Exploring the phase structure of quantum chromodynamics (QCD) is one of the primary goals of ultrarelativistic heavy-ion collisions. It is generally believed that at sufficiently high temperature there should be a transition from ordinary hadronic matter to a chirally symmetric plasma of quark and gluons. The order parameter for this phase transition is the quark-antiquark condensate. At temperature of about 150 MeV, Lattice QCD calculations indicate that this symmetry is restored. The order of the phase transition seems to depend on the mass of the non-strange u and d quarks, \( m_u \approx m_d \), and the mass of the strange quark \( m_s \), and at the temperature on the order of 150 MeV, heavier quark flavors do not play an essential role.

For \( N_f \) massless quark flavors, the QCD Lagrangian possesses a chiral \( U(N_f)_R \times U(N_f)_L \) symmetry. The linear sigma model with \( U(1)_A \) symmetry will be effectively restored as well as \( SU(N_f)_R \times SU(N_f)_L \) symmetry for \( N_f \gg 1 \). Consequently, one of the \( N_f^2 - 1 \) Goldstone bosons becomes massive, leaving \( N_f^2 - 1 \) Goldstone bosons. The \( SU(N_f)_R \times SU(N_f)_L \times U(1)_A \) symmetry in the chiral limit allows for existence of topological string defects, the formation and evolution of such defects and their possible observable effects in ultrarelativistic heavy-ion collisions as well as in the early universe transition have been invoked in Refs. \cite{10,11}. In particular, signals for detecting the effective restoration of the \( U(1)_A \) chiral symmetry in ultrarelativistic heavy-ion collisions have been invoked in Refs. \cite{12,13,14}. Recently, the issue of finding signals for the restoration of chiral symmetry in ultrarelativistic heavy-ion collisions has received considerable attention. For example, the signals for the restoration of the \( SU(2) \) chiral symmetry associated with the \( \sigma \) meson have been proposed in Refs. \cite{10,11}. In particular, signals for detecting the effective restoration of the \( U(1)_A \) chiral symmetry in ultrarelativistic heavy-ion collisions have been invoked in Refs. \cite{12,13,14}.

On the other hand, in QCD, spontaneous symmetry breaking of \( U(1)_A \) symmetry allows for existence of topological string defects, the formation and evolution of such defects and their possible observable effects in ultrarelativistic heavy-ion collisions as well as in the early universe transition have been invoked in Refs. \cite{15,10,13}. In this letter, we study the effects from effective restoration of the \( U(1)_A \) symmetry by using the \( U(N_f)_R \times U(N_f)_L \) linear sigma model with stored in addition to \( SU(N_f)_R \times SU(N_f)_L \). Since the chiral condensate \( \langle \bar{q} q \rangle \neq 0 \) also breaks the \( U(1)_A \) symmetry, there are only two possibilities: either the \( U(1)_A \) symmetry is restored at a temperature much greater than the \( SU(N_f)_R \times SU(N_f)_L \) symmetry or the two symmetries are restored at (approximately) the same temperature. Recent lattice gauge theory computations have demonstrated a rapid dropping of the topological susceptibility around the chiral phase transition, seemingly suggesting that the simultaneous restoration exists, this is also supported by the random matrix model. On the other hand, the fate of the \( U(1)_A \) anomaly in nature is not completely clear since instanton model calculations indicate that the topological susceptibility is essentially unchanged at \( T_c \), also Lattice results obtained from the \( SU(3) \) pure gauge theory show that the topological susceptibility is approximately constant up to the critical temperature \( T_c \), it has a sharp decrease above the transition, but it remains to be different from zero up to \( T_c \). Additionally, other lattice computations which measure the chiral susceptibility find that the \( U(1)_A \) symmetry restoration is at or below the 15% level.

Recently, the issue of finding signals for the restoration of chiral symmetry in ultrarelativistic heavy-ion collisions has received considerable attention. For example, the signals for the restoration of the \( SU(2) \) chiral symmetry associated with the \( \sigma \) meson have been proposed in Refs. \cite{10,11}. In particular, signals for detecting the effective restoration of the \( U(1)_A \) chiral symmetry in ultrarelativistic heavy-ion collisions have been invoked in Refs. \cite{12,13,14}. In this letter, we study the effects from effective restoration of the \( U(1)_A \) symmetry by using the \( U(N_f)_R \times U(N_f)_L \) linear sigma model with
two massless flavors.

