GENERALIZED p -FORMS ELECTRODYNAMICS IN CLIFFORD SPACES

C. Castro

Center for Theoretical Studies of Physical Systems
Clark Atlanta University, Atlanta, Georgia 30314, USA

April, 2003 . Revised November, 2003

Abstract

Using Clifford algebraic methods we describe how to generalize Maxwell theory of Electrodynamics asociated with ordinary point-charges to a generalized Maxwell theory in Clifford spaces involving *extended* charges and p -forms of arbitrary rank. Clifford algebras contain the appropriate algebraic-geometric features to implement the principle of polydimensional transformations (branes of different dimensionality are rotated into each other) that could possibly lead to a proper formulation of string and M theory.

1. Introduction : Clifford Algebras in Physics

Clifford algebras display a rich mathematical structure which is very useful for a unified description of geometry and physics [1,2,3,4,5,6,9]. For example, a geometric approach to the physics of the Standard Model in terms of Clifford algebras was advanced by [3] . Important applications have also been found in string theory, extended objects, gravity, QFT.... Using Clifford-algebraic methods, and related ideas, the bosonic p -brane propagator, in the quenched-reduced minisuperspace approximation, was constructed in [10]; the logarithmic corrections to the black hole entropy based on the geometry of Clifford space were obtained in [12]; a higher derivative gravity action with torsion was derived from a geometric action starting from the scalar curvature in Clifford-spaces [11]. Many recent important aspects of Clifford algebra are described in the monographs [5].

In [7,8,9] it was proposed that every physical quantity could in fact be assigned to a *polyvector*, or multivector, that is a Clifford number or a Clifford aggregate. Since spinors are the members of left or right minimal ideals of Clifford algebras, this fact may provide the framework for a deeper understanding of supersymmetries, i.e., the transformations relating bosons and fermions. The Fock-Stueckelberg theory of relativistic particle can also be embedded in the Clifford algebra of spacetime [9] .

In this new physical theory the arena for physics is no longer the ordinary spacetime, but a more general Clifford manifold of Clifford algebra valued objects, polyvectors. Such a manifold has been called a pan-dimensional continuum [8] or C -space [7]. The latter describes on a unified basis the objects of various dimensionality: not only points, but also closed lines, surfaces, volumes,..., called 0-loops (points), 1-loops (closed strings) 2-loops (closed membranes), 3-loops, etc.. It is a sort of a *dimension* category, where the role of functorial maps is played by C -space transformations which reshuffle a p -brane history for a p' -brane history or a mixture of all of them, for example.

Technically those transformations in C -space that reshuffle objects of different dimensions are generalizations of the ordinary Lorentz transformations, that mix space with time, to C -space. Since Clifford algebras contain the appropriate algebraic-geometric features to implement this principle of polydimensional transformations and are deeply ingrained in the algebraic properties of quaternions and octonions [17], it is very plausible that this may lead to the proper formulation of string and M theory [11] within the framework of Projective Geometry [18] .

The main purpose of this letter is to describe how to generalize Maxwells theory of Electrodynamics asociated with ordinary point-charges to a Generalized Maxwell theory in Clifford spaces involving *extended* charges and p -forms of arbitrary rank. Clifford algebras implement naturally the idea of electric-magnetic duality and incorporate Dirac monopoles in a straightfoward fashion [13]. Other recent approaches to the generalization of Maxwell's Electrodynamics has been based in the study of gerbes or higher-gauge theories [15]. For a detailed discussion of the role of Clifford algebras and Electrodynamics see [4,14].

1.2. Extending Relativity from Minkowski spacetime to C -space

In order to generalize Maxwell theory of Electrodynamics we review briefly the extended relativity theory in C-spaces [7,9,11] by starting with a natural generalization of the notion of a space-time interval in Minkowski space ds^2 to C-space. This is given by the scalar part of the Clifford geometric product of two infinitesimal displacements :

$$dX^2 = \langle dX^+ dX \rangle_{\text{scalar}} = d\Omega^2 + dx_\mu dx^\mu + dx_{\mu\nu} dx^{\mu\nu} + \dots \quad (1.1)$$

where the operation X^+ is the analog of the matrix transpose (hermitian) conjugation operation and represents the *reversed* Clifford multivector obtained by reversing the ordering position of all the basis vectors γ^μ present in the expansion of X .

The Clifford valued multivector:

$$X = X^M E_M = \Omega I + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + \dots \quad (1.2)$$

denotes the position of a polyparticle in a manifold, called Clifford space or C -space.

The coordinates $x^{\mu_1\mu_2}, x^{\mu_1\mu_2\mu_3}, \dots$ are the holographic areas, volumes, ...projections of the nested family of p -loops onto the embedding spacetime coordinate planes/hyperplanes. In order to match units in the expansion of the multivector X it is required to introduce a length scale. Such scale was chosen to be the minimal Planck scale [7,16] and can be set to unity.

For example, if we take differential dX of X and compute the scalar product $\langle dX dX \rangle$ we obtain the C-space analog of the particle proper time interval :

$$d\Sigma^2 = (d\Omega)^2 + \Lambda_P^{2D-2} dx_\mu dx^\mu + \Lambda_P^{2D-4} dx_{\mu\nu} dx^{\mu\nu} + \dots \quad (1.3)$$

where in (1-3) one has introduced explicitly the Planck scale Λ_P in order to fuse together objects of different dimensionality : 0-loops (points) , 1-loops (closed strings) , ..., p -loops (closed p -branes). Einstein introduced the speed of light as a universal absolute invariant in order to “unite” space with time (to match units) in the Minkowski space interval:

$$ds^2 = c^2 dt^2 - dx_i dx^i. \quad (1.4)$$

A similar unification is needed here to “unite” objects of different dimensions, such as x^μ , $x^{\mu_1\mu_2}$, etc... The minimal Planck scale then emerges as another universal invariant in constructing an extended scale relativity theory in C-spaces [7, 16] .

