Nucleosynthesis and the variation of fundamental couplings

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We determine the influence of a variation of the fundamental “constants” on the predicted helium abundance in Big Bang Nucleosynthesis. The analytic estimate is performed in two parts: the first step determines the dependence of the helium abundance on the nuclear physics parameters, while the second step relates those parameters to the fundamental couplings of particle physics. This procedure can incorporate in a flexible way the time variation of several couplings within a grand unified theory while keeping the nuclear physics computation separate from any model-dependent assumptions.

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I. INTRODUCTION

The possible time variation of the fundamental couplings touches a basic cornerstone of our understanding of particle physics [4, 2]. It is a characteristic feature for cosmological models of quintessence – combining the dependence of couplings on the value of the cosmological scalar field with the time variation of this “cosmon” field induces a time variation of the couplings with unfortunately unknown strength [3, 4, 5].

Recent observations of quasar absorption spectra by Webb et al. [6, 7, 8] have suggested that the electromagnetic fine structure “constant” might vary over cosmological timescales, $\Delta \alpha_{em}/\alpha_{em} = -0.54(12) \times 10^{-5}$ for $z \approx 2$. However, other groups did exclude such a variation with high statistical significance [10, 11, 12]. Also systematic effects, such as the evolution of isotope ratios [13, 14] could have an impact on these measurements. While the reality of a variation of $\alpha_{em}$ in QSO absorption lines is still in dispute we need to gain an overview of other possible effects of a variation of the fundamental couplings on cosmological observations.

A central element of modern cosmology is Big Bang Nucleosynthesis (BBN). Actually, the bounds on the variation of $\alpha_{em}$ at $z \sim 3$ do not say much about the possible size of a variation $\Delta \alpha_{em}/\alpha_{em}$ at the time of BBN, around $z \sim 10^{10}$. Furthermore, a major issue is the complex interplay of the variation of several couplings on the outcome of the element synthesis. While there are a number of recent investigations into the bounds of a variation of $\alpha_{em}$ or other single parameters for BBN [15, 16, 17, 18, 19, 20], we will follow a different approach which determines element abundances in a model independent way. For a review of current limits on fundamental couplings see [21].

To determine light element abundances in the absence of time varying couplings, one needs to know the particle masses and the reaction rates of the relevant nuclear processes (from laboratory experiments). The time evolution of the scale factor $a(t)$ only depends on the number of relativistic particles or, more precisely, on the ratio $\rho/T^4$ for energy density and temperature during BBN. Excellent numerical codes for BBN abundance prediction exist [22, 23, 24, 25]. An essential parameter for these computations is the baryon to photon ratio $\eta$. Taking the value from WMAP measurements [26], $\eta = 6.14 \pm 0.25 \times 10^{-10}$, yields a prediction of the helium abundance $Y_{He} = 0.2484^{+0.0001}_{-0.0006}$ [27]. The observational determination varies among different groups. Izotov and Thuan [28] quote two different values for two different equivalent width samples of spectra. The one is $Y_{He} = 0.2421 \pm 0.0021$ and the other $Y_{He} = 0.2444 \pm 0.0020$. If we were to calculate $\eta$ from those samples we would get $\eta = 3.4^{+0.7}_{-0.6} \times 10^{-10}$ for the first and $\eta = 4.0^{+1.1}_{-0.5} \times 10^{-10}$ for the second quote. Another estimate for $Y_{He}$ was obtained byFields and Olive [29] $Y_{He} = 0.238 \pm 0.002 \pm 0.005$, where the second is the systematic error (not quoted by Izotov and Thuan).

These results show some tension between theory and observation. This discrepancy is likely due to systematic errors which are not fully understood. For instance the assumptions made about the extragalactic HII regions differ among several groups.

Increasing the number of light species which are effective at BBN (e.g. more neutrinos) would enhance $Y_{He}$ and only worsen the discrepancy. (This also holds for the possible presence of early dark energy [30, 31].) If a mechanism for decreasing $Y_{He}$ has to be found the time variation of fundamental couplings seems to be a particularly plausible candidate [2]. Effects of the variation of the weak and strong scales or some dimensionless coupling on BBN have been discussed long ago [3, 32, 33]. One may therefore try to estimate the allowed variation of couplings at a very early time in cosmology.

Confidence limits on the variation of couplings or parameters in the framework of BBN always assume an underlying model. However, the confidence regions determined from a model where only $\alpha_{em}$ varies are meaningless if one wants to employ a model where other couplings, such as the weak scale, are allowed to change. In a Grand Unified Theory (GUT) framework, not only does the electromagnetic interaction vary, but also weak and strong interactions. The details of how these are connected depend on the specific GUT and the varia-
tion of the unified couplings and mass scales of spontaneous symmetry breaking. The present BBN limits on time varying couplings are difficult to compare due to this strong implicit model dependence. It is therefore essential to formulate the BBN estimates in a way that is as model independent as possible. This should facilitate the comparison between different assumptions on the time variation of fundamental couplings.