The Lagrangian of the $U(N_f)_R \times U(N_f)_L$ linear sigma model is given by\footnote{For simplicity we consider here the configurations which are specified by the $U(1)_A$ phase only. In considering non-abelian phases, there is another class of topological defects known as non-abelian strings\cite{17}, the pion strings, which can also exist during the chiral phase transition\cite{18}.}

$$
\mathcal{L}(\Phi) = \text{Tr}(\partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi - \lambda_1 (\text{Tr}(\Phi^\dagger \Phi))^2 - \lambda_2 \text{Tr}(\Phi^\dagger \Phi)^2 + c[\text{det}(\Phi) + \text{det}(\Phi^\dagger)]) + \text{Tr}[H(\Phi + \Phi^\dagger)].
$$

(1)

Where $\Phi$ is a complex $N_f \times N_f$ matrix parametrizing the scalar and pseudoscalar mesons,

$$
\Phi = T_a \phi_a = T_a (\sigma_a + i \pi_a),
$$

(2)

with $\sigma_a$ being the scalar ($J^P = 0^+$) fields and $\pi_a$ being the pseudoscalar ($J^P = 0^-$) fields. The $N_f \times N_f$ matrix $H$ breaks the symmetry explicitly and is chosen as

$$
H = T_a h_a,
$$

(3)

where $h_a$ are external fields, $a = 0, 1, \ldots, N_f^2 - 1$ and $T_a, a \neq 0$ are a basis of generators for the $SU(N_f)_L$ Lie algebra. $T_0 = 1$ is the generator for the $U(1)_A$ Lie algebra.

In the above model, the determinant terms correspond to the $U(1)_A$ anomaly, as shown by 't Hooft\footnote{For simplicity we consider here the configurations which are specified by the $U(1)_A$ phase only. In considering non-abelian phases, there is another class of topological defects known as non-abelian strings\cite{17}, the pion strings, which can also exist during the chiral phase transition\cite{18}.}, they arise from instantons. These terms are invariant under $SU(N_f)_R \times SU(N_f)_L \cong SU(N_f)_V \times SU(N_f)_A$, but break the $U(1)_A$ symmetry of the Lagrangian explicitly. The last term in Eq. (1) which is due to nonzero quark masses breaks the axial and possibly the $SU(N_f)_V$ vector symmetry explicitly.

When $h_a = 0, c = 0$, for $m^2 < 0$ the global $SU(N_f)_V \times U(N_f)_A$ symmetry is broken to $SU(N_f)_V$, and $\langle \Phi \rangle$ develops a non-vanishing vacuum expectation value, $\langle \Phi \rangle = T_0 \sigma_0$. Spontaneously breaking $U(N_f)_A$ leads to $N_f^2$ Goldstone bosons which form a pseudoscalar, $N_f^2$ dimensional multiplet. However when $h_a = 0$, and $c \neq 0$, the $U(1)_A$ is further broken to $Z(N_f)$ by the axial anomaly, and $SU(N_f)_V \times SU(N_f)_A$ is still the symmetry of the Lagrangian. A nonvanishing $\langle \Phi \rangle$ spontaneously breaks the symmetry to $SU(N_f)_V$, with the appearance of $N_f^2 - 1$ Goldstone bosons which form a pseudoscalar, $N_f^2 - 1$ dimensional multiplet. The $N_f^2$th pseudoscalar meson is no longer massless, because the $U(1)_A$ symmetry is already explicitly broken, e.g for $N_f = 2$, the $\eta$ meson is massless compared to other pseudoscalar mesons. All these symmetry are in addition explicitly broken by non-zero quark masses making all the Goldstone bosons massive.

