energy excitation $\omega(p)$ are fixed by minimizing the energy;

3) $\vec{Q}$ is orthogonal to $\vec{\theta}$ and $Q$ is small for large $\theta$;

4) the excitation spectrum can be approximated by

$$\omega(\vec{p}) = p^2 + M^2(\vec{p})$$

(3)

where the function $M(\vec{p})$ will be discussed later.

The previous results, obtained by variational methods in the static limit, represent the solutions of a set of self-consistent minimization equations of the effective potential in the Hartree-Fock approximation for the parameters $A, Q, M$. As discussed in detail in [19], since $Q$ is small, the inhomogeneous background is a smooth function of $x$ and then, the breaking of translational (smooth) and rotational invariance is approximated by a translational invariant propagator with a momentum dependent mass term.

In the particular case $\theta_{ij} = \epsilon_{ijk} \theta^k$ with $\vec{\theta} = (0, 0, \theta)$, $\vec{Q} = (Q/\sqrt{2}, Q/\sqrt{2}, 0)$ and large values of $\theta \Lambda^2$ ($\Lambda$ is the UV cut-off), it turns out that

$$\frac{Q^2}{\Lambda^2} = \left( \frac{\lambda}{24\pi^2} \right)^{1/2} \frac{1}{\theta \Lambda^2}$$

(4)

and that $M(\vec{p})$, for small $p$, is given by

$$M^2(\vec{p}) \bigg|_{p \to 0} \sim \alpha + \frac{\lambda}{6\pi^2} \frac{1}{|\vec{p} \times \vec{\theta}|^2}$$

(5)

where $\alpha$ is a constant and $\times$ indicates the usual vector product.

The peculiar behavior for small $p$ of the last term in the previous equation is due to the IR/UV connection of the non-commutative field theory and gives a divergent mass term in the IR region and a minimum of the irreducible two-point function. However the effective theory has a natural self-generated IR cut-off $Q$ where it is more correct to cut the small momenta also because one is neglecting the phonon branch. Then the excitation spectrum, which is related to the phase transition, should be correctly identified by Eqs. (4) and (5) with $p \geq Q$ and $Q$ has a roton-like dip at a typical scale of order $Q$.

It should be clear that the Moyal-deformed term in Eq. (1) can mimic effective interactions which are non-local and globally repulsive. Then, within this simple model, the previous results of the non-commutative theory can describe some interesting physical effects of the Bose-Einstein condensates when the repulsive interaction is dominant and one focuses only on the rotonic part of the spectrum because the spontaneous symmetry breaking with a single scalar field gives a gap.

For example, for a system trapped along the $z$ direction, with a typical dimension $L$, where atoms interact by a dipole-dipole interaction, it has been shown by standard condensed matter methods [18], that the excitation spectrum has different features according to the value of the momentum $q$ in the $x-y$ plane: if $qL \ll 1$ there is a phononic behavior; if $qL >> 1$, due to the dependence of the coupling strength on the momentum, there is a roton excitation. All the dipoles are oriented along the $z$ axis and for a tight confinement one has (almost) a dipole system in the $x-y$ plane with a repulsive dipole-dipole interaction.

We shall see that an analogous result on the roton spectrum can be obtained by the non-commutative $\lambda \phi^4$ theory. Moreover, for a dipole-dipole interaction, there is another reason why the system can be associated with a non-commutative dynamics. According to [18] it easy to show that in the $x-y$ plane a system of interacting dipole-like objects, compound by two bound opposite charge particles in a strong magnetic field $B$ along the $z$ axis, follows a non-commutative dynamics

$$[x, y] \simeq -i/B$$

(6)

Although this formal analogy with the usual dipole-dipole interaction should be considered in a restricted sense, mainly because in the non-commutative dynamics the ”dipole” dimension depends on its center mass momentum, the previous discussion suggests that some features of the previous BEC with trapped atoms with
magnetic moment can be reproduced by considering the \( \lambda \phi^4 \) field theory with the non-commutative parameter in the same direction of the dipole orientation

\[
[x, y] = i\theta \tag{7}
\]

Indeed, by Eqs. (4), (5) and (7), in the small \( p \) region, it is easy to verify that the non-commutative \( \lambda \phi^4 \) theory, in the \( x - y \) plane, has a roton-like spectrum, similar to Figure 1 of [13] with a minimum at momentum \( p_x^2 + p_y^2 = 2Q^2 \) (with \( p_x^2 + p_y^2 \rangle \geq \lambda^2 \)). The roton spectrum of the non-commutative theory is associated to a spatially modulated background in the \( x - y \) plane according to point 1), quoted above, and one expects that a similar oscillating behavior should be present in the dipolar atomic BEC. This aspect has not been analyzed in [13], but an oscillating condensate was obtained by solving the Gross-Pitaevskii equation with laser induced dipole-dipole interaction in an elongated cigar-shaped BEC [15] and, in [14], it has been shown that, in the same geometrical configuration, the dipolar interaction gives a roton dip in the excitation spectrum.

This dynamical correlation between roton-like spectrum and non-uniform background is not limited to dipole-dipole (induced) interactions. Indeed in [17] it has been suggested that a roton spectrum also occurs in a BEC close to a Mott-insulating phase with a macroscopic population of the states with momenta equal to the reciprocal lattice vector. The above examples confirm, in our opinion, the suggestion given in [3] that the roton dynamic configuration, the dipolar interaction gives a roton dip in the excitation spectrum.