Our concept to solve this problem is as follows. First we describe our general assumption, namely that the deviations are small, and explain how we linearize the problem (Section II). Rather than using a numerical computation we will use an analytic approximation to determine the variation of $Y_{\text{He}}$ – this will help us to better understand how $Y_{\text{He}}$ depends on fundamental couplings. We emphasize that relative errors for the relative variation below 10% are acceptable in contrast to the much higher required precision for the total abundance. In the first step of our analysis we estimate (Section III) how the results of a BBN calculation depend on seven parameters $X_i$ characterizing nuclear physics (Eq. (1)).

In a second step (Section IV) we determine the dependence of the parameter $X_i$ on the relevant dimensionless “fundamental” particle physics parameters $G_k$ (Eq. (2)), which characterize the standard model of electroweak, strong and gravitational interactions. The connection between both is formulated in form of a “transfer matrix” $f_{ik}$. The advantage of this separation is the possibility to compute $f_{ik}$ without invoking BBN whereas the first step does not use any assumption about the particle physics – nuclear physics connection. The two issues can therefore be dealt with independently. Any improvement on the estimate of the dependencies in the first step can be propagated to the fundamental couplings without repeating the whole calculation. In the same spirit one may discuss a third step which relates the standard model parameters to the GUT parameters $G_k$. Typically, this induces relations between the relative variations of the standard model parameters $\Delta G_k/G_k$. As an example, we estimate in Section V the variation $\Delta Y_{\text{He}}/Y_{\text{He}}$ for two models for the connection between the $\Delta G_k/G_k$ within GUT models. The size of the effect depends strongly on these models. Keeping the ratio between the characteristic scales for the weak and strong interactions fixed we obtain $\Delta Y_{\text{He}}/Y_{\text{He}} = 35.0 \Delta \alpha_{\text{em}}/\alpha_{\text{em}}$ whereas for a fixed ratio between the weak scale and the GUT scale one finds $\Delta Y_{\text{He}}/Y_{\text{He}} = 130 \Delta \alpha_{\text{em}}/\alpha_{\text{em}}$.

II. LINEARIZATION

The success of BBN motivates the basic assumption of this paper, namely that the relative time variation of the fundamental constants between nucleosynthesis and the present epoch is small. We can then linearize in the relative variation of the fundamental parameters $\Delta G_k/G_k$ and use for $G_k$ the values extracted from laboratory experiments. We express the relative change of the helium abundance as

$$\frac{\Delta Y_{\text{He}}}{Y_{\text{He}}} = \frac{Y_{\text{He}}(G + \Delta G) - Y_{\text{He}}}{Y_{\text{He}}} = \sum_k \left(\frac{\Delta G_k}{G_k}\right) X_k.$$  \hspace{1cm} (1)

Here $Y_{\text{He}}$ corresponds to the helium abundance computed in absence of a cosmological time variation of couplings, assuming that only standard model particles (with three neutrinos) contribute to the energy density at BBN. Our aim is to determine the coefficients $c_k^{(G)}$ which relate the fundamental parameters to the change in $Y_{\text{He}}$.

In this paper we will consider the effects of the variation of six dimensionless quantities

$$G_k = \left(\frac{M_P}{\Lambda_{\text{QCD}}}, \frac{\alpha_{\text{em}}}{\Lambda_{\text{QCD}}}, m_e/\Lambda_{\text{QCD}}, m_q/\Lambda_{\text{QCD}}, \Delta m/\Lambda_{\text{QCD}}\right).$$ \hspace{1cm} (2)

Here $\Lambda_{\text{QCD}}$ is the characteristic mass scale of the strong interactions which dominates the mass of the nucleons and the strong interaction rates whereas $\langle \phi \rangle$ is the Fermi scale (vacuum expectation value of the Higgs field) relevant for the weak interactions. The strength of the gravitational interactions is given by the (reduced) Planck mass $M_P$ and $m_e$ is the electron mass. The up- and down-quark masses $m_u, m_d$ are reflected in $m_q = (m_u + m_d)/2$ and $\Delta m = m_d - m_u$. In combination with $\langle \phi \rangle/\Lambda_{\text{QCD}}$ the three last mass ratios could be replaced by the relevant Yukawa couplings $h_e, h_u, h_d$. We emphasize that only ratios of mass scales are observable and have cosmological significance $\nu \nu \nu$.

For a given model for the time variation of the fundamental constants the variations $\Delta G_k/G_k$ are typically related to each other. For example, we may assume a unified theory (GUT) and vary only the gauge coupling at the unification scale $M_{\text{GUT}}$, while keeping $G_{3,4,5,6}$ fixed. This results in $\nu \nu \nu(\Delta G_{3,4,5,6} = 0)$

$$\frac{\Delta(M_P/\Lambda_{\text{QCD}})}{M_P/\Lambda_{\text{QCD}}} = -\frac{\pi}{11} \frac{\Delta \alpha_{\text{em}}}{\alpha_{\text{em}}} = \frac{\pi}{11} \frac{\langle \phi \rangle_{\text{BBN}}}{\alpha_{\text{em}}^2} - \alpha_{\text{em}}^2.$$ \hspace{1cm} (3)

Then only a single independent varying coupling is left that we may choose as $\Delta \alpha_{\text{em}}/\alpha_{\text{em}}$.

