On the large N limit, Wilson Loops, Confinement and Composite Antisymmetric Tensor Field theories

Carlos Castro
Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, GA. 30314
April 2002, Revised February 2004

Abstract

A novel approach to evaluate the Wilson loops associated with a $SU(\infty)$ gauge theory in terms of pure string degrees of freedom is presented. It is based on the Guendelman-Nissimov-Pacheva formulation of composite antisymmetric tensor field theories of area (volume) preserving diffeomorphisms which admit $p$-brane solutions and which provide a new route to scale symmetry breaking and confinement in Yang-Mills theory. The quantum effects are discussed and we evaluate the vacuum expectation values (vev) of the Wilson loops in the large $N$ limit of the quenched reduced $SU(N)$ Yang-Mills theory in terms of a path integral involving pure string degrees of freedom. The quenched approximation is necessary to avoid a crumpling of the string world-sheet giving rise to very large Hausdorff dimensions as pointed out by Olesen. The approach is also consistent with the recent results based on the AdS/CFT correspondence and dual QCD models (dual Higgs model with dual Dirac strings). More general Loop wave equations in C-spaces (Clifford manifolds) are proposed in terms of generalized holographic variables that contain the dynamics of an aggregate of closed branes ($p$-loops) of various dimensionalities. This allows us to construct the higher-dimensional version of Wilson loops in terms of antisymmetric tensor fields of arbitrary rank which couple to $p$-branes of different dimensionality.

I. Introduction

It has been believed for a long time that QCD confinement is supposed to be a non-perturbative solution to QCD in four dimensions; i.e. to $SU(3)$ Yang-Mills theory [1]. A formal proof of the colour confinement amounts to a derivation of the area-law for a Wilson loop associated, for example, with the world lines of a quark-antiquark pair joined in by a string. The area law in the Euclidean regime is $W(C) \sim \exp[-TA]$ where $T$ is the string tension and the (Euclidean) area is $A = l t_E$. The colour-electric potential rises linearly with the length of the string separating the quark-antiquark and blows up in the $l \rightarrow \infty$. This would be a signal of (colour-electric field lines) confinement, an infinite amount energy would be required to separate the quarks.

Many attempts have been explored to solve this problem, in particular those based on the so-called string ansatz [2-3]:

$$W[C] \sim \int_{\Sigma(C)} [DX] \exp(iS_{\text{string}}).$$

which says that the effective (collective) infrared degrees of QCD at strong coupling are given by string configurations whose worldsheets have for boundary the loop $C$. The Schwinger-Dyson equations for QCD can be reformulated as an infinite chain of equations for the Wilson loops that simplify drastically in the large $N$ limit giving the single equation known as the Makeenko-Migdal loop equation [4].

In light of the Maldacena AdS/CFT correspondence formulated by many authors [5] as a relation between partition functions, Maldacena and others proposed that the average value of a Wilson loop in the large $N$ limit, for $N = 4$ $SU(N)$ SYM was given by the partition function of a world-sheet string action which ends along the loop $C$ in the four-dim boundary. Another approach has been based on the dual formulation of QCD [6] (in the infrared limit) given by a $U(1)$ gauge theory adjoined by a dual Higgs model with dual Dirac strings [7] (where the quarks live at their end-points). The average value of the Wilson loop in this dual phase obeys the area-law fall-off. For other approaches to solve the confinement problem based on Skyrmions and others methods see [8].

The outline of this work goes as follows. In section 2 we will present a novel approach to evaluate the Wilson loops associated with a gauge theory of area-preserving diffeomorphisms in terms of the (area
dimensional knots and algebraic are nothing but high-dimensional Knots. For the mathematical intricacies of Chern-Simons branes, high-
p, p
are embeddings of a

The latter condition is more closely related to the EM duality among two Chern-Simons

volume-preserving diffs is 

and confinement occurs in 

followed by the brane/wave duality principle [34] which permits us to show how scale-symmetry breaking and confinement occurs in p-branes solutions of composite antisymmetric tensor field theories of area (volume) preserving diffs through the introduction of a preferred scale.

In section 3 we review the derivation [11] showing how the quenched large N SU(N) Yang-Mills theory admits strings, membranes and bag excitations. We also explain why the quenched approximation is necessary to avoid a crumpling of the string world-sheet giving rise to very large Hausdorff dimensions as pointed out by Olesen [25] and the collapse (clustering) of eigenvalues [24]. In section 4 we evaluate the vacuum expectation values (v.e.v) of the Wilson loops in the large N limit of the quenched reduced SU(N) Yang-Mills theory in terms of a path integral involving pure string degrees of freedom.

This Wilson loop average is consistent with the recent results based on the AdS/CFT correspondence and dual QCD models (dual Higgs model with dual Dirac strings). Finally we present more general Loop wave equations in C-spaces (Clifford manifolds) [12,20] than those considered so far. These loop equations are given in terms of the holographic variables associated with an aggregate of closed branes (p-loops) of various dimensionalities. This allows us to construct the higher-dimensional version of Wilson loops in terms of antisymmetric tensor fields of arbitrary rank which couple to p-branes of different dimensionality.

II

2.1 Branes as composite antisymmetric tensor field theories

In this section we will review the construction of p'-brane solutions to the rank p + 1 composite antisymmetric tensor field theories [9] developed by Guendelman, Nissimov and Pacheva [9] when the condition $D = p + p'$ is satisfied. These field theories possess an infinite-dimensional group of volume-preserving diffeomorphisms of the target space of the scalar primitive field constituents. The role of local gauge symmetry is traded over to an infinite-dimensional global Noether symmetry of volume-preserving diffs. The study of the Ward identities for this infinite-dim global Noether symmetry to obtain non-perturbative information in the mini-QED models (the composite form of QED) was analysed in [9].

The starting Lagrangian is defined [10]:

$$ L = - \frac{1}{g^2} F_{\mu_1 \mu_2 ... \mu_{p+1}}^2, \quad F = dA = \epsilon_{a_1 a_2 ... a_{p+1}} \partial_{\mu_1} \phi^{a_1} ... \partial_{\mu_{p+1}} \phi^{a_{p+1}}. \quad (2.1) $$

the rank p + 1 composite field strength is given in terms of p + 1 scalar fields $\phi^a(x), \phi^2(x), ... \phi^{p+1}(x)$. Notice that the dimensionality of spacetime where the field theory is defined is greater than the number of primitive scalars $D > p + 1$. An Euler variation w.r.t the $\phi^a$ fields yields the following field equations, after pre-multiplying by a factor of $\partial_{\mu_{p+2}} \phi^{a_1}$ and using the Bianchi identity $dF = 0$:

$$ \partial_{\mu_{i}} \frac{\delta L}{\delta (\partial_{\mu_1} \phi^{a_{i+2}})} = 0 \quad \Rightarrow \quad F_{\mu_{p+2} \mu_2 ... \mu_{p+1}} \partial_{\mu_{1}} F^{\mu_{1} \mu_2 ... \mu_{p+1}} = 0. \quad (2.2) $$

Notice that despite the Abelian-looking form $F = dA$ the infinite-dimensional (global) symmetry of volume-preserving diffs is not Abelian. The theory we are describing is not the standard YM type.