In the present study, since we only concentrate on the effects of the effective restoration of the $U(1)_A$ symmetry, we can ignore the possible effects of the restoration of $SU(2)_R \times SU(2)_L$, this implies that we can forget $\pi$ and $a_0$ fields, keeping only the $\sigma$ and $\eta$ mesons which are usually specified by the $U(1)_A$ phase. With this restriction on $\Phi$, the effective Lagrangian we adopt here is

$$
\mathcal{L}(\Phi) = \text{Tr}(\partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi - \lambda_1 (\text{Tr}(\Phi^\dagger \Phi))^2 - \lambda_2 \text{Tr}(\Phi^\dagger \Phi)^2 + c[\text{det}(\Phi) + \text{det}(\Phi^\dagger)]) + \text{Tr}[H(\Phi + \Phi^\dagger)].
$$

(4)

where $\Phi = \frac{1}{2}(\sigma + i \eta)\mathbb{I}$. In the following, we demonstrate that both a static string-like solution of the $\eta$ string and a static kink-like solution of the domain wall are expected to be produced during the chiral phase transition.$^1$

The $\eta$ string is a static configuration of the Lagrangian of Eq. (1) with $c = 0$. In this case, during chiral symmetry breaking, the field $\sigma$ takes on a nonvanishing expectation value, which breaks $U(2)_R \times U(2)_L$ down to $U(2)_V$. This results in a massive $\sigma$ and four massless Goldstone bosons.

In our discussion of the $\eta$ string and domain walls it is convenient to define the new fields

$$
\phi = \frac{\sigma + i \eta}{\sqrt{2}}.
$$

(5)

The linear sigma model in Eq. (1) with $c = 0$ now can be rewritten as

$$
\mathcal{L} = (\partial_\mu \phi^\ast (\partial^\mu \phi) - \lambda(\phi^\ast \phi - \frac{v^2}{2})^2,
$$

(6)

where $v^2 = \frac{\lambda}{\lambda_1 + \lambda_2}$ and $\lambda = \lambda_1 + \frac{\lambda_2}{2}$. For static configurations, the energy functional corresponding to the above Lagrangian is

$$
E = \int d^3 x [\nabla \phi^\ast \nabla \phi + \lambda(\phi^\ast \phi - \frac{v^2}{2})^2],
$$

(7)

and the time independent equation of motion is

$$
\nabla^2 \phi = 2\lambda(\phi^\ast \phi - \frac{v^2}{2})\phi.
$$

(8)

The $\eta$ string solution extremising the energy functional of Eq. (7) is given in Refs.\footnote{For simplicity we consider here the configurations which are specified by the $U(1)_A$ phase only. In considering non-abelian phases, there is another class of topological defects known as non-abelian strings\cite{17}, the pion strings, which can also exist during the chiral phase transition\cite{18}.} (13), (19).

$$
\phi = \frac{\rho(r)}{\sqrt{2}}\exp(i\theta),
$$

(9)

where $\rho(r) = [1 - \exp(-\mu r)]$, the coordinates rand are polar coordinates in the $x - y$ plane, the $\eta$ string is assumed to lie along the $z$ axis and $\mu^2 = \frac{\lambda_2 + \lambda_1}{2}v^2$. The energy per unit length of the string is

$$
E = [0.75 + \log(\mu R)]v^2.
$$

(10)

For global symmetry in general the energy density of the string solution is logarithmically divergent, $R$ is introduced as a cutoff which is taken to be $O(fm)$ in the following numerical calculation.

In the case of $c \neq 0$, during chiral symmetry breaking, the field $\sigma$ takes on a nonvanishing expectation
value, which breaks $SU(2)_R \times SU(2)_L$ down to $SU(2)_V$. This results in a massive $\sigma$ and three massless Goldstone bosons, in the same time the $\eta$ meson is massive compared to other pseudoscalar mesons. Then the determinant term in Eq. 4 cannot be simply neglected during the chiral phase transition in nature, so that one of the appropriate description is no longer one of the $\eta$ strings, but one of domain walls. Then in the following discussion we only consider the possible effects of domain walls and ignore the possible effects of the $\eta$ string in the ultrarelativistic heavy-ion collisions. With the definition of new fields in Eq. 5, the Lagrangian of Eq. 4 can be simplistically expressed as

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi + c \text{Re}(\phi^2) - \lambda (\phi^* \phi)^2.$$  \tag{11}