For practical reasons we will work in a frame where we keep the strong scale $\Lambda_{\text{QCD}}$ fixed. This can be achieved by an appropriate Weyl scaling $\nu \nu \nu \nu$, and will result in a time dependence of the reduced Planck mass $M_P$. This particular frame can be understood as a rescaling of the cosmological “clock” $M_P$ which compensates for the constant strong interactions. In a frame with fixed $M_P$ the strong interaction scale $\Lambda_{\text{QCD}}$ would vary with time.

As mentioned before our computation of the coefficients $c_k^{(G)}$ proceeds in two steps. We first consider the dependence of $Y_{\text{He}}$ on the characteristic quantities for nuclear decays and reactions, also referred to as “nuclear physics parameters”

$$X_i = (M_P, \alpha_{\text{em}}, \langle \phi \rangle, m_e, \tau_\nu, Q, B_d),$$ \hspace{1cm} (4)
according to
\[
\frac{\Delta Y_{\text{He}}}{Y_{\text{He}}} = \sum_i c_i(X) \frac{\Delta X_i}{X_i}.
\] (5)

Here, \(\tau_n\) is the neutron lifetime, \(Q\) the neutron proton mass difference and \(B_d\) the deuteron binding energy. (We keep \(\Lambda_{\text{QCD}}\) fixed – otherwise the dimensionful parameters have to be multiplied by appropriate powers of \(\Lambda_{\text{QCD}}\).) We emphasize that at this stage the effect of the variation of, say, \(\alpha_{em}\) is computed at fixed values of \(X_{1,3,4,5,6,7}\). The computation of the coefficients \(c_i(X)\) involves the details of nuclear physics, i.e. reaction rates etc. .

A second step translates \(\Delta X_i/X_i\) into the variation of the fundamental couplings
\[
\frac{\Delta X_i}{X_i} = \sum_k f_{ik} \frac{\Delta G_k}{G_k},
\] (6)

with
\[
f_{ik} = \frac{\partial \ln X_i}{\partial \ln G_k}.
\] (7)

This step involves the connection between nuclear physics and particle physics, namely the dependence of \(\tau_n, Q\) and \(B_d\) on the couplings \(G_{2,3,4,5,6}\). (Obviously, one has \(f_{ik} = \delta_{ik}\) for \(i = 1...4\) and \(k = 1...6\).) For known \(f_{ik}\) the coefficients \(c_k^{(G)}\) follow from \(c_i^{(X)}\) as
\[
c_k^{(G)} = \sum_i c_i^{(X)} f_{ik}.
\] (8)

Before proceeding to estimates of the various coefficients we list our results for the dependence of the helium abundance on the nuclear physics parameters (Section III) in Table I

<table>
<thead>
<tr>
<th>variable</th>
<th>(M_P)</th>
<th>(\alpha_{em})</th>
<th>(\langle \phi \rangle)</th>
<th>(m_e)</th>
<th>(\tau_n)</th>
<th>(Q)</th>
<th>(B_d)</th>
</tr>
</thead>
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<tr>
<td>coeff.</td>
<td>-0.81</td>
<td>-0.043</td>
<td>2.4</td>
<td>0.024</td>
<td>0.24</td>
<td>-1.8</td>
<td>0.53</td>
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</table>

The dependence on the particle physics parameters is shown in Table II. This Table allows for a quick inspection of the impact of various parameter variations. We observe a particularly strong influence of a possible variation of \(\langle \phi \rangle\) and \(\Delta m\) [2].

<table>
<thead>
<tr>
<th>variable</th>
<th>(M_P)</th>
<th>(\alpha_{em})</th>
<th>(\langle \phi \rangle)</th>
<th>(m_e)</th>
<th>(m_q)</th>
<th>(\Delta m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>coeff.</td>
<td>-0.81</td>
<td>1.94</td>
<td>3.36</td>
<td>0.389</td>
<td>-1.59</td>
<td>-5.358</td>
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</tbody>
</table>

### III. HELIUM ABUNDANCE

In this section we want to derive a semi-analytic estimate for the primordial helium abundance \(Y_{\text{He}}\). In doing so we will follow the approach of Esmailzadeh, Starkman and Dimopoulos [33], hereafter ESD. They estimate the BBN abundances via quasi static equilibrium and fixed point conditions. This approach should be sufficient for an understanding of the effect of small variations. Of course, the estimate of the coefficients \(c_i^{(X)}\) as well as the determination of the corresponding errors would benefit from a systematic numerical investigation using the BBN codes.

