Conformal transformations near Naked Singularities - I

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Abstract

We show that the behaviour of the outgoing radial null geodesic congruence on the boundary of the trapped region (suitably defined as a four dimensional region) is related to the property of nakedness in spherical dust collapse. The argument involves a conformal transformation which justifies the difference in the Penrose diagrams in the naked and covered dust collapse scenarios.

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1 Introduction

Consider a cloud of matter (regular initial Cauchy data) collapsing indefinitely under its own gravity. A singularity eventually develops in the spacetime and it is indicated by the divergence of the Kretschmann scalar. In the advanced stages of collapse trapped regions are formed [1], [2] and there exists a null ray which marginally escapes to infinity (event horizon). It is not clear whether the singular boundary is entirely surrounded by the trapped region. In other words, it is not known if a portion of the singular boundary is exposed in the untrapped region and non-spacelike geodesics can emanate from it (naked singularities). In fact, exact solutions to Einstein’s field equations with certain kind of source terms are known to exhibit both naked and covered singularities depending upon the sort of regular initial data chosen (See [3] for details).

The complexity of the general problem lies in the fact that the Cauchy initial value problem for the Einstein field equations with sources is less tractable. The systematics available about the initial value problem is far too small for any implication for questions like formation of naked singularities. For instance, even the well-posedness of the problem is not self evident and has to be proved independently for different types of sources [4], [5], [6]. It would be indeed difficult, for example, to find or even expect a conserved or monotonically behaving function of the Cauchy surface with respect to its evolution, which could be expected to provide insight into the process of creation of naked singularities.

As a result of this difficulty, a large number of investigations that have been carried out, have been concerning certain exact solutions or numerical simulations ([3] and references cited in them). Issues like strength of the singularities, genericity, behaviour with respect to change of source etc. have been studied in examples like dust, null dust, perfect and imperfect fluids and scalar fields. However, they do not suggest any typical geometrical feature which could be expected to arise before a naked singularity forms (In case of a singularity the singularity theorems make use of a typical geometrical feature viz. trapped surfaces to prove its existence). The lack of indication of existence of such a feature is evident in the fact that one is forced to check for the existence of naked singularities in a direct manner whenever required. To be precise, one simply checks if non space-
like geodesics emerge from the singular boundary using differential geometry and the form of the metric in the example.

This paper is a first step towards an indirect criterion. Preferably, the criterion should be applicable away from the singular boundary. That is a non-local problem and given the difficulties of Cauchy evolution, there is no indication available for what the criterion could be. It would be perhaps appropriate in such a situation to examine regions near the singularity. There are also parts of the singularity from which geodesics cannot escape. Some criterion applicable to such a portion would also be significant. It would indicate that the information about the exposure of a part of the boundary is contained elsewhere on the boundary.

This work is the first part in which the example of spherical dust collapse is considered. This example is well studied and the initial data which leads to naked singularities is known.

2 Spherical Dust Collapse

We illustrate the main features of spherical dust collapse in radial coordinates (figures 1 and 2) and in Penrose diagrams (figures 3 and 4).

Point O is defined as the point where the apparent horizon\(^2\) meets the singularity. In figure 3, there is no portion of the boundary beyond O that is exposed. In figure 4, however, there is a null portion. In this paper, we examine if O would yield the information about the existence of an exposed portion. O is a covered point of the singular boundary (with or without a naked portion). Mathematically, one works with points on the apparent horizon in the approach to the concerned point O. This could be looked upon as a property of the marginal trapped regions which constitute the apparent horizon. One is therefore working with outgoing null congruences with zero expan-

\(^2\)In this work we define a trapped region as the union of all possible closed trapped surfaces (See [5] or [4] for the definition of a closed trapped surface). This is in general a four dimensional region. The boundary of this region within spacetime is referred to as ‘apparent horizon’ in this work. However, according to the usual definition (See [5] and [4]), a section of this boundary by a Cauchy surface would be called an apparent horizon. We make this change for lack of established terminology for the ideas we use (which are independent of any Cauchy surface). See also figure 60 of [5] and [10].
If a singularity meets such a region, the point in the strict sense will not have any non-spacelike geodesics emerging from itself and will therefore be covered.