After defining $c = \alpha m^2$, the potential takes the form

$$V(\phi) = \lambda (\phi^* \phi)^2 - m^2 (\alpha \text{Re}(\phi^2) - \phi^* \phi).$$  \tag{12}

The limit $\alpha \to \infty$ corresponds to the maximum explicit $U(1)_A$ symmetry breaking. In this limit, for realistic values of the $\sigma$ meson and the $\pi$ meson mass (i.e., $m^2 - c = \text{constant}$), the $\eta$ and $a_0$ mesons become infinitely heavy and are thus removed from the spectrum of physics excitations, and $U(2)_R \times U(2)_L$ is identical to the $O(4)$ model, there has no $\eta$ strings and domain walls. For the chiral symmetry spontaneously breaking to occur, we always require $\alpha > 1$. In the following numerical calculation, we take $c = (386.79 \text{MeV})^2$, for other parameters we have $\lambda_1 = -31.51$, $\lambda_2 = 82.77$ and $m^2 = (263.83 \text{MeV})^2$ corresponding to $m_\sigma = 400 \text{MeV}$ and $m_\eta = 547 \text{MeV}$.

For static configuration in Eq. 11, the energy functional is given by

$$E = \int d^3 x \left[ \nabla \phi^* \nabla \phi + \lambda (\phi^* \phi) + m^2 (\phi^* \phi) - \alpha m^2 \text{Re}(\phi^2) \right].$$  \tag{13}

The corresponding equation of motion for the field $\phi$ is

$$\nabla^2 \phi + m^2 (\alpha \phi^* - \phi) - 2 \lambda |\phi|^2 |\phi| = 0,$$  \tag{14}

which accepts the static symmetric kink solution $^{19}$ $^{20}$

$$\sigma = m \sqrt{\frac{(\alpha - 1)}{\lambda}} \tanh \left[ \sqrt{\frac{(\alpha - 1)}{2m}} x \right],$$  \tag{15}

$$\eta = 0.$$  \tag{16}

The thickness of this wall is approximately

$$\delta \sim (m \sqrt{\alpha - 1})^{-1} \simeq 0.7 \text{fm},$$  \tag{17}

and the mass per unit area of the walls is

$$\omega = \frac{2 \sqrt{2} m^3}{3 \lambda} (\alpha - 1)^{\frac{3}{2}} \simeq (129.273 \text{MeV})^3.$$  \tag{18}

The stability becomes a consequence of a topological conservation law. The topological current from which this law is derived $j^\mu = \epsilon^{\mu \nu \rho \sigma} \partial_\nu \phi$, the associated charge of a configuration is $N = \int d^3 x j_0^\mu = \phi|_{x = +\infty} - \phi|_{x = -\infty}$, the presence of a kink with $\phi$ in different vacuum at $x = \pm\infty$, gives rise to a non-zero charge $N$ and consequently indicates the stability of the configuration. Moreover, the form of the potential implies that the symmetric wall solution (within the domain wall the $\eta = 0$) is dynamically stable. We consider infinitesimal perturbations of the field $\eta$ and check if the variation in the energy is positive or negative. Discarding terms of cubic and higher orders in $\eta$, we find

$$E = E_{\text{(domain wall)}} + \delta E,$$  \tag{19}

where

$$\delta E = \int d^3 x [\frac{1}{2} \varphi \tilde{\eta} \tilde{\eta} + \frac{1}{2} (\alpha + 1) m^2 \eta^2 + \frac{\lambda}{4} \sigma^2 \eta^2].$$  \tag{20}

From the above equation, the term $\delta E$ in Eq. 19 is always positive, therefore, the domain walls of the Lagrangian 11 is topologically stable and dynamically stable.

In the Sine-Gorden model, the kink solutions are absolutely stable and such a stable domain wall will immediately rule out by the cosmological constraint in general. In our case, the domain wall is only metastable in full theory since there are other dynamical fields corresponding to the remaining $SU(2)$ generators (such as $\pi$ and $a_0$ fields). However, one can show that these dynamical fields do not contribute to the domain wall background but simply remain in their vacuum states. Their fluctuations affect the overall energy density, but do not affect the properties of the domain wall such as the surface tension and so we can neglect their effects 21. Then the domain wall can still be taken as classically stable object, and therefore, it decays through the quantum tunnelling process with exponentially large lifetime which is longer than the characteristic scale existing in the ultrarelativistic heavy ion collisions 22. Then all the pions which are eventually emitted from such an object will be completely incoherent with the rest of pions.