We are going to find now p'-brane solutions to eq-(2), where $D = p + p' + 2$. These brane solutions obeyed the classical analogs of S and T-duality [10]. Ordinary EM duality for branes requires $D = p + p' + 4$. The latter condition is more closely related to the EM duality among two Chern-Simons p, p'-branes which are embeddings of a p, p'-dimensional object into $p + 2; p' + 2$ dimensions. These co-dimension two objects are nothing but high-dimensional Knots. For the mathematical intricacies of Chern-Simons branes, high-dimensional knots and algebraic, K, L theory see [13].

A special class of (non-Maxwellian) extended solutions to eqs-(2) requires a dualization procedure [10]:

$$ G = * F \quad \Rightarrow \quad G_{\mu_1 \mu_2 ... \nu_{p' +1}}^a(\phi(x)) = \epsilon^{\mu_1 \mu_2 ... \nu_{p+1} \nu_{p' +1}} F_{\mu_1 \mu_2 ... \nu_{p+1}}(\phi(x)). \quad (2.3) $$
After this dualization procedure the eqs-(2) are recast in the form:

\[ G^{\mu_1 \nu_2 \ldots \nu_{p'+1}} \partial_{\mu_1} G_{\nu_2 \nu_3 \ldots \nu_{p'+2}} (\tilde{\phi}(x)) = 0. \]  

(2.4)

The dualized equations (4) have a different form than eqs-(2) due to the position of the indices (the index contraction differs in both cases). Extended \( p' \)-brane solutions to eqs-(4) exist based on solutions to the Aurilia-Smailagic-Spallucci local gauge field theory reformulation of extended objects given in [14]. These solutions are [10]:

\[ G^{\nu_1 \nu_2 \ldots \nu_{p'+1}} (\tilde{\phi}(x)) \big|_{x=X} = T \frac{\{X^{\nu_1}, X^{\nu_2}, \ldots, X^{\nu_{p'+1}}\}}{\sqrt{-\frac{1}{(p'+1)!} \{X^{\mu_1}, X^{\mu_2}, \ldots, X^{\mu_{p'+1}}\} \{X^{\nu_1}, X^{\nu_2}, \ldots, X^{\nu_{p'+1}}\}}. \]  

(2.5)

where \( T \) is the \( p' \)-brane tension and the Nambu-Poisson bracket w.r.t the \( p' + 1 \) world-volume variables is defined as the ordinary determinant /Jacobian:

\[ \{X^{\nu_1}, X^{\nu_2}, X^{\nu_3}, \ldots, X^{\nu_{p'+1}}\}_{NPB} = \varepsilon^{\sigma_1 \sigma_2 \sigma_3 \ldots \sigma_{p'+1}} \partial_{\sigma_1} X^{\nu_1} \partial_{\sigma_2} X^{\nu_2} \ldots \partial_{\sigma_{p'+1}} X^{\nu_{p'+1}}. \]  

(2.6)

All quantities are evaluated on the \( p' + 1 \)-dim world-volume support of the \( p' \)-brane; i.e. one must restrict the dual-scalar solutions \( \tilde{\phi}(x) \) to those points in the \( D \)-dimensional spacetime which have support on the brane given by \( x = X(\sigma^1, \sigma^2, \ldots) \). Solutions to all of the \( D \)-dim spacetime region can be extended simply by using delta functionals \( \delta(x - X(\sigma)) \).

### 2.2 Wilson Loops in terms of Composite Antisymmetric Tensor Fields

In this section we are going to study the string solutions (\( p' = 1 \)) to the rank two (\( p + 1 = 2 \)) composite antisymmetric tensor field theories of area-preserving diffs in \( D = 4 = p + p' + 2 = 2 + 2 \). The Wilson loop associated with the composite gauge field is defined:

\[ \exp [i \int_C A_\mu(\phi^a) \, dx^\mu]. \quad A_\mu(\phi) \equiv \varepsilon_{ab} \phi^a(x) \partial_\mu \phi^b(x). \]  

(2.7)

Due to the Abelian-looking form of the composite field strength (as we said earlier, the algebra of volume-preserving diffs is not abelian) one can nevertheless use Stokes law:

\[ F = dA \Rightarrow F_{\mu \nu}(\phi) \equiv \{\phi^1, \phi^2\} = \varepsilon_{ab} \partial_\mu \phi^a \partial_\nu \phi^b. \quad a, b = 1, 2. \]  

(2.8)

after using Stokes law the exponential can be written as:

\[ \exp [i \int_{\Sigma(C)} F_{\mu_1}(\phi) dx^\mu \wedge dx^\nu]. \]  

(2.9)

where the flux is evaluated through a surface \( \Sigma(C) \) whose boundary is \( C \).

If one evaluates all these quantities along the points \( x \) whose support lie on the string-world sheet \( x = X \) one may use the string solutions above to the composite antisymmetric tensor field theory given by the previous equations (5):

\[ G(\tilde{\phi}) = \Pi =^{*} F(\phi) \Rightarrow \]

\[ G^{\nu_1 \nu_2}(\tilde{\phi}) \big|_{x=X} = \Pi^{\nu_1 \nu_2}(X) = \frac{T \{X^{\nu_1}, X^{\nu_2}\}}{\sqrt{-\frac{1}{2} \{X^{\mu}, X^{\nu}\} \{X^{\mu}, X^{\nu}\}}} = \varepsilon^{\nu_1 \nu_2 \mu_1 \mu_2} F_{\mu_1 \mu_2}(\phi) \big|_{x=X}. \]  

(2.10)

where \( T \) is the string’s tension and one is using now ordinary Poisson brackets.

The quantity \( \Pi^{\mu \nu} \) is the area-conjugate momentum of the string obeying the Hamilton-Jacobi equation for the string analog of a point particle momentum. Hamilton-Jacobi equations for strings and branes have
been given in \[14\]. Using these relations above (10) allows one to rewrite the flux (after inserting the product of two spacetime epsilon tensors \(\epsilon_{\mu_1\mu_2\mu_3\mu_4}\)) as:

\[
\frac{1}{4!} \epsilon^{\mu_1\mu_2\mu_3\mu_4} F_{\mu_1\mu_2}(\phi) \epsilon_{\mu_1\mu_2\mu_3\mu_4} dx^{\mu_1} \wedge dx^{\mu_2} = G^{\nu\mu_3\mu_4}(\tilde{\phi}) d\Sigma_{\mu_3\mu_4}. \tag{2.11}
\]

For those self-dual string configurations, the following relations among the Poisson brackets are obeyed:

\[
\text{Self Dual Strings} \Rightarrow d\Sigma = \ast d\Sigma \Rightarrow \{X_{\mu_3}, X_{\mu_4}\}_{PB} = \epsilon_{\mu_1\mu_2\mu_3\mu_4}\{X^{\mu_1}, X^{\mu_2}\}_{PB}. \tag{2.12}
\]