We proceed as follows. Consider a congruence of outgoing null
geodesics. Let us parametrize each geodesic using an affine parameter. We investigate the tangent vectors $\xi$ to geodesics in the congruence at the apparent horizon ($\xi^\mu = dx^\mu/ds$ where $s$ is an affine parameter along the null geodesic of the congruence).

$O$ is a part of the singularity which is marginally trapped (expansion of any outgoing congruence will be zero here) and so there will be no outgoing geodesic emerging from $O$. Any metric which confirms to that is referred to in this paper as having the ‘correct’ causal structure at the boundary.

In the limit of approach to $O$, the vector $\xi$ is expected to vanish if the metric used depicts the appropriate causal structure at $O$. $O$ is a part of the singularity which is marginally trapped (expansion of any outgoing congruence will be zero here) there will be no outgoing geodesic emerging from $O$. Any metric which confirms to that is referred to in this paper as having the ‘correct’ causal structure at
the boundary. We examine this in the case of spherical dust collapse where the metric is evolved with regular initial Cauchy data. It is found that the condition is met at O for the collapse leading to covered singularities. However, the tangent vector stays non-zero in the limit for the naked case. This is remedied by performing a conformal transformation which diverges at O. The appropriate causal structure is then restored. Thus, in case of spherical dust collapse, this behaviour of the tangent vector is related to the property of nakedness of the singularity. Could this be true in general? It would mean that a diverging conformal transformation is necessitated at the singular boundary in the naked case as against the covered case. Some theorems in the case of general collapse are provable and work is in progress.

Thus, it is suggested that in a general collapse, the tangent vector field of the congruence on the apparent horizon (precisely, its limiting behaviour at the trapped part of the singularity) contains the information about whether the singular boundary has a naked portion.

The plan of the rest of the paper is as follows. We wish to investigate the tangent vector on the apparent horizon. We choose the
radial null geodesic congruence in spherical symmetry. The general expression for tangent vector is not available in a closed analytic form. However, one can calculate it in a simpler special case of dust, the self similar model, and subsequently show how to generalize the results. The next section describes the self similar dust model, where we discuss the tangent vector and demonstrate the connection with nakedness. In the next section we show that the results can be extended to the general dust case. We then turn to the mentioned conformal transformation in the next section. It leads to the Penrose diagram for the naked case from the usual metric of Tolman and Bondi in spherical co-ordinates. We show that the conformal transformation should diverge at the singular boundary. This results from the fact that the tangent vector ought to vanish at O in the correct causal metric.

3 Self Similar Dust Model

The collapse of a spherical cloud of pressureless fluid is given by the following metric [7].

\[
\begin{align*}
\text{ds}^2 &= \text{dt}^2 - \frac{R'^2}{1 + f(r)} \text{dr}^2 - R^2 d\Omega^2
\end{align*}
\]

where t and r are the co-moving time and radial co-ordinates respectively. \(R(t, r)\) is called the ‘area radius’ and a closed expression for this quantity which results in the dust case has enabled substantial progress in understanding the model.

Two free functions arise viz. \(F(r)\), called the mass function since it is the total mass to the interior of a shell of radius \(r\) and the total energy function \(f(r)\) which is called so because of the constraint below which resembles a relation between kinetic and gravitational potential energies of a shell.

\[
\dot{R}^2 = \frac{F}{R} + f
\]

The source energy momentum tensor is \(\text{diag}[\rho(t, r), 0, 0, 0]\).

The solution for \(R\) mentioned above is

\[
t - t_0(r) = \frac{-R^{3/2}}{\sqrt{F}} G \left( \frac{-Rf}{F} \right)
\]
where a singularity boundary is formed at \( t = t_0(r) \). The function \( G \) is defined as follows.

\[
G(y) = \begin{cases} 
\arcsin\left(\frac{\sqrt{1-y}}{y}\right), & 1 \geq y > 0, \\
\frac{2}{3}, & y = 0, \\
\arcsinh\left(\frac{\sqrt{1-y}}{-y}\right), & 0 > y \geq -\infty.
\end{cases}
\]

The central shell focussing singularity, which is the limit as \( r \to 0 \) along this locus is of interest and turns out to be naked for some initial data.