In the ultrarelativistic heavy-ion collisions, domain walls are expected to be produced during the chiral phase transition. If a bubble wall is produced 22, it exists for some lifetime and then decays into its underlying fields, the $\sigma$ fields. We make the assumption that the size of the bubble wall should be around the size of the QGP formed at ultrarelativistic heavy-ion collisions. The experimental observation of the domain wall bubbles can be carried out by using the Hanbury-Brown-Twiss (HBT) intensity interferometry of pions 23 24. As pointed by Shuryak and Zhitnitsky in Ref. 24, if a bubble exists for enough long time ($\sim$5 fm) and then decays the bubble can be taken as a long-lived object. Therefore the pions from the bubble lead to the same effect of not producing an HBT peak in two-pion spectra which is just as that of the long-lived hadronic resonances. To see this,
an effective intercept parameter, $\lambda_{\text{eff}}$, is introduced in Bose-Einstein correlation function

\[ C_2(k, K) = \frac{N_2(p_1, p_2)}{N_1(p_1)N_2(p_2)} = 1 + \lambda_{\text{eff}}(p)R_c(k, K), \quad (21) \]

where the effective intercept parameter and the correlator are given by

\[ \lambda_{\text{eff}}(p) = \left[ \frac{N_c(p)}{N_c(p) + N_h(p)} \right]^2 \quad (22) \]

and

\[ R_c(k, K) = \frac{|S_c(k, K)|^2}{|S_c(k = 0, K = p)|^2}, \quad (23) \]

where $k = (p_1 - p_2)$, $K = (p_1 + p_2)/2$, $N_c(N_h)$ is the one-particle invariant momentum distribution of the “core” (and “halo”) decayed pions respectively. $S_c$ is the Fourier transform of the one-boson emission function. The produced bubbles would give an additional factor $\beta$ to the effective intercept.

\[ \lambda'_{\text{eff}} = \left[ \frac{N_c}{N_c + N_h + N_{\text{domain-wall}}} \right]^2 \]

\[ \approx \left[ 1 + \frac{N_{\text{domain-wall}}}{N_c + N_h} \right]^2 \left[ \frac{N_c}{N_c + N_h} \right]^2 \]

\[ = \beta \lambda_{\text{eff}}, \quad (24) \]

where $N_{\text{domain-wall}}$ is the number of pions from the decay of domain wall bubbles. In RHIC Pb-Pb collisions if we take the radius of QGP phase as the domain wall bubble radius $R \sim 6 \text{ fm}$, then the domain wall bubble energy is about $E_{\text{domain-wall}} \approx 4\pi R^2 \omega = 25 \text{ GeV}$, if all the energy accumulated in the wall will lead to the production of the $\sigma$ mesons (which will result in additional $\sim 60$ mesons per event) one should expect a $40 \pi^+$ (or $\pi^-$) to be produced from the bubble wall in the central rapidity region. At RHIC energy the total number of pions is about 1500, so the factor is about $\beta \sim 0.85$. In the case of LHC Pb-Pb collisions the QGP radius is about 10 fm, this gives out $\beta \sim 0.7 - 0.8$. Thus we can use pion interferometry as a sensitive tool to detect this possible increase of the $\sigma$ production in ultrarelativistic heavy-ion collisions.

In summary, we have discussed the possible effects of the restoration of the axial $U(1)_A$ symmetry during the chiral phase transition by using the $U(N_f)_R \times U(N_f)_L$ linear sigma model with two massless quark flavors. It is emphasized that if the axial $U(1)_A$ symmetry is to be restored above the certain temperature, it is the domain wall rather than the $\eta$ string that is expected to be produced and has a long lifetime then the time scale existing in the ultrarelativistic heavy-ion collisions. These domain walls will decay into the $\sigma$ mesons, and the increase of the $\sigma$ mesons can be viewed as a signal of restoration of the axial $U(1)_A$ symmetry in ultrarelativistic heavy-ion collisions.

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