Self-dual strings automatically obey the string equations of motion as a result of the Jacobi identities for the Poisson brackets:

\[
\{X^\nu, \{X_\mu, X_\nu\}\} = \epsilon_{\mu\nu\rho\tau}\{X^\rho, \{X_\tau, X^\nu\}\} = 0. \tag{2.13}
\]

The vanishing of the second term of the last equation is due to the Jacobi identities of the Poisson bracket. Upon evaluation of the flux through the (self-dual) string world sheet, whose boundary is \(C\), and restricting to self-dual string configurations allows finally to yield the explicit relationship between the Wilson loop for the field \(A_\mu(\phi)\) and the Dirac-Nambu-Goto string action, in terms of the string coordinates \(X^\mu(\sigma, \tau)\), and whose worldsheet boundary is \(C\):

\[
W(C) = \exp \left[ i \int_C A_\mu(\phi) dx^\mu \right]_{z=\chi} = \exp \left[ iT \int \Sigma(C) d\sigma d\tau \sqrt{-\{X^\mu, X^\nu\}\{X_\mu, X_\nu\}} \right]. \tag{2.14}
\]

since the determinant of the induced worldsheet metric as a result of the string’s embedding onto the (flat) target spacetime is:

\[
det [h_{ab}] = det [\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu] = \{X^\mu, X^\nu\}\{X_\mu, X_\nu\}. \tag{2.15}
\]

Therefore, we have proven, on-\textit{shell}, that the Wilson loop associated with the composite antisymmetric tensor field theory of area-preserving diffeomorphisms, after using Stokes law, equals the exponential of the self-dual string action (an \textit{area}) whose worldsheet boundary is \(C\).

Some important remarks are in order. Firstly, one must not confuse the physical closed string, a loop, with the rectangular Wilson loop associated with the static quark-antiquark world lines at the ends of an open string of length \(l\). It is not difficult to verify that the rectangular (Euclidean) Wilson loop spanned by the quark-antiquark world lines in a (Euclidean) time \(t_E\), the string of length \(l\) and of area \(lt_E\), does not obey the self-dual string equations of motion. Secondly, the physical loop \(C\) coincides with the spatial-boundary of a closed-string world-sheet (world-tube) associated with a closed string history. For the confinement of 3 quarks located inside a \textit{closed} string using these composite models (composite measures) based on a dynamical tension generation for strings and branes see \[15\].

The third remark is the importance of the self-dual string condition to be able to obtain and recast the exact on-shell relation for the Wilson loop in terms of the self-dual string configurations. Thus, expressing explicitly the Wilson loop in terms of self-dual string configurations via the composite-antisymmetric tensor field theories of area (volume) preserving diffs is the first important result of this work.

### 2.3 On Scale Symmetry Breaking and Confinement in Yang-Mills

A new approach to scale symmetry breaking and confinement in Yang-Mills was advanced in \[33\] by showing there is a way to achieve it at the classical level in certain models. In particular, 4-dim scale-invariant theories which contain a rank-four field strength give rise to scale symmetry breaking in Yang-Mills at the classical level as a result of the appearance of a \textit{constant} of integration in the solution to the classical equations of motion. This constant of integration amounts to introducing a \textit{preferred} scale in the equations of motion inducing an spontaneous scale symmetry breakdown.

The authors \[33\] have shown that one can rewrite the standard pure Yang-Mills action

\[
S = \int d^4x \left( -\frac{1}{4} \text{trace} F_{\mu\nu} F^{\mu\nu} \right). \tag{2.16}
\]
which is invariant under the scale symmetry: \( x \rightarrow \lambda x \) and \( A_\mu \rightarrow \lambda A_\mu(\lambda x) \) in terms of an auxiliary field \( \omega \) in the form:

\[
S = \int d^4x \left[ -\frac{1}{4} \omega^2 + \frac{1}{2} \omega \sqrt{-\text{trace } F_{\mu\nu} F^{\mu\nu}} \right].
\] (2.17)

By eliminating the auxiliary field \( \omega \) via its equations of motion and plugging its value back into the action (2-17) one recovers the original Yang-Mills action (2-16).

The scale symmetry breaking occurs when one considers instead the replacement \( \omega \rightarrow \epsilon^{\mu\nu\rho\tau} \partial_\mu A_\nu^{\rho\tau} \) in (2-17) and solves the new equations of motion obtained from the variation of the rank three antisymmetric tensor field \( A_\nu^{\rho\tau} \):

\[
\epsilon^{\gamma\delta\alpha\beta} \partial_\beta \left( \omega - \sqrt{-\text{trace } F_{\mu\nu} F^{\mu\nu}} \right) = 0.
\] (2.18)

The new equation of motion (2-18) is now solved by:

\[
\omega = \sqrt{-\text{trace } F_{\mu\nu} F^{\mu\nu}} + M.
\] (2.19)

where \( M \) is a constant of integration which has units of \( (mass)^2 \), the same units as a string tension, and induces spontaneous scale symmetry breaking.

A variation of the action (2-17) w.r.t the Yang-Mills field after plugging back the solution (2-19) obtained for \( \omega \) yields, finally, the sought-after equations of motion for the field \( A_\mu \):

\[
D_\mu \left[ (\sqrt{-\text{trace } F_{\mu\nu} F^{\mu\nu}} + M)(\frac{F_{\mu\nu}}{\sqrt{-\text{trace } F_{\mu\nu} F^{\mu\nu}}}) \right] = 0.
\] (2.20a)

These equations of motion (2-20a) can also be obtained directly from the effective Lagrangian:

\[
L_{\text{eff}} = -\frac{1}{4} \text{trace } F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} M \sqrt{-\text{trace } F_{\mu\nu} F^{\mu\nu}}
\] (2.20b)

One may notice that a restricted class of solutions to the equations of motion (2-20a) obtained from the effective Lagrangian (2-20b) exists when a preferred scale is chosen:

\[
M = \sqrt{-\text{trace } F_{\mu\nu} F^{\mu\nu}} \neq 0 \Rightarrow F_{\mu\nu} \neq 0.
\] (2.20c)

and such that the equations of motion (2-20a), subject to the constraint (2-20c), reduce precisely to the same Yang-Mills equations of motion \( D_\mu F^{\mu\nu} = 0 \), in the absence of sources. In this special case, when \( M = \sqrt{-\text{trace } F_{\mu\nu} F^{\mu\nu}} \neq 0 \), the effective action (2-20b) is minimized for the Yang-Mills solutions \( D_\mu F^{\mu\nu} = 0 \) despite that \( F_{\mu\nu} \neq 0 \). Hence, the scale \( M \) induces an spontaneous scale-symmetry breakdown and a generation of confining behaviour leading to a scalar potential that has the same form as the confining potential:

\[
V = -\frac{Mr}{\sqrt{2}}.
\] (2.20b)

When \( M = 0 \), there is no scale-symmetry breaking and no confinement occurs. For further details how this confining potential and the Cornell confining potential (obtained from the addition of Coulomb charges) is generated we refer to [33].