The self-similar model is the one in which \( F(r) = \lambda r \) where \( \lambda \) is a constant (which decides if the central singularity will be naked or not) and \( f(r) = 0 \). \(^3\)

We choose the scaling \( t_0(r) = r \). A self similar co-ordinate \( z = t/r \) is introduced. We note the expressions for \( R \) and \( R' \) which will be useful in the subsequent analysis.

\[
R = r\lambda^{-2/3} (3/2 (z - 1))^{2/3}
\]

\[
R' = \left(\frac{2\lambda/3}{z - 1}\right)^{1/3} \left(\frac{z - 3}{2}\right)
\]

We cast the metric into double null co-ordinates. It is not difficult to show that

\[
ds^2 = r^2 \left( z^2 - R'^2 \right) \ du \ dv
\]

where

\[
du = \frac{dr}{r} + \frac{dz}{z - R'(z)}
\]

\[
dv = \frac{dr}{r} + \frac{dz}{z + R'(z)}
\]

The double null form \((ds^2 = C^2(u,v)dudv)\) turns out to be useful when affine parameters along null geodesics are to be calculated. For instance, along an outgoing radial null geodesic \((du = 0)\), the affine parameter is \( \int_{u=\text{constant}} C^2 dv \) up to a multiplicative and an additive constant.

Now let us turn to calculating the tangent vector to the outgoing null radial geodesic congruence, which is our primary interest.

\(^3\)The model is referred to as self similar since there exists a homothetic Killing field \( t \frac{\partial}{\partial t} + r \frac{\partial}{\partial r} \).
Assume the vector to be of the form

$$\xi = (Q(t,r), Q(t,r)\sqrt{1 + f/R'}, 0, 0) \quad (9)$$

where $Q$ is obtained from the geodesic equation which $\xi$ has to satisfy. That constraint turns out to be

$$Q\dot{Q} + QQ'\sqrt{1 + f/R'} + Q^2 \dot{R}'/2R' = 0 \quad (10)$$

We have provided the expressions for the most general dust case here. One may read off the expressions for the self similar case by setting $f$ to zero and using equation (5) for $R'$.

The equation above takes the form

$$1/Q = \int_{u=\text{constant}} \frac{\dot{R}'}{2R'} \, dk + A(u) \quad (11)$$

where $k$ is an affine parameter along outgoing radial null geodesics and $u$ is the retarded null co-ordinate. $A$ is an arbitrary function of $u$ resulting because of the partial integration.

As indicated earlier,

$$dk = r^2 (z^2 - R'^2) \, dv \quad (12)$$

keeping $u$ fixed.

Using this in equation (11) we obtain

$$1/Q = \int_{u=\text{constant}} \frac{r}{3} \frac{(z - 3)^2}{z^2 - 1} \left[ \frac{2\lambda/3}{z - 1} \right]^{1/3} \, dz \quad (13)$$

or using the fact that $du = 0$ from equation (7) \footnote{The expression for $r$ thus obtained in terms of $z$ has a positive multiplicative constant. Whether $r$ vanishes or not (and later whether $1/Q$ vanishes or is non-zero) is of importance in the calculations. So setting this constant to 1, which is done here, will not affect the result.}

$$1/Q = \int_{u=\text{constant}} \frac{1}{3} \frac{(z - 3)^2}{z^2 - 1} \left[ \frac{2\lambda/3}{z - 1} \right]^{1/3} e^{\left[-\int \frac{db}{b-R'(u)}\right]} \, dz \quad (14)$$

The integral over $z$ is to be evaluated from $r = 0$ to the apparent horizon, where we shall be interested in evaluating the tangent vector.
The latter can be shown to be the curve $R = F$ and turns out to be the locus $z = 1 - 2\lambda/3$.

The integral being over $z$, it is important to know which value of $z$ along the outgoing null curve yields $r = 0$, the lower limit of the integral. This issue as it is shown further leads to the difference in the behaviour of $Q$ in the naked and covered cases.