In the primordial universe, at temperatures above several MeV, the abundances of protons and neutrons are in thermal equilibrium. Protons are converted into neutrons and vice versa. The latter process has a reaction rate given by [39]
\[
\Gamma_{n\rightarrow p} = A \int dx \ x^2 \left( 1 - \frac{m_e^2}{(Q + x)^2} \right)^{\frac{1}{2}} (Q + x)^2 (1 + e^{(x/T)} - 1)^{-1} (1 + e^{-(Q+x)/T})^{-1}.
\] (9)

The integral runs from \(-\infty\) to \(+\infty\) with an energy gap between \(-Q - m_e\) and \(-Q + m_e\) with \(Q\) being the proton neutron mass difference. Here \(A \sim \langle \phi \rangle^{-4}\) is the 4 point transition probability in Fermi-theory which depends on the axial and vector couplings \(c_V\) and \(c_A\). For simplicity we will work with constant \(c_V\) and \(c_A\).

We now assume that this reaction freezes out at a temperature \(T_n^*\) when the Hubble expansion is comparable to this reaction rate, i.e.
\[
\Gamma_{n\rightarrow p}(T_n^*) = b H(T_n^*).
\] (10)

We note that an equally well justified assumption would be to include the reaction rate \(\Gamma_{p\rightarrow n}\) in this condition. The insufficient information of our simple approach (as compared to a more complete treatment by a solution of the Boltzmann equation) is accounted for by the unknown factor \(b\). At the end we fix \(b\) so that we obtain the same \(Y_{\text{He}}\) as predicted by full numerical codes [1]. The Hubble parameter \(H\) is given by the Friedman equation for a radiation dominated universe [2]
\[
H^2 = \frac{\rho}{3M_P^2}, \quad \rho = g_\ast \frac{\pi^2}{30} T^4,
\] (11)

with an effective number of degrees of freedom \(g_\ast = 10.75\) before positron-electron annihilation. The freeze out

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1 For \(b = 1\) we obtain a \(^4\)He abundance that deviates by about 10 percent from the value \(Y_{\text{He}} = 0.2484\) found with a fully numerical computation using the WMAP value for \(\eta\) [23]. In order for our analytic approximation to yield the \(Y_{\text{He}}\) predicted numerically we use \(b = 1.22\).
2 We do not consider changes in the expansion rate due to changes in baryon or electron masses.
temperature of the neutrons, with no change in fundamental couplings, obtains as $T_n^* = 0.77$ Mev and the neutron concentration at freeze out can then be calculated as

$$Y_n^* = \frac{1}{1 + e^{Q/T_n^*}} = 0.158. \quad (12)$$

Following the “freeze out” of the neutron to proton ratio the neutrons decay, thereby further changing $Y_n$ for $T < T_n^*$. After a short time the synthesis of deuterium and tritium starts which subsequently leads to the production of helium. Since almost all existing neutrons end up in helium, we need to know how many neutrons remained when helium was synthesized in appreciable amounts. We will assume that the neutrons decay freely until a time $t_f$ when helium formation starts to dominate over the neutron decay process, i.e.

$$2\dot{Y}_{He}(t_f) = -\dot{Y}_n(t_f). \quad (13)$$

The final $^4$He abundance is then estimated by

$$Y_{He} = \frac{1}{2} Y_n(t_f) = \frac{1}{2} Y_n^* e^{-(t_f/\tau_n)}. \quad (14)$$

It depends on the couplings via $Q$, $T_n^*$, $\tau_n$ and $t_f$. In turn, $T_n^*$ depends on $A \sim \langle \phi \rangle^{-3}$, $Q$, $m_n$ and $M_P$ via Eqs. (9), (10) and (11).

We need an estimate of $t_f$. The by far dominant process for helium production is the reaction (4)

$$d + t \rightarrow ^4\text{He} + n. \quad (15)$$

To write down the equations governing the abundances comprised of several reactions we will adopt the notation of ESD who abbreviate a reaction rate

$$\alpha + \beta \rightarrow \gamma + \delta \quad (16)$$

as $[\alpha \beta \gamma \delta]$. The condition for the time until which the neutrons decay is given by

$$2Y_d Y_{Y}[dtn\alpha] = \frac{1}{\tau_n} Y_n^* e^{-(t_f/\tau_n)}. \quad (17)$$

To compute the time when this relation is satisfied we need to know the abundance of deuterium($Y_d$) and tritium($Y_d$) as well as the reaction rate $[dtn\alpha]$. In the temperature range we are considering, deuterium can be assumed to be in thermal equilibrium and hence its abundance is given by the Saha equation (11)

$$Y_d = 8.15 \left( \frac{T}{m_n} \right)^{3/2} \eta e^{B_d/T} Y_n Y_p, \quad (18)$$

with the proton abundance being $Y_p \approx (1 - Y_n)$, $m_n$ the neutron mass.