Now we are going to re-interpret these findings [33] in terms of the composite-antisymmetric tensor field theories (CATF) of area (volume) preserving-diffs [9] that inspired the work of [33]. The rank-two composite antisymmetric tensor field (CATF) strength may be written in terms of a symplectic (antisymmetric) two-form \( \omega_{ab} \) [9, 10] as:

\[
F^{(\text{CATF})}_{\mu\nu} = \omega_{ab} \partial_\mu \phi^a \partial_\nu \phi^b.
\] (2.21)

The next step is to relate the \( F^{(\text{CATF})}_{\mu\nu} \) in terms of string degrees of freedom associated with the large \( N \) limit of \( SU(N) \) Yang-Mills and Born-Infeld models in the quenched reduced approximation derived in [11,
Another procedure to achieve this is by invoking the string (brane) analog of wave-particle duality for point particles which has been coined brane-wave duality in the literature by [34]. For point-particles one has standard QM wave-particle correspondence:

\[ x^\mu(\tau) \leftrightarrow \phi(x^\mu). \quad \frac{\partial x^\mu}{\partial \tau} \leftrightarrow \partial_\mu \phi(x^\mu). \quad (2.22) \]

Given the relativistic constraint \( p_\mu p^\mu + m^2 = 0 \), upon quantization one recovers the Klein-Gordon equation

\[ (\partial_\mu \partial^\mu + m^2)\phi(x) = 0. \quad (2.23) \]

Thus, standard quantization is the basis for the particle-field (wave) correspondence (2.22).

The brane/wave duality [34] is just a generalization of the point particle case:

\[ X^\mu(\sigma^a) \leftrightarrow \phi^a(x^\mu). \quad a = 1, 2. \quad (2.24) \]

but **encoded** via the string (brane) kinetic terms given in terms of the Poisson brackets (Nambu-Poisson Brackets) of the string (brane) coordinates \( X^\mu(\sigma^a) \) as follows:

\[
\{X_\mu, X_\nu\}_PB = \omega^{ab} \partial_\sigma X_\mu(\sigma) \partial_\tau X_\nu(\sigma) \leftrightarrow \\
F^{CATF}_{\mu\nu}[\phi] = \omega^{ab} \partial_\mu \phi^a(x) \partial_\nu \phi^b(x). \quad (2.25)
\]

The evaluation of the Poisson bracket with respect to the variables \( \sigma^1, \sigma^2 \) requires to use the inverse \( \omega^{ab} \) (a 2 \times 2 matrix) of the symplectic non-degenerate two-form \( \omega_{ab} \) associated with a 2-dim phase space. Symplectic-diffs are area-preserving.

The above equation (2.25) expresses the composite antisymmetric tensor field strength/string correspondence by interpreting the scalar fields \( \phi^a \) as the *generalized* world-sheet variables \( \sigma^a \) associated with the 2-dim world-sheet swept by a string for \( a = 1, 2 \). Therefore, the string/wave duality expressed by eqs-(2.25) is basically a world-volume/target space duality since \( \phi^a(x) \) represent the mappings (immersions) from spacetime \( x^\mu \) to a field-space \( \phi^a \). Whereas the inverse maps from the field-space \( \phi^a \) variables to the spacetime \( x^\mu \) variables is what the maps/embeddings of the string’s world-volume into the target spacetime background represent. The latter are the maps \( X^\mu(\sigma^a) \). Hence, eq-(2.25) is just the correspondence between a two-dim area measure of the string’s world-sheet and the CATF strength \( F_{\mu\nu}[\phi] \). It can be generalized to branes as well.

Since this last *correspondence* between branes and CATF theories, based on brane-wave duality, seems too *heuristic* in the next section we will *show* rigorously why the Poisson bracket has a correspondence to the Lie-algebra commutator of the large \( N \) limit of \( SU(N) \) Yang-Mills theory in the *quenched* reduced approximation:

\[
\{X_\mu, X_\nu\}_PB \leftrightarrow [A_\mu, A_\nu]. \quad (2.26)
\]

In particular, we will see why the large \( N \) limit of Yang-Mills theory admits strings, membranes and bag excitations [11].

After this discussion about brane/wave duality, ... one still needs to find the nexus to the novel scale symmetry breaking and confinement approach to Yang-Mills by the authors in the large \( N \) limit [33]. To achieve this we must write the corresponding analogous action to the action given by eq-(2.17) in terms of the field-strength \( F^{CATF}_{\mu\nu}[\phi] \) as follows:

\[
S_{CATF} = \int d^4x \left[ -\frac{1}{4} \omega^2 + \frac{1}{2} \omega \sqrt{-F^{CATF}_{\mu\nu}F^{\mu\nu}_{CATF}} \right]. \quad (2.27)
\]

The solutions to the equations of motion of the \( \omega \) auxiliary variable in (2.27) after implementing the brane/wave duality correspondence between CATF theories and branes, become:

\[
\omega = \sqrt{-F^{CATF}_{\mu\nu}F^{\mu\nu}_{CATF}} + M \leftrightarrow \sqrt{-\{X_\mu, X_\nu\}_PB [X^\mu, X^\nu]_PB} + M. \quad (2.28)
\]
where $M$ is the constant of integration which breaks scale symmetry. From the world-volume point perspective the coordinates $X^\mu(\sigma)$ are dimensionless scalars so the Poisson bracket has units of $(\text{length})^{-2} = (\text{mass})^2$, exactly like $M$.

After implementing the brane/wave duality correspondence via the CATF theory represented by the action (2-27), the effective string Lagrangian which corresponds to the effective Lagrangian given in eq-(2-20b) is then:

$$L_{\text{eff}} = \frac{1}{4} \{ X_\mu, X_\nu \}_{PB} \{ X^\mu, X^\nu \}_{PB} - \frac{1}{4} M \sqrt{-\{ X_\mu, X_\nu \}_{PB} \{ X^\mu, X^\nu \}_{PB}}. \quad (2.29)$$

It is not difficult to verify that the equations of motion associated with the effective Lagrangian (2-29) reduce precisely to the same equations of motion for the Eguchi-Schild string action $\{ X_\mu, X_\nu, X_\rho \} = 0$ when a preferred scale is chosen so that $M \sim \sqrt{-\{ X_\mu, X_\nu \}_{PB}} \neq 0$. When the latter constraint on the "area-norms" is implemented in eq-(2-29), the equations of motion for both the Eguchi-Schild string action and (2-29) agree and are minimized for nontrivial (nonzero) areas. This is how these nontrivial areas vacuum solutions of the action associated with the effective Lagrangian in eq-(2-29) induce a preferred scale; i.e. an spontaneous scale-symmetry breaking occurs as a result of the constant of integration $M$ in (2-28) which induces confinement in the large $N$ Yang-Mills case as shown by [33].

Thus the whole point of this subsection has been to implement the results of [33] pertaining to novel approaches to scale symmetry and confinement in Yang-Mills in the large $N$ limit to the case of composite antisymmetric tensor field theories of area (volume) preserving-diffs, and their $p$-brane duals, via the brane-wave duality correspondence principle [34]. These same results could have been obtained directly by using the rigorous procedure of [11,17] based on Moyal quantization techniques where strings, membranes and bag actions were obtained from the large $N$ limit of quenched $SU(N)$ Yang-Mills and Born-Infeld actions.