After the indications obtained for bosonic systems, let us now consider the informations coming from non-commutative effective field theories for fermionic system with a transition from a homogeneous phase, where \( \langle \psi \bar{\psi} \rangle \) is constant, to an inhomogeneous one, where \( \langle \psi \bar{\psi} \rangle \) is \( \vec{x} \) dependent. The simplest field theoretical approach to chiral symmetry breaking, corresponding to superconductivity, is the GN model with four fermion interaction described by the lagrangian

\[
L(x) = i\bar{\psi}(D \psi) + g(\bar{\psi}\psi)^2 . \tag{8}
\]

In 4 dimensions, for \( g \) larger than some critical value, \( g_c \), the chiral symmetry is broken and the dynamical generated mass is constant \( m_0 \sim g < \bar{\psi}\psi > 0 \).

In [8] the transition from homogeneous to inhomogeneous phase has been obtained by generalizing the GN model to the non-commutative case with lagrangian

\[
L(x) = i\bar{\psi} \partial \psi + g\bar{\psi}_\alpha \psi_\alpha + \bar{\psi}_\beta \psi_\beta - g\bar{\psi}_\alpha \psi_\beta \psi_\alpha \psi_\beta . \tag{9}
\]

For \( g \) larger than some critical value, one finds again chiral symmetry breaking but, this time, in an inhomogeneous phase where the pair correlation function has a dependence on a total momentum, \( \vec{P} \) of the ("Cooper") pair, with \( P \sim (1/\lambda^2) \).

The order parameter turns out to be an oscillating function of \( \vec{x} \) and one has the breaking of translational, rotational and chiral invariance:

\[
\psi(\vec{x}) = (1 + c P^2 \cos(P x)) \psi \psi > 0 , \tag{10}
\]

where \( \psi \psi > 0 \) is the constant order parameter of the commutative case and \( c \) is a numerical constant.

Since the translational invariant breaking effects are small for large \( \theta \) in the analysis of the non-commutative GN model in [8], the static two-point function has been computed to order \( O(P^4) \) and the background, \( \psi(\vec{x})\psi(\vec{x}) \), turns out to be a slowly changing function. Then one can identify the quasi-particle spectrum as \( \omega^2(\vec{k}) = k^2 + m^2(\vec{k}) \). As in the scalar case, for small momenta, \( k \geq P \), the quasiparticle spectrum has a leading contribution due to the IR/UV connection of the non-commutative theory, and the self-consistent IR and UV behavior of the gap equation has been interpolated by

\[
m(k) = m_0 \left[ 1 + \frac{g}{\pi^2} \frac{1}{|k| \times \hat{\theta}|^2} \right] \tag{11}
\]

where the constant \( m_0 \) has been fixed by the minimization of the energy and reduces to \( m_0 \) in the planar limit \( \theta \lambda^2 \to \infty \), where the non-commutative effects disappear. Also in this case the spectrum has a roton-like dip in the plane orthogonal to \( \theta \) and one recovers the dynamical relation with the non-uniform ground state [8].

The previous field theoretical model is then analogous to a condensed matter system with a non-local four-fermion interaction, with an inhomogeneous phase where the particle-hole (p-h) pairs have a non-zero total momentum. From this point of view the model mimics a non-s-wave superconductor in a strong coupling regime.

A kind of p-h instability has been originally proposed in [14] for specific electronic materials (for a review see [20]) and is known as spin density wave. An analogous phase, known as chiral density wave [21], has also been analyzed in QCD at finite density (for a review see [22, 23]). In [24] it has been suggested the possibility of a color magnetic superconducting phase, above a critical density, by breaking the chiral symmetry in a chiral density wave background.

The results of the non-commutative models, where the anisotropic behavior is due to \( \theta \), indicate that in condensed matter and in QCD systems in analogous conditions, i.e., with a pairing for \( \vec{P} \neq 0 \) and in strong coupling regime, there is a transition to an inhomogeneous phase and the excitation spectrum should have a small roton-like dip.

In the standard treatment of inhomogeneous superconductivity the anisotropic behavior of the gap is usually associated only with a dependence on the angular variables of the quasiparticle momentum, because its module
is fixed at the Fermi momentum. In our case there is a dependence on the momentum and a deformed Fermi surface. 

Then, also for fermionic condensates, the effective field theory approach suggests that a roton dip in the spectrum should be related to a non-uniform background. In fact this result has been for example obtained by analyzing the relation between chiral symmetry breaking and magnetism in \[24\] with isoscalar and isovector chiral density wave background. The lower energy spectrum is given by

\[ E_\beta^2 = \vec{p}^2 + M^2 + \frac{\vec{q}^2}{4} - \sqrt{(\vec{p} \cdot \vec{q})^2 + M^2 \vec{q}^2} \]  

(12)

where \( \vec{q} \) is the momentum of the chiral density waves. For \( \vec{p} \) parallel to \( \vec{q} \), \( E_\beta \) has a minimum for \( p = \sqrt{q^2/4 - M^2} \). The condition \( q^2/4 - M^2 \geq 0 \) is fulfilled for all values of \( q \) and \( M \) which minimize the energy in the range \( \mu_1 \leq \mu \leq \mu_2 \) of the chemical potential where the transition occurs.

Finally, the field theoretical results indicate that the phase transitions to inhomogeneous condensates are first order and one expects similar behavior for the corresponding condensed matter systems. This conclusion could be a by-product of the self-consistent approach \[23\] \[24\] but, at least for the scalar case, it has been confirmed by lattice calculations \[4\]. Points (a)-(d) summarize the results of this letter, pointing out the general relation between a minimum in the self-energy and the inhomogeneous background, due to non-contact interactions, as the common underlying ingredient shared by the various physical systems and set-ups analyzed above.

We acknowledge R. Jackiw, R. Casalbuoni, M. Baldo for useful suggestions and we are also grateful to G. Angilella, F. Cataliotti, M. Consoli for frequent discussions during the preparation of the paper and to G. Nardulli for reading the manuscript.