The estimate of $Y_d$ is more involved and also requires knowledge of the abundance $Y_3$ for $^3$He. The tritium concentration is established by the reactions

$$^3\text{He} + n \rightarrow p + t$$
$$d + d \rightarrow p + t \quad (19)$$

creating and

$$t + d \rightarrow ^4\text{He} + n \quad (20)$$

annihilating tritium. Other reactions are subdominant by at least 2 orders of magnitude (as can be verified from (10) and are therefore neglected. Close to thermal equilibrium the fixed point condition (15) leads us to an equation for $Y_1$ of the form:

$$Y_1 = \frac{Y_n Y_3 [n3p]\gamma + Y_d Y_d [d3p]}{Y_d [dtn\alpha]}. \quad (21)$$

Likewise, we can write down the dominant processes for the $^3$He abundance. Invoking the fixed point condition yields

$$Y_3 = \frac{Y_d Y_p [p3\beta \gamma] + Y_d Y_d [d3p]}{Y_d [d3p\alpha] + Y_n [n3p]} \quad (22)$$

From Eqs. (15), (21) and (22) we can determine the abundance of deuterium, tritium and helium-3 as a function of $T$ and $Y_n$. In turn, temperature and time are related by the background cosmology and $Y_n = Y_n^* e^{-t/\tau_n}$. Eq. (15) now determines $t_f$.

The dependence of $Y_{He}$ on the various parameters cannot be solved analytically. In the linear approximation, however, the computation of the response coefficients $e^{t_1}$ is straightforward. For this purpose we assume that all strong interaction rates are determined by the strong interaction scale $\Lambda_{QCD}$. At this point we benefit from our particular frame with constant $\Lambda_{QCD}$ which implies that we can use constant rates $[dtn\alpha]$ etc., except for small electromagnetic effects.

The results of this computation can be found in Table I. They are plausible in the sense that they resemble what one would expect from simple arguments. Increasing the Planck mass gives a slower expansion rate, resulting in a later freeze out of weak interactions, hence less neutrons are available for helium production. Increasing the decay time $\tau_n$ of the free neutrons leaves more neutrons to be converted into helium since effectively all neutrons end up being bound in helium. Increasing $\langle \phi \rangle$ results in a decrease of the Fermi interaction $G_F$, hence weak interactions freeze out earlier resulting in an increase in $Y_{He}$. Changing $Q$ results in a different neutron-proton mass ratio at freeze out and also in modified weak rates due to changes in the available phase space. If we exclude the changes in phase space volume, the coefficient is $-1.4$ instead of $-1.8$. Thus, helium abundance is a decreasing function of the proton-neutron mass difference $Q$ as anticipated. Increasing the binding energy of the deuteron, $B_d$, results in earlier formation of helium and reduces the amount of neutrons decaying into protons. The influence of the electron mass is only through the phase space volume in the weak rates which is a very small effect for our purposes.

Changes in $\alpha_{em}$ affect the nuclear reaction rates with the main effects being variations in the Coulomb barrier for charge-induced reactions, final-state interactions,
radiative capture and mass differences. We use the procedure of Bergström, Iguri and Rubinstein (BIR) \(^{12}\) for computing the impact of varying \(\alpha_{em}\) on all nuclear reaction rates used in our computation, including the improvements of Nollet and Lopez \(^{43}\). We use the rates of the NACRE compilation \(^{14}\) where available, otherwise we use those of Smith, Kawano and Malaney (SKM) \(^{40}\). For the process \(^3\text{He}(n, p)\)\(^t\) we use the fit of Cyburt, Fields and Olive \(^{45}\). Since the analytic NACRE rate fits have a different expansion in terms of \(T\) we have fitted the rates to the SKM functional form for use of the BIR treatment as described in \(^{15}\).

Except for electromagnetic effects we have not taken into account any other effect that may change the reaction rates.

IV. FUNDAMENTAL COUPLINGS

In this section we describe the relation between the fundamental couplings \(G_k\) and the nuclear physics parameters \(X_i\). This relation was expressed in the form of a matrix equation \(^{17}\). We will now discuss what effects we took into account by explaining each row of the matrix \(f_{ik}\) (see Table III). Each coefficient \(f_{ik}\) describes the response of the “nuclear physics parameter” \(X_i\) when one varies a single parameter \(G_k\), while keeping the other \(G_{1\neq k}\) fixed. For \(i = 1,...,4\) the parameters appear both in the lists of \(X_i\) and \(G_k\) and \(f_{ik} = \delta_{ik}\) by virtue of our definition. Also \(\tau_n, Q\) and \(B_d\) do not depend on \(M_P\) implying \(f_{ik} = \delta_{ik}\). The nontrivial coefficients \(f_{ik}\) for \(i = 5, 6, 7\) account for the dependence of \(\tau_n, Q\) and \(B_d\) on \(\alpha_{em}, \langle \phi \rangle\), \(m_e, m_q\) and \(\Delta m\).