The findings of this subsection are the second important finding of this work. So far eq-(2-14) has been a classical (on-shell) result expressing the Wilson loop associated with the composite antisymmetric tensor field theories of area-preserving diffs in terms of their classical self-dual string solutions eq-(2-10). It is warranted to study the full quantum theory (off-shell extension). In particular to evaluate the average Wilson loop $W(C)$ and show, in fact, that it can be expressed by a path-integral in terms of string variables.

This we shall do next by following a different route than the one based on composite antisymmetric tensor field theories (CATF) of area-preserving diffs and which is given by the large $N$ limit of Yang-Mills theories in the quenched approximation. Hoppe long ago [22] proved that the large $N$ limit (which is basis dependent, it is not unique) of $SU(N)$ is isomorphic to the algebra of area-preserving diffs of a sphere. The topology of the surface is important since other algebras like $w_\infty$ and $w_{1+\infty}$ associated with $w_\infty$ strings, higher conformal spins, are the area-preserving diffs of a plane and cylinder respectively. Since the Guendelman-Nissimov-Pacheva formulation of antisymmetric tensor field theories involve area-preserving (volume) diffs, it is very natural to study now the $SU(N)$ YM theories in the large $N$ limit and its relation to strings, membranes and bags [11].

3. Strings, Membranes and Bags from the large $N$ limit of quenched Yang-Mills

3.1 The reduced Eguchi-Kawai Model

To our knowledge, Eguchi and Kawai [23] were the first to propose a reduction of the dynamical degrees of freedom of Yang-Mills by arguing that Yang-Mills theory on a $D$-dim spacetime is equivalent, in the large $N = \infty$ limit, to a reduced model based on the action:

$$S_{\text{EK}} = \frac{1}{2g^2\Lambda} \text{trace} [A_\mu, A_\nu]^2. \quad (3.1)$$

Namely, this model amounts to a dimensional-reduction of the Yang-Mills theory to a "point", where $A_\mu$ are a collection of $D$ space-time independent matrices and $\Lambda$ is a dimensional parameter related to the inverse lattice scale.

However, it turned out that strictly speaking the Wilson-loop average:
< \frac{1}{N} \text{trace Path } e^{i \int dx^\mu A_\mu} > = \int [DA_\mu] \left[ \frac{1}{N} \text{trace Path } e^{i \int dx^\mu A_\mu} \right] e^{iS[A]}.

(3.2)

using the full-fledged Yang-Mills action $S_{YM}[A]$ in the path-integral is not equal to the average using the reduced Eguchi-Kawai action, $S_{EK}[A]$, except in $D = 2$ [24, 26].

One reason is that the shift-symmetry invariance of the reduced Eguchi-Kawai action, symmetry were the matrices $A_\mu^\gamma$ are shifted by a diagonal matrix $a_\mu \delta^{0\gamma}$, can be broken spontaneously in the large $N$ limit [24]. Such large $N$ limit plays the role of an statistical averaging and phase transitions may occur in perturbation theory of the reduced model in $D \geq 3$ [24]. It happens that the path integral measure for $D \geq 3$ is singular and the eigenvalues collapse (cluster) leading to a breakdown of the shift-symmetry in perturbation theory.

Related to the stringy picture of the large $N$ Yang-Mills reduced models, another interpretation of the shift-symmetry breakdown phenomenon has been given by Olesen [25] by showing that the dynamics of the reduced Eguchi-Kawai action is not able to suppress the higher-modes thereby resulting in a very crumpled (fractal-like) world-sheet with a very high Hausdorff dimension. A smooth string world-sheet would require a dominance of the lower-modes in the path integral. For this to occur one could introduce a quenching procedure in the Eguchi-Kawai action to be able to suppress the higher-modes. It has been shown that the equivalence between the large $N$ limit of Yang-Mills theory on a whole space and the reduced Eguchi-Kawai model is valid provided a quenching prescription is introduced [27] such that the $D$-dim planar graphs associated with the large $N$ Yang-Mills theory are truly reproduced by the reduced, quenched Eguchi-Kawai model. An alternative reduction procedure based on a twisting prescription was advocated by [32] which is also equivalent to the large $N$ Yang-Mills theory on the lattice and in the continuum.

Another alternative to quenching is to introduce supersymmetry. In the supersymmetric case there is an extra contribution in the path integral measure coming from the fermions. There is no measure-singularity in $D = 4$ for either Majorana or Weyl fermions, and in $D = 10$ for Majorana-Weyl fermions [24]. The shift-symmetry is not broken in this case and no quenching is needed in the supersymmetric case in those cases [29]. Another physical interpretation is based on the fact that supersymmetry makes the strings more smooth and removes the tachyon in the ground state (vacuum) sector. The tachyon produces crumpling [28]. It was conjectured by the authors [30] that the large $N$ reduced model of the 10-dim super Yang-Mills theory is equivalent to the Green-Schwarz superstring.

For these reasons in the next subsection we will discuss the quenching procedure of the large $N$ limit of $SU(N)$ Yang-Mills to avoid the problems of the spontaneous shift-symmetry breakdown and the crumpling of the string world-sheet due to the presence of higher-modes in perturbation theory. We will review [11] to show how the quenched large $N$ limit of the Yang-Mills action is given precisely by the Eguchi-Schild string action. Our proof (very different from those in the literature) is based on a Moyal deformation quantization procedure of the Yang-Mills action where the $N \to \infty$ limit is identified with the "classical" $\hbar = 0$ limit, by setting $\hbar = 2\pi/N$, such that the Moyal brackets reduces to the Poisson brackets in that limit [11]. The supersymmetric Yang-Mills case requires to introduce a generalized super-Moyal bracket [31] leading to the supersymmetric version of the Eguchi-Schild string in the large $N$ limit.

Thus, at the classical level, we will see that there is indeed an equivalence between the quenched large $N$ limit of Yang-Mills and the Eguchi-Schild string. The question still remains if this classical equivalence also holds at the quantum level. Question which is also related to the problems pointed out earlier about world-sheet crumpling, collapse (clustering) of eigenvalues, spontaneous symmetry breakdown of the shift-symmetry. Hence, only in the quenched-approximation, one may argue that the Wilson-loop expectation values agree when one employs the large $N$ limit of the Yang-Mills action and the Eguchi-Schild string action, respectively, in the evaluation of the path-integrals to compute Wilson-loop averages. This would require also to restrict the integration measure $[DA_\mu]$ in the path integral to the sector of spacetime-independent (zero-mode sector) $A_\mu$ field configurations, which allows us then to implement the gauge field/string coordinates correspondence $A_\mu \to X_\mu$, in the large $N$ limit, as we shall explain next.