Consider then the equation

$$r = e^{-\int \frac{db}{b - R'(b)}}z$$

which we examine for $r = 0$. That will happen when the integral in the square bracket diverges positively. Two cases can be immediately seen to arise.

Case(i) $b - R'(b) = 0$ has no real root.

The integrand therefore does not diverge anywhere and also remains positive (or entirely negative) all over the real line. It can be checked that $b - R'(b) > 0$ for any one real $b$ which would be sufficient to claim that the integrand is positive. Also, $b - R'(b)$ is bounded since $b$ is to be limited to the nonsingular region $z < 1$ ($z = 1$ is the singularity curve itself). So, in order that the integral diverge, the range of integration should be infinite. We have chosen to limit the final point to the apparent horizon $z = 1 - 2\lambda/3$ and hence the initial point must be $z = -\infty$.

In fact a shorter intuitive argument is possible. It is known from the Tolman Bondi dust model that central singularity forms at $(t_0(0),0)$. If one assumes the Penrose diagram for the covered case (which is indeed what this case turns out to be), the null rays crossing the apparent horizon begin at the centre at $t < t_0$ which makes $z = -\infty$ there.

Case(ii) $b - R'(b) = 0$ has at least one real root.

In this case, the equation implies that $r = 0$ at the value of $z$ for which the integral in the exponent diverges. The range of integration for equation (14) would then be limited at the lower end by that value of $z$. This will be the root (in fact the one closest to $1 - 2\lambda/3$, the apparent horizon).

That this is so is seen as follows. Equation (15) can then be recast using an expansion for the integrand in the exponent as outlined below.
Expanding $R'(b)$ using the Taylor series about the root (called $z_-$) it can be shown that the leading order behaviour of $r$ is as follows

$$r = (z - z_-)^{1/\alpha} + O(z - z_-)$$  \hspace{1cm} (16)

where

$$\alpha = \left(\frac{dR'(b)}{db}\right)_{b=z_-}$$  \hspace{1cm} (17)

(It can be easily checked that $\alpha < 0$)

At $z = z_-$, therefore, $r$ vanishes.

Thus in conclusion of this analysis, we note that the lower limit of integral for $1/Q$ differs. It is $-\infty$ when $R'(b) - b = 0$ can never have a real solution and is the root (closest to apparent horizon) when a solution exists.

This observation plays the key role in further analysis. Making note of this consider equation (14). Analyzing the various factors in the integrand one finds that the integrand would diverge if $z = -1$ ( $z \neq 1$ since we are not on the singular boundary).

$z$ will take the value $-1$ in case i. In case ii, the following takes place. Consider $b - R'$ using equation (5). It is easy to see that $b - R' < 0$ for all $b < 0$. So, the root $b_-$ cannot be negative. Hence it is certainly greater than $-1$. Thus $z$ cannot take the value $-1$ in case ii in the integral for $1/Q$.

Thus the integrand diverges as $1/(z + 1)$ in case i and is finite in case ii.

Expanding the rest of the integrand factor in a Taylor series about $z = -1$, one can easily check that the integral diverges logarithmically in case i while staying finite in case ii.

Thus, $Q$ vanishes in case i and stays non zero (and finite) in case ii.

Returning now to equation (9), we can now see that $\xi$ behaves in different ways in case i and case ii on the apparent horizon, in particular as one approaches the point O on the Penrose diagrams shown (figures 3 and 4)\(^5\). It can be checked that the factor $\sqrt{1 + f/R'}$ tends to is a non-zero finite quantity on the apparent horizon.

So, $\xi$ vanishes in case i and tends to a non-zero quantity in case ii.

\(^5\)Figure 4 is a diagram of a locally naked singularity. The self similar cloud which we examine here turns out to be globally naked. However, the structure near O is the same as any locally naked case and figure 4 can be used.
Thinking of $\xi$ as ‘flux density’ of the congruence, we may interpret that the congruence tends to cluster in case ii as against case i.

From previous analysis of naked singularities (self similar cases) using analysis for emergence of geodesics (roots analysis), it can be checked that case i corresponds to the covered case and case ii corresponds to the naked singular metric.