The nucleon masses and nuclear binding energies depend on the quark masses and \(\alpha_{em}\). The dependence of the neutron-proton mass difference on the fundamental couplings is given by (see \(^{17}\)):

\[
Q = \left[ -0.76 \left( 1 + \frac{\Delta \alpha_{em}}{\alpha_{em}} \right) + 2.05 \left( 1 + \frac{\Delta (\Delta m)}{\Delta m} \right) \right] \text{MeV} .
\]

From this we can determine \(f_{62}\) and \(f_{66}\). Recent studies have suggested that the deuteron binding energy \(B_d\) may increase with decreasing pion mass \(^{17, 18}\). We may parametrize the dependence of \(B_d\) on \(m_q\) at fixed \(\langle \phi \rangle\) by a linear fit \(^{18}\) and neglect the dependence on \(\langle \phi \rangle\) at fixed \(m_q, \Delta m\). For the electromagnetic part we use the Monte Carlo simulation data of Pudliner et al. \(^{42}\). Hence the deuteron binding energy may be expressed in terms of the pion mass \(m_\pi \propto \alpha_{em}^{1/2}\) as:

\[
B_d = B_d^0 \left[ (r + 1) - r \frac{m_q}{m_q^0} \right] - 0.018 \frac{\Delta \alpha_{em}}{\alpha_{em}} \text{MeV} ,
\]

where \(r\) is a parameter that varies between 6 and 10 and \(B_d^0 = 2.225\) MeV is the deuteron binding energy as measured in the laboratory today.

The neutron lifetime is changed due to variations in the weak scale \(\tau_n \propto G_F^{-2} \propto \langle \phi \rangle^4\). Furthermore, a change in the phase space volume \(f\) of free neutron decay

\[
f = \int_{m_n}^{Q} dq \ q^2 (Q - q)^2 (1 - \frac{m_n^2}{q^2})^{1/2} ,
\]

results in a dependence of \(\tau_n\) on \(Q\) and the electron mass \(m_e\). Because \(Q\) also depends on \(\alpha_{em}\) and \(\Delta m, \Delta \tau_n\) will also have contributions from the variation of those parameters. A linear analysis then yields the corresponding entries for \(\tau_n:\)

\[
\frac{\Delta \tau_n}{\tau_n} = \frac{3.86}{\alpha_{em}} \frac{\Delta \alpha_{em}}{\langle \phi \rangle} + 4 \frac{\Delta \langle \phi \rangle}{\langle \phi \rangle} + 1.52 \frac{\Delta m_e}{m_e} - 10.4 \frac{\Delta (\Delta m)}{\Delta m} .
\]

The entries of the matrix \(f_{ik}\) can be found in Table III. We have quoted the coefficients for the effects discussed above. When there is no contribution at all a zero is written. For some relations between the \(G_k\) and the \(X_i\) small effects are present but with negligible coefficients. To distinguish those from the others, we have left the matrix entry empty. Having determined the transfer matrix we can calculate the dependence of \(\Delta Y_{He}/Y_{He}\) on the fundamental parameters (see Eq. 5). The results are shown in Table III above.

### TABLE III

<table>
<thead>
<tr>
<th>Parameter (^a)</th>
<th>(M_P)</th>
<th>(\alpha_{em})</th>
<th>(\langle \phi \rangle)</th>
<th>(m_e)</th>
<th>(m_q)</th>
<th>(\Delta m)</th>
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\(^a\)The parameters are dimensionless, but we omitted the scaling by \(\Lambda_{QCD}\).

V. TWO GUT EXAMPLES

In this section we present two examples based on GUTs for which we have expressed the changes in the fundamental parameters by the variation of only one independent coupling and computed the resulting change in \(Y_{He}\). The variation of the couplings is assumed to be due to a scalar field \(\chi\) called the cosmon \(^{30}\). There are good arguments \(^{5}\) that this field plays the role of quintessence \(^{30}\) and its present potential and kinetic energy can be associated with the dark energy of the universe. For our considerations, however, we will not need any particular details of the evolution of the cosmon, except that its value at the time of nucleosynthesis was different from its present value.
For the details of the derivation of how the fundamental constants change in a GUT scheme we refer the reader to Ref. [50]. Merely quoting the results, to one loop order the fundamental couplings as functions of the cosmon field \( \chi \) are given by

\[
\begin{align*}
\alpha_e^{-1}(M_W) &= \frac{4\pi Z_F(\chi)}{g^2} + \frac{7}{2\pi} \ln \zeta_w(\chi), \\
\alpha_w^{-1}(M_W) &= \frac{4\pi Z_F(\chi)}{g^2} + \frac{5}{3\pi} \ln \zeta_w(\chi), \\
\alpha_{em}^{-1}(M_W) &= \frac{32\pi Z_F(\chi)}{3g^2} - \frac{5}{3\pi} \ln \zeta_w(\chi),
\end{align*}
\]

where the W-Boson mass is \( M_W(\chi) = \zeta_w(\chi) \chi \) and \( Z_F(\chi) \) determines the renormalized grand unified gauge coupling (\( g_\text{R}^2 = g^2 / Z_F, \ g \) fixed). We normalize \( \chi \) such that \( M_{\text{GUT}}(\chi) = \chi \). In Eqs. (27)-(29) we can replace \( M_W = g_w(\phi) / \sqrt{2} \) by \( \phi \). The relative variation of \( g_w \) (or \( \alpha_w \)) induces only a correction of higher order in these relations.