3.2 The Eguchi-Schild String from the large $N$ limit of quenched-reduced Yang-Mills

We will briefly review [11] how the large $N$ limit of 4D Yang-Mills theory in the quenched approximation, furnishes the Eguchi-Schild string action. The quenched action reduced to a "point" in $D = 4$ is:
\[ S = - \frac{1}{4} \left( \frac{2\pi}{a} \right)^4 N g_{YM} \text{Tr} \left( F_{\mu\nu} F^{\mu\nu} \right). \]  

(3.3)

\[ \text{Tr} \ F_{\mu\nu} F^{\mu\nu} = \text{Tr} \ [A_\mu(0), A_\nu(0)] [A^\mu(0), A^\nu(0)]. \]  

(3.4)

Notice that the reduced-quenched action is defined at a "point" \( x_0 = 0 \). This is attained by neglecting the off-diagonal components of the matrices ("fast moving modes") and absorbing the full space-time dependence of the gauge fields into a unitary translational operator given by a plane-wave diagonal \( N \times N \) matrix \( U(x) = \exp[ip_\mu x_\mu] \). The \( N \) entries of the plane-wave elements along the diagonal are evaluated in terms of \( N \) distinct eigenvectors \( p_\mu^a \), \( a = 1, 2, 3...N \). Notice that the matrix \( U(x) \) is diagonal but is not proportional to the unit matrix in general. Thus, one can absorb the space-time dependence of the gauge fields as follows:

\[ A_\mu(x) = U^\dagger(x) A_\mu(0) U(x) \Rightarrow F_{\mu\nu}(x) = U^\dagger(x) F_{\mu\nu}(0) U(x). \]  

(3.5)

Due to the cyclic property of the trace and the unitary condition \( U^\dagger U = 1 \), after simple algebra the trace of the field-strength squared reduces simply to \( \text{Tr} [A_\mu(0), A_\nu(0)]^2 \). Therefore, there are no \( \partial_\mu A_\nu \) terms since one has reduced the theory to a "point", the origin \( x_0 = 0 \). For simplicity we have omitted the matrix \( SU(N) \) indices in eq-(3-5).

The Weyl-Wigner-Groenewold-Moyal (WWGM) quantization establishes a one-to-one correspondence between a linear operator \( D_\mu = \partial_\mu + A_\mu \) acting on the Hilbert space \( \mathcal{H} \) of square integrable functions in \( R^D \) and a smooth \( c \)-number function \( A_\mu(x, y) \) which is the Fourier transform of \( A_\mu(q, p) \). The latter quantity is obtained by evaluating the trace of the \( D_\mu = \partial_\mu + A_\mu \) operator summing over the diagonal elements with respect to an orthonormal basis in the Hilbert space. Under the WWGM correspondence, in the quenched-reduced approximation, the matrix operator product \( A_\mu A_\nu \) is mapped into the noncommutative Moyal star product of their symbols \( A_\mu \ast A_\nu \) and the commutators are mapped into their Moyal brackets:

\[ \frac{1}{\hbar} [A_\mu, A_\nu] \Rightarrow \{A_\mu, A_\nu\}_{MB}. \]  

(3.6)

Replacing the Trace operation with an integration w.r.t. the internal phase space variables, \( \sigma \equiv q, p \) gives:

\[ \frac{(2\pi)^4}{N^2} \text{Trace} \rightarrow \int d^2 \sigma. \]  

(3.7)

The WWGM deformation quantization of the quenched-reduced orginal actions is:

\[ S^* = - \frac{1}{4} \left( \frac{2\pi}{a} \right)^4 N g_{YM} \int d^2 \sigma F_{\mu\nu}(\sigma) \ast F^{\mu\nu}(\sigma). \]  

(3.8a)

\[ F_{\mu\nu} = \{iA_\mu, iA_\nu\}. \]  

(3.8b)

By performing the following gauge fields/coordinate correspondence:

\[ A_\mu(\sigma) \rightarrow \left( \frac{2\pi}{N} \right)^{1/4} X_\mu(\sigma) \]  

(3.9)

\[ F_{\mu\nu}(\sigma) \rightarrow \left( \frac{2\pi}{N} \right)^{1/2} \{X_\mu(\sigma), X_\nu(\sigma)\}. \]  

(3.10)

And, finally, by setting the Moyal deformation parameter "\( \hbar = 2\pi/N \) of the WWGM deformed action (3-8 a), to zero; i.e by taking the classical \( \hbar = 0 \) limit, which is tantamount to taking the large \( N = \infty \) limit, one can see that the quenched-reduced YM action in the large \( N \) limit will become:

\[ S = - \frac{1}{4g_{YM}^2} \left( \frac{2\pi}{a} \right)^4 \int d^2 \sigma \{X_\mu, X_\nu\}_{PB} \{X^\mu, X^\nu\}_{PB}. \]  

(3.11)
due to the fact that the Moyal brackets collapse to the ordinary Poisson brackets in the $\hbar = 2\pi/N = 0$ limit (large $N$ limit). Therefore, finally, we have obtained the Eguchi-Schild string action (3-11) which is invariant under area-preserving reparametrizations from the quenched-reduced large $N$ Yang-Mills theory via the WWGM quantization procedure.

The large $N$ limit of 4D Yang-Mills theory in the quenched approximation, and supplemented by a topological theta term can be related through a Weyl-Wigner Groenelwold Moyal (WWGM) quantization procedure also to a bag model, i.e., to an open domain of the 3-dim disk $D^3$ [11]. The bulk $D^3 \times R^1$ is the interior of a hadronic bag and the boundary is the world volume $S^2 \times R^1$ of a Chern-Simons membrane of topology $S^2$ (a codimension two object). Hence, we have an example where the world-volume of a boundary $S^2 \times R^1$ is the boundary of the world-volume of an open 3-brane of topology $D^3$ such $\partial(D^3 \times R^1) = S^2 \times R^1$ (setting aside the points at infinity). The boundary dynamics is not trivial despite the fact that there are no transverse bulk dynamics associated with the interior of the bag. This is due to the fact that the 3-brane is spacetime filling: $3 + 1 = 4$ and therefore has no transverse physical degrees of freedom.

To obtain the 3-brane (bag) action form the WWGM quantization of Yang-Mills, one must enlarge the two-dim phase space to a four-dim one: $q^1, p^1, q^2, p^2$ and to repeat the same procedure as before. The trace becomes now an integration w.r.t the $q^i, p^i$ variables that have a correspondence to the four world-volume $\sigma^a$ variables. The large $N$ limit of quenched Yang-Mills theory is obtained: Chern-Tsirakian action for a 3-brane (bag) in the conformal gauge [11] once the correspondence $A_\mu(\sigma^a) \to X^\mu(\sigma^a)$ is made.