4 Extension to the general dust case

In the general dust case, the equation (10) yields no closed analytic solution which would have clearly been useful. However, we note that we are interested only in the behaviour of $Q$ in the limit of approach to point O on the apparent horizon.

To this end the following observation plays an important role. It is shown that given a dust solution, one can construct a modified dust solution (modified distribution) which in a suitable limit approaches the given dust solution [8]. The key result that makes this construction useful is that it is proved that naked modified distributions reproduce naked dust solutions given and covered modified distributions reproduce covered ones. One can then work with the modified distribution for the given dust solution and take the limit which preserves naked or covered nature. We outline the construction in [8] below

a) Marginally bound case ($f = 0$)

Imagine a shell of radius $r_c$ in the given Tolman Bondi dust model. Replace the interior of the shell by a self similar dust metric, matching the first and second fundamental forms at the interface $r = r_c$. It can be shown that this restricts the self similarity parameter $\lambda$ which appears in the mass function. This specifies the self similar solution completely. Now taking the limit as $r_c$ tends to zero, one can show [8] that the matching constraint does imply that the interior self similar solution stays naked in the limit if the original dust solution was naked and likewise in the covered case.

b) Non Marginally bound case ($f \neq 0$)

The construction is similar in this case except for an additional interface. Two shells, $r_c1$ and $r_c2$ (say $r_c1 < r_c2$) are now considered. To the interior of $r_c1$, we replace by a self similar metric. Between the two shells, we replace with a dust portion having $f$ so behaved that it increases smoothly from zero at $r_c1$ to $f(r_c2)$ of the original
dust metric. The $F$ function for this extra portion of dust however is the same as that of the original dust metric. We match the first and second fundamental forms at each of the interfaces. As before, this can be shown to constrain the interior self similar solution uniquely given $r_c \leq 1$ and the original dust solution. Again, the property of being naked or covered is preserved in the limit ($r_c \rightarrow 0$) like the previous case [8].

We now consider $Q$ in the modified distribution for any given dust solution. In the self similar part of the latter, results of the previous section apply. Since the congruence of outgoing geodesics is smooth, so is $Q$. This makes $Q$ continuous across the interface/s in the modified distribution. Now imagine the given dust solution as the limiting case of the modified distribution. In the limit of approach to point O on the apparent horizon, one has to evaluate $Q$ in the self similar part. Because of continuity of $Q$, the same behaviour will continue to hold in the limit of the interface/s tending to zero when the original dust solution is reproduced. Making use of the fact that the property of being naked or covered is preserved in this limit, one concludes that the behaviour of $Q$ (and hence that of $\xi$)in the self similar naked and covered cases continues to hold in the general dust scenario as well.

5 Conformal transformation and Penrose diagram

The issue about the metric being causally appropriate is related to the behaviour of $\xi$ by the simple argument below.

The point O being the intersection of the apparent horizon and the singular boundary implies that any outgoing null congruence ought to have zero expansion in the limit of approach to O. So O ought not to have any outgoing geodesics emanating from itself. Hence $\xi$ ought to vanish if this causal property is correctly reflected in the metric. If the metric is not the correct one, then one performs a conformal transformation which diverges at O. This makes the $\xi$ vanish and yields the causally correct metric.

We now show that the under a conformal transformation which diverges in the limit of O, $\xi$ which tends to a non-zero limit transforms to a vector field which vanishes in the limit.

Recall that we defined $\xi$ for any geodesic congruence using an affine
parametrization. Under conformal transformations, affine parameters along null geodesics change (unlike timelike geodesics which do not remain geodesic curves, null geodesics do stay so provided the affine parameter changes appropriately). Infinitesimal parameter $ds$ transforms to $\Omega^2(x^\mu) ds$ [4], where $\Omega^2$ is a conformal transformation. Thus it is obvious that $\xi^\mu = dx^\mu / ds$ if finite and non-vanishing in the limit will vanish under $\Omega^2$ transformation provided the latter diverges there.