As mentioned before, we will work in a frame in which the scale of the strong interaction is fixed such that the strong interaction rates are constant for our BBN estimate. We will consider two particularly simple scenarios where \( M_P(\chi) / M_{\text{GUT}}(\chi) = \text{const.} \) with

\[
\frac{\Delta M_P / \Lambda_{\text{QCD}}}{M_P / \Lambda_{\text{QCD}}} = - \Delta \ln \zeta_w + \Delta \ln (\phi / \Lambda_{\text{QCD}}).
\] (30)

Furthermore, we also neglect the variation of the Yukawa couplings and hence the variations in \( m_e, m_q \) and \( \Delta m \) obey

\[
\begin{align*}
\frac{\Delta m_e}{m_e} &= \frac{\Delta (\Delta m)}{\Delta m}, \\
\frac{\Delta m_q}{m_q} &= \frac{\Delta (\phi)}{\langle \phi \rangle}.
\end{align*}
\] (31)

The effect of the variation of the cosmon field \( \chi \) can now be expressed as a variation in the renormalized grand unified gauge coupling expressed by \( Z_F \) and a variation in \( \ln \zeta_w \).

At this stage the two unknown quantities \( \Delta \ln Z_F \) and \( \Delta \ln \zeta_w \) contain all relevant information about the unknown coupling of the cosmon to matter and radiation. For the present investigation we can simply use the relative variation of the GUT-coupling \( \Delta \ln Z_F \) and the ratio between weak and GUT scale \( \Delta \ln \zeta_w \) as free parameters. For the running of \( \alpha_{em} \) at \( \mu < M_W \) we have the relation

\[
\alpha_{em}(\mu)^{-1} = \alpha_{em}(M_W)^{-1} + \frac{2}{3\pi} \sum_i Q_i^2 \ln \frac{M_W}{\mu},
\] (32)

where the \( Q_i \) are the charges of the particles with masses in the range between \( M_W \) and \( \mu \). In our case this is given by five quarks (top lies above \( M_W \)) in three colours plus 3 leptons, i.e. \( \sum_i Q_i^2 = 3 \times (8/9 + 3/9) + 3 \). Similarly, for the running of \( \alpha_s \) below \( M_W \) we include five quarks and associate \( \Lambda_{\text{QCD}} \) with the scale where the one loop expression for \( \alpha_s(\mu) \) diverges.

We can now express \( \alpha_{em} = \alpha_{em}(m_e) \) and \( \Lambda_{\text{QCD}} \) in terms of \( \alpha_{em}(M_W) \) and \( \alpha_s(M_W) \). Thus they relate \( \Lambda_{\text{QCD}} / \chi \) and \( \alpha_{em} \) to \( Z_F \) and \( \ln \zeta_w \). The specific relation between the variations of \( \Lambda_{\text{QCD}} / \Lambda_{\text{QCD}} \) and \( \alpha_{em} \) depends on the variation of the weak scale \( \ln \zeta_w \). Our two examples will either keep \( \ln \zeta_w \) or \( \ln (\phi / \Lambda_{\text{QCD}}) \) fixed.

The first example is as simple as possible – we also keep the weak scale fixed w.r.t. the strong scale. This will result in a \( \chi \) independent ratio \( \langle \phi / \Lambda_{\text{QCD}} \rangle \sim \text{const.} \) and leads to

\[
\frac{\Delta \alpha_{em}(M_W)}{\alpha_{em}^2(M_W)} = - \frac{88\pi}{7} \frac{\Delta Z_F}{g^2}
\] (33)

and

\[
\frac{\Delta M_P / \Lambda_{\text{QCD}}}{M_P / \Lambda_{\text{QCD}}} = - \frac{\pi}{11} \frac{\Delta \alpha_{em}(M_W)}{\alpha_{em}^2(M_W)}.
\] (34)

Since the weak scale is fixed we set \( \langle \phi / \Lambda_{\text{QCD}} \rangle = 0 \) in Eq. (1).