This Moyal deformation approach also furnishes dynamical membranes (a QCD membrane) when one uses the spatial quenching approximation to a line (one dimension), instead of quenching to a point. In this fashion we constructed what is called a QCD membrane [11]. Basically, a Moyal quantization takes the operator $A_\mu(x^\mu)$ into $A_\mu(x^\mu; q, p)$ and commutators into Moyal brackets. A dimensional reduction to one temporal dimension (quenching to a line) brings us to functions of the form $A_\mu(t, q, p)$, which precisely corresponds to the membrane coordinates $X_\mu(t, \sigma^1, \sigma^2)$ after identifying the $\sigma^a$ variables with $q, p$. The $\hbar = 0$ limit turns the Moyal bracket into a Poisson one. Upon the identification of $\hbar = 2\pi/N$, the classical $\hbar = 0$ limit is tantamount to the $N = \infty$ limit and it is in this fashion how the large $N$ $SU(N)$ matrix model bears a direct relation to the physics of membranes. The Moyal quantization explains this in a straightforward fashion without having to use $\infty \times \infty$ matrices!

To obtain superstrings and supersymmetric branes from the large $N$ limit of $SU(N)$ supersymmetric Yang-Mills theory via deformation quantization requires the supersymmetric extension of the Moyal brackets [31] which is a nontrivial problem.

4. The Wilson-loop average via the Quenched large $N$ Yang-Mills

In this last section we will compute the vacuum expectation value of the Wilson loop in the infinite colour limit via a Moyal deformation quantization procedure. We have reviewed in the last section that a Moyal deformation quantization allows to study the large $N$ limit of $SU(N)$ YM theories in the quenched approximation leading to the Eguchi-Schild string. In addition to strings, it was also shown in [11] that $SU(N)$ reduced-quenched gauge theories admit hadronic bags and Chern-Simons (dynamical boundaries) membranes excitations in the large $N$ limit by enlarging the two-dim phase space to a four-dim one and after identifying the four phase space coordinates with the four world-volume coordinates. This Moyal deformation approach also furnishes dynamical membranes (a QCD membrane) when the quenching is performed along a line, instead of a point [11].

These results can be extended to more general $p$-brane actions given by Dolan-Tchirakian (Skyrme type actions) starting from Generalized Yang Mills theories in the large $N$ limit; i.e, branes are roughly speaking Moyal deformations of Generalized Yang-Mills theories [11]. In particular, it was later shown how Nambu-Goto strings can be obtained directly from $SU(N)$ Born-Infeld models in the large $N$ limit [17].

To illustrate the power of these approaches we will show how one can obtain the celebrated Maldacena relation relating the size of the $AdS_5$ throat $\rho$ to the $'t$ Hooft coupling $N_{YM}^2$ and the Planck scale $L_{Planck}^4 \sim (\alpha)^2$ (the inverse string tension squared) from a Moyal deformation approach to quenched-reduced large $N$ QCD. The bag constant, $\mu$, of mass dimension, was related to the bag tension as [11]:

$$L_{Planck}^4 \sim (\alpha)^2$$
\[ T_{\text{bag}} = \mu^4 \sim \frac{1}{a^4 g_{YM}^2}. \quad (4.1) \]

where \( a \) was related to the lattice spacing of the large \( N \) quenched, reduced QCD given by \( (2\pi/a) = \Lambda_{\text{QCD}} = 200 \text{ Mev}. \)

Based on the known result that a stack of \( N \) coincident \( D3 \) branes ( whose world volume is four-dimensional ) in the large \( N \) limit is related to black \( p = 3 \) branes solutions to closed type \( IIB \) string theory in \( D = 10 \), and whose near-horizon geometry is given by \( AdS_5 \times S^5 \), one may set the lattice spacing \( a \) associated with large \( N \) quenched, reduced \( SU(N) \) YM in terms of the Planck scale \( L_P \) to be \( a^4 = NL_P^4 \).

This merely states that we are setting the hadronic bag scale to be \( a = N^{1/4} L_P \). Inserting this relationship into the expression for the bag tension gives:

\[ T_{\text{bag}} = \mu^4 \sim \frac{1}{a^4 g_{YM}^2} = \frac{1}{NL_P^4 g_{YM}^2} \Rightarrow \mu^{-4} \sim (N g_{YM}^2) L_P^4. \quad (4.1) \]

which has a similar form as the Maldacena relation if one identifies the size of the \( AdS_5 \) throat to the bag scale \( \mu^{-1} \).

Finally, we turn now to the average Wilson loop which is defined:

\[ < W_A[C] >_{vev} = \int [DA] W_A(C) e^{iS_{YM}[A]}. \quad (4.2) \]

The Wilson loop for \( SU(N) \) YM is defined as:

\[ W[C] = \frac{1}{N} \text{trace Path exp}[i \int_C A_\mu dx^\mu]. \quad (4.3) \]

In the quenched-reduced approximation, defined at a "point", the Wilson loop shrinks to zero size \( C \rightarrow 0 \) and hence the exponential reduces to unity since the integral has collapsed to zero. So then we get

\[ W[C] \rightarrow W[C = 0] = \frac{1}{N} \text{trace } 1_{N \times N} = \frac{N}{N} = 1. \quad (4.4) \]

Notice how important the factor of \( 1/N \) is in eq-(4-4) to cancel the \( N \) factor stemming from taking the trace of the unit \( N \times N \) matrix \( 1_{N \times N} \).

As we have shown above in eqs- (3-10, 3-12 ) the quenched-reduced YM action in the large \( N \) limit becomes the Eguchi-Schild action for the string after using the \( A_\mu(\sigma) \rightarrow X_\mu(\sigma) \) correspondence via the WWGM quantization method. This gauge field/string coordinate correspondence given in eq-( 3-11 ) has been known for some time [18] but it was based on different arguments than the WWGM quantization in phase spaces [11]. Thus, we have the following results:

\[ [DA]_{\text{quenched}} \rightarrow [DX] \quad W(C) \rightarrow W(0) = 1. \quad e^{iS_{YM}}_{\text{quenched}} \rightarrow e^{iS_{\text{string}}} \quad (4.5) \]

Under these conditions the quenched-reduced \( SU(N) \) Yang-Mills in the large \( N \) limit allows to compute the vacuum expectation values ( vev) of the Wilson loop purely in terms of string degrees of freedom using the Eguchi-Schild action for the string, the square of the Poisson brackets, which is area-preserving diffs invariant; i.e. in the quenched-reduced large \( N \) Yang-Mills, the spacetime-independent sector ( zero modes ) of the gauge fields \( A_\mu \) have a one-to-one correspondence to the string coordinates \( X_\mu \) as explained earlier in the previous section, leading finally to:

\[ < W[C] >_{\text{quenched}} = \int [DA]_{\text{quenched}} W(0) e^{iS_{YM}(A)} = \int [DA]_{\text{quenched}} e^{iS_{YM}(A)} = \int_{\Sigma(C)} [DX] e^{iS_{\text{string}}} \equiv \Psi_o[C]. \quad (4.6) \]

the state \( \Psi_o[C] \) is the vacuum wave functional representing the creation of a string ( a loop ) from the vacuum ( a point ) and sweeping in the process a world-sheet \( \Sigma \) whose boundary is the loop configuration.
C. The path integral involves a summation over all string embeddings $X$ subject to the condition that the boundary of $\Sigma$ is $C$. We have not included the bag-action in (4-6) in the evaluation of the Wilson-loop averages in large $N$ limit since the bag action is devoid of bulk transverse degrees of freedom. The bag in 4-dim is spacetime filling giving a trivial action equal to the four-dim spacetime-volume of the bag. For this reason we concentrate solely on the string-degrees of freedom (areas) in (4-6). This last expression (4-6) relating the vev of the Wilson loop of the large $N$ limit of $SU(N)$ Yang-Mills, in the quenched approximation, in terms of string degrees of freedom (areas) is the third important result of this work.