Thus we find that in the dust case, one requires a conformal transformation which diverges on the apparent horizon in the limit of approach to the singularity in the naked case as against the covered case where the usual metric given in equation (1) is appropriate to describe the singularity structure.\(^6\)

This justifies the difference in the structure of the singular boundary near O in figures 3 and 4. In the naked dust case, when one uses the metric used in the previous sections (figure 2), it can be shown that several radial null geodesics appear to emerge from the central singularity with the same tangent vector [9]. Thus the limiting behaviour is not correct for the set of outgoing radial null geodesics to form a congruence. i.e. the condition that there exists only one curve of the set passing through any point is violated in the limit of approach to the central singularity. The causally appropriate structure of the central singularity in the naked case is therefore not a point, but a one dimensional curve. The diverging conformal transformation does precisely that leading from figure 2 to figure 4. The central singularity being transformed to the one dimensional naked singular portion shown) Thus this naked point on the central singularity is depicted as a curve, to obtain the Penrose diagram. No such divergence is required, in the covered case while going from figure 1 to figure 3 as is also implied by the tangent vector vanishing at O in the usual Tolman Bondi metric of figure 1.

\(^6\)It is evident from the definitions that the difference in the behaviour of tangent vector is due to difference in the metrics, one being conformally related to the other. This raises the question of the tangent vector being an appropriate characterization of the causal structure. It certainly is not appropriate at any event within the spacetime but in the limit of approach of the boundary its behaviour indicates if a metric with the correct limiting causal structure has been employed in its calculation.
6 Arbitrariness in the definition of $\xi$

As defined earlier, $\xi$ is a tangent vector, defined using an affine parameter, to a congruence of null geodesics the congruence being outgoing to a marginal trapped surface in the apparent horizon. Thus, the $\xi$ field is not unique. The arbitrariness is due to two reasons. Firstly, the congruence is not uniquely defined. Secondly, the affine parameter used in the definition is arbitrary up to a multiplicative and additive constant.

In spherical symmetry, the first difficulty is remedied by choosing a null congruence which is radial on the apparent horizon. This choice is not possible in general. It will be shown in the sequel to this paper that vanishing (or nonvanishing) of the vector field at $O$ is independent of the choice of the null congruence.

The second difficulty brings in an arbitrariness of a scalar multiple of the vector field. This scalar is constant along any null geodesic of the congruence. In the approach to the boundary point $O$, we claim that this will not affect the vanishing (or nonvanishing) of the field. The argument is as follows.

Consider a vector field $\xi_1$ which does not vanish at any boundary point $P$. Let $\phi(x)$ be the value of the scalar at any point $x$ on the manifold which multiplies this vector field (because of the multiplicative arbitrariness of the affine parameter). $\phi$ is constant along any null geodesic of the congruence. We assume that $P$ has a neighbourhood which intersects the physical spacetime (spacetime without boundary) in a convex normal neighbourhood. So, $\xi_1$ not vanishing at $P$ would imply the existence of a null geodesic beginning from $P$\footnote{This is similar to Thm 8.2.1 of [4]}. Now, if $\phi$ vanishes at $P$ then it should vanish all along the portion of the null geodesic in the convex normal neighbourhood. This would make the multiplicative scaling of the affine parameter zero all along the null geodesic, which is impossible. So $\phi$ cannot vanish in the approach to $P$. This completes the argument.

7 Summary and Conclusion

The tangent vector field to an affinely parametrized null geodesic congruence is examined for behaviour on the apparent horizon (defined...
as the boundary of the four dimensional trapped region as in the first section) in the approach to the singularity (point O) in the dust collapse model. There is a correlation with the property of nakedness with this behaviour. Demanding that the vector vanishes at the covered point O forces the divergence of the conformal transformation at O which leads to the Penrose diagram for the naked scenario. Since the vector vanishes in the covered case, there is no such divergence and hence the Penrose diagrams in the two cases differ.

Thus, we have shown that the information about whether the singularity formed in collapse is naked is contained at the intersection of the boundary of the (four dimensional) trapped region and singular boundary in the spherical dust case.

The procedure of checking if an appropriate conformal transformation is necessary does not involve checking for emergence of causal curves from the singularity. So far, the latter has been the only method for checking if a singularity formed in collapse is naked. The work presented in this paper suggests an alternative method and demonstrates its validity in spherical dust collapse.

References