In our first example one has \( \Delta \alpha_{em}^{-1}(m_e) = \Delta \alpha_{em}^{-1}(M_W) \) which is related to the only unknown parameter \( \Delta Z_F \) by Eq. (33). Eq. (34) results in

\[
\frac{\Delta M_P / \Lambda_{\text{QCD}}}{M_P / \Lambda_{\text{QCD}}} = - 39.1 \frac{\Delta \alpha_{em}}{\alpha_{em}}.
\] (35)

The only nonvanishing entries in Eq. (1) from \( \Delta G_1 \) and \( \Delta G_2 \) are therefore related and \( \Delta Y_{\text{He}} / Y_{\text{He}} \) can be expressed in terms of a single parameter that we may choose as \( \Delta \alpha_{em} / \alpha_{em} \). In order to get an idea of the sensitivity we compute the value \( \Delta \alpha_{em} / \alpha_{em} \) which would be needed in order to obtain a helium abundance \( Y_{\text{He}} = 0.24 \) for \( \eta \) corresponding to the central WMAP value. For our first example we find

\[
\frac{\Delta \alpha_{em}(m_e)}{\alpha_{em}(m_e)} = -1.0 \times 10^{-3}.
\] (36)

In the second example we will set \( \langle \phi / \chi \rangle = \text{const.} \). For this setting we obtain

\[
\frac{\Delta \alpha_{em}(M_W)}{\alpha_{em}^2(M_W)} = - \frac{32\pi}{3} \frac{\Delta Z_F}{g^2}
\] (37)

and

\[
\frac{\Delta M_P / \Lambda_{\text{QCD}}}{M_P / \Lambda_{\text{QCD}}} = - \frac{\pi}{12} \frac{\Delta \alpha_{em}(M_W)}{\alpha_{em}^2(M_W)}.
\] (38)

We also need to relate the different mass scales via Eq. (22) giving

\[
\frac{\Delta \alpha_{em}(m_e)}{\alpha_{em}^2(m_e)} = \frac{\Delta \alpha_{em}(M_W)}{\alpha_{em}^2(M_W)} [1 + \frac{1}{18} \sum_i Q_i^2],
\] (39)

\footnote{We have put \( r = 6 \). The change in \( Y_{\text{He}} \) is negligible if we choose \( r = 10 \).}
where $\sum_i \tilde{Q}_i^2 = 2$ runs only over the three light quarks whose effect on the running of $\alpha_{em}$ is cut off at $\mu \sim \Lambda_{QCD}$.

In Eq. (41) we now have nonvanishing entries from $\Delta G_{3,4,5,6}$ as well, related by Eqs. (31) and (38) to $\Delta \alpha_{em}/\alpha_{em}$ giving

$$\frac{\Delta (M_p/\Lambda_{QCD})}{M_p/\Lambda_{QCD}} = \frac{\Delta (\phi)}{\langle \phi \rangle} = -32.3 \frac{\Delta \alpha_{em}}{\alpha_{em}}. \quad (40)$$

The variation of $\Delta \alpha_{em}$ which would be needed for $Y_{He} = 0.24$ is now reduced as compared to the first model,

$$\frac{\Delta \alpha_{em}(m_e)}{\alpha_{em}(m_e)} = -2.7 \times 10^{-4}. \quad (41)$$

VI. DISCUSSION AND CONCLUSION

Big Bang nucleosynthesis offers an excellent testing ground for the time variation of the fundamental couplings, probing directly their values at a time close to the big bang or at high redshift. This paper presents a general analysis how the primordial helium abundance depends on six “fundamental couplings” as summarized in Table III. In grand unified (GUT) models the variations of the various couplings are interrelated. Typically, the dominant effect comes from a variation of the ratios $M_p/\Lambda_{QCD}$ or $\langle \phi \rangle/\Lambda_{QCD}$ rather than from the direct influence of a variation of the fine structure constant $\alpha$. This is easily seen by comparing Eqs. (35) and (40) with Table I. As noted before $\alpha$, the variation of the weak interaction scale $\langle \phi \rangle$ or the quark masses can play a very substantial role. We have checked that the impact of a variation of the strong and electromagnetic interaction rates is a rather minor effect. Discussing two different models gives us a certain handle to investigate the effect of cancellations – compare Eqs. (35) and (40). Extending our analysis to the abundances of deuterium and lithium may be a way to (partially) lift the degeneracies between the variations of various couplings.

Excluding very particular cancellations we may infer from the approximate agreement between the WMAP-prediction and the observations of $Y_{He}$ a bound $|\Delta \alpha_{em}/\alpha_{em}(z = 10^10)| < a$ few times $10^{-3}$. A typical size of a coupling variation that could explain the present discrepancy between WMAP and the observed $Y_{He}$ would be in a range $\Delta \alpha_{em}/\alpha_{em} \approx (2 - 10) \times 10^{-4}$.

Obviously our treatment can be improved. The coefficients quoted in Table III could be estimated with higher accuracy by using a full numerical code instead of our analytic estimate. Likewise, looking at Table III one can see that there are some small effects contributing to the matrix $f_{ik}$ which we have not included. Also, we have only investigated the change in the helium abundance. Stringent bounds on BBN are obtained from the primordial abundances of deuterium and lithium and it would be worth extending our analysis to these other light elements.

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