The physical meaning of this relation can be envisaged as follows. As we shrink the Wilson loop to a point, the subsequent large $N$ limit procedure amounts to introducing an extra dependence on the phase space variables $(q,p)$, that later are identified as the string coordinates. The $SU(N)$ fiber sitting at the point $P$ becomes the area world-sheet of the string in the large $N$ limit. Hence the Wilson loop which had initially shrunk to a point re-emerges as an internal loop living in the $SU(N)$ fiber that was sitting at the point $P$. This is compatible with the area-preserving diffs invariant nature of the Eguchi-Schild action. Roughly speaking, since areas are preserved, as we shrink the Wilson loop to a point (to zero) it must re-emerged along the fibers in order to preserve the area.

Hence we have obtained an exact result consistent with those given in the literature since (by definition) the vacuum wave-functional $\Psi[C]$, appearing in the r.h.s, is defined by a path integral over all world-sheets whose boundary is $C$. The latter is the quantum amplitude for a closed string to emerge from the vacuum (a "point") and sweep a world-sheet whose boundary is $C$. The topology is given by a disc. A perturbative evaluation of the path integral requires summing over surfaces of all genera. For more general actions one must restrict the measure of integration modulo the volume of the world-sheet diffs group and the group of Weyl diffs for Polyakov-Howe-Tucker type of actions.

Nonperturbative effects are never seen in perturbation theory, like the contribution of self-dual string configurations (Euclideanized world-sheet). The latter have a dominant weight in the path integral since they saturate the lower bound of the (Euclidean) action. Hence it is not too surprising that the on-shell value of the Wilson loop for composite antisymmetric tensor field theories was given by the exponential of the self-dual string action.

The propagator from one loop configuration $C_1$ to another $C_2$ is given by the path integral:

$$\Delta(C_1,C_2) = \int_{C_1}^{C_2} [DX] \exp \{iS_{\text{string}}\}. \tag{4.7}$$

where the path integral involves summing over all surfaces $\Sigma(C_1,C_2)$ bounded by the two loops $C_1,C_2$.

In [19] an explicit expression for the string representation of a quantum loop $C$, in terms of a full phase space string path integral based on the Eguchi-Schild string action, was given. The wavefunctional was a complicated expression: $\Psi[x,A,\sigma^{\mu\nu}]$, where $\sigma^{\mu\nu}(C)$ were the holographic area projections of a loop onto the coordinate planes; $A$ was the Eguchi temporal-area variable associated with the closed loop and $x$ were the coordinate variables of the loop boundary. What was left open was to see what type of Loop wave equations the functional $\Psi[x,A,\sigma^{\mu\nu}]$ obeyed. Loop wave equations for closed-strings and closed $p$-branes were given in [20]. Dirac-like wave equations were also obtained by a suitable "square-root" procedure which generalized the Hosotoni string Dirac-like equations [21].

One can extend the loop wave equations [20] to C-spaces (Clifford manifolds) [12] by writing the more general loop equations for a nested family of $p$-loops (closed-$p$-branes) $p = 0,1,2,3,\ldots,p_{\text{max}}$ where the maximum value of $p_{\text{max}}$ corresponds to a spacetime "filling" brane such that the $p_{\text{max}} + 1$-dim world-tube (a cylinder) swept by the closed $p_{\text{max}}$-brane is embedded into a $D$-dim target spacetime background, where $D = p_{\text{max}} + 2$. The interior region of the $p_{\text{max}} + 1$-dim world-tube (cylinder) has $p_{\text{max}} + 2 = D$ dimensions.

The wavefunction:

$$\Psi[\Omega,x^{\mu};\sigma^{\mu\nu},\ldots] \tag{4.8}$$

is a Clifford-algebra valued object (a polyvector) where the Clifford-valued coordinate is defined as:

$$X = \Omega \Gamma^\mu + x^\mu \gamma^\mu + \sigma_{\mu\nu} \gamma^\mu \wedge \gamma^\nu + \ldots \tag{4.9}$$
after setting the Planck scale to unity that is required in order to combine objects of different dimensionality [12 ].

The C-space wave equations is:

\[
\frac{\delta^2 \Psi}{(\delta \Omega)^2} + \frac{\delta^2 \Psi}{(\delta x_\mu)^2} + \frac{\delta^2 \Psi}{(\delta \sigma_{\mu\nu})^2} + \frac{\delta^2 \Psi}{(\delta \sigma_{\mu\nu\rho})^2} + \ldots = E^2 \Psi. \tag{4.10}
\]

The more fundamental problem is to see if these C-space wave equations have a direct relation to the Maakenko-Migdal loop equations in the infinite colour limit of YM. The fact the the wave equations in C-spaces incorporate automatically the closed p-brane/ p-loops holographic coordinates is compatible with the AdS/CFT correspondence. The loop transform (the analog of the Fourier transform) is defined:

\[
\Psi[C] = \int [DA] \left[ \exp \left[ i \oint_C A_\mu dx^\mu \right] \right] \Psi[A]. \tag{4.11}
\]

Conversely, the inverse loop transform will yield \( \Psi[A] \) in terms of \( \Psi[C] \). The main question is to see if the loop equation obeyed by \( \Psi[C] \) agrees in with those given in [20].

One can generalized these results to p-loops \( C_p \); i.e a closed p-brane enclosing a \( p+1 \)-dimensional region \( \Sigma_{p+1}(C_p) \). The p-loop transform is defined in terms of an integral involving the rank \( p \) antisymmetric tensor field \( A_p \):

\[
\Psi[C_p] = \int [DA_p] \exp \left[ i \oint_{C_p} A_{\mu_1 \mu_2 \ldots \mu_p} d\Sigma^{\mu_1 \mu_2 \ldots \mu_p} \right] \Psi[A_p]. \tag{4.12}
\]

The \( p \)-rank antisymmetric tensor couples to the p-brane. One can follow a similar procedure to evaluate the average of the generalized Wilson loop; \( W_{A_p}[C_p] \) and express it as a path integral over a \( p \)-brane action such that the boundary of the \( p+1 \)-dim region associated with the p-brane's worldvolume is given by \( C_p \).

This remains a difficult problem, more details about this will be given in a future publication.

Acknowledgements

We wish to thank the referee for very valuable insights and suggestions to improve the paper. To T. Smith and J. Mahecha for their invaluable help and to M. Bowers, J. Boedo for their hospitality in California where this work was completed.

References

3-S. Ketov, "Quantum Nonlinear Sigma Models" (Springer Verlag 2000).
7-V. Ivanova, N. Troitskaya, "On the Wilson loop in the dual representation within the dual Higgs model with dual Dirac strings" [arXiv: hep-th/0112060].