To continue along the same path of special Relativity in Minkowski space, one considers the analog of Lorentz transformations in C-spaces which transform a polyvector X into another polyvector X' given by $X' = R X R^{-1}$ with

$$R = e^{\theta^A E_A} = \exp [(\theta I + \theta^\mu \gamma_\mu + \theta^{\mu_1\mu_2} \gamma_{\mu_1} \wedge \gamma_{\mu_2} \dots)]. \quad (1.5)$$

and

$$R^{-1} = e^{-\theta^A E_A} = \exp [-(\theta I + \theta^\nu \gamma_\nu + \theta^{\nu_1\nu_2} \gamma_{\nu_1} \wedge \gamma_{\nu_2} \dots)]. \quad R R^{-1} = 1. \quad (1.6)$$

where the theta parameters:

$$\theta; \theta^\mu; \theta^{\mu\nu}; \dots \quad (1.7)$$

are the C-space version of the Lorentz rotations/boosts parameters.

Since a Clifford algebra admits a matrix representation, one can write the norm of a polyvectors in terms of the trace operation as: $\|X\|^2 = \text{Trace } X^2$ Hence under C-space Lorentz transformation the norms of poly-vectors behave like follows:

$$\text{Trace } X'^2 = \text{Trace } [R X^2 R^{-1}] = \text{Trace } [R R^{-1} X^2] = \text{Trace } X^2. \quad (1.8)$$

These norms are invariant under C-space Lorentz transformations due to the cyclic property of the trace operation and $R R^{-1} = 1$. This completes the very brief review of relativity in C-spaces [7, 9, 11] .

2. C-space Generalized Maxwell Electrodynamics

In this section we will generalize Maxwell Electrodynamics to C-spaces ; i.e. construct the Clifford algebra-valued extension of the Abelian field strength $F = dA$ associated with ordinary vectors A_μ . Based on the standard definition of the Abelian field strength $F = dA$ we shall use the same definition in terms of polyvector-valued quantities and differential operators in C-space :

$$A = A_N E^N = \Phi I + A_\mu \gamma^\mu + A_{\mu\nu} \gamma^\mu \wedge \gamma^\nu + \dots \quad (2.1)$$

the first component in the expansion Φ is a scalar field; A_μ is the standard Maxwell field A_μ , the third component $A_{\mu\nu}$ is a rank two antisymmetric tensor field...and the last component of the expansion is a pseudo-scalar. The fact that a scalar and pseudo-scalar field appear very naturally in the expansion of the C-space polyvector valued field A_N suggests that one could attempt to identify the latter fields with the dilaton and the axion field, respectively. Once again, in order to match units in the expansion (2-1), it requires the introduction of suitable powers of a length scale parameter, the Planck scale which is conveniently set to unity.

The differential operator is the generalized Dirac operator :

$$d = E^M \partial_M = I \partial_\Omega + \gamma^\mu \partial_{x_\mu} + \gamma^\mu \wedge \gamma^\nu \partial_{x_{\mu\nu}} + \dots \quad (2.2)$$

the polvector valued indices $M, N, \dots$ range from $1, 2, \dots, 2^D$ since a Clifford algebra in D -dim has 2^D basis elements. At this point we will not be concerned with the representations of Clifford algebras in (r, s) dimensions. Where r, s denotes the temporal and spatial dimensions respectively [2] .

The generalized Maxwell field strength in C-space is :

$$\begin{aligned} F = dA &= E^M \partial_M (E^N A_N) = E^M E^N \partial_M A_N = \frac{1}{2} \{E^M, E^N\} \partial_M A_N + \\ &\frac{1}{2} [E^M, E^N] \partial_M A_N = \frac{1}{2} F_{(MN)} \{E^M, E^N\} + \frac{1}{2} F_{[MN]} [E^M, E^N]. \end{aligned} \quad (2.3)$$

where one has *decomposed* the Field strength components into a symmetric plus antisymmetric piece by simply writing the Clifford geometric product of two polyvectors $E^M E^N$ as the sum of an anticommutator plus a commutator piece respectively .

$$F_{(MN)} = \frac{1}{2} (\partial_M A_N + \partial_N A_M). \quad (2.4)$$

$$F_{[MN]} = \frac{1}{2} (\partial_M A_N - \partial_N A_M). \quad (2.5)$$

The standard Maxwell action :

$$S(A) = \int [\mathcal{D}X] F_{[MN]} F^{[MN]}. \quad (2.5)$$

for the measure in C-space :

$$[\mathcal{D}X] = (d\Omega)(dx^0 dx^1 \dots dx^D)(dx^{01} dx^{02} \dots)(dx^{012} \dots) \dots (dx^{0123 \dots D}). \quad (2.6)$$

is automatically gauge invariant under the transformations $\delta A_M = \partial_M \Lambda$ since $\delta F_{[MN]} = 0$ due to the antisymmetry condition $[\partial_M, \partial_N] \Lambda = 0$.

We will show below how the most general C-space Maxwell action is up to a numerical factor given by :

$$\int [\mathcal{D}X] \langle F^\dagger F \rangle_{\text{scalar}} = \int [\mathcal{D}X] (F_{(MN)} F^{(MN)} + F_{[MN]} F^{[MN]}). \quad (2.7)$$

is invariant (up to total derivatives) under *infinitesimal* gauge transformations provided the symmetric part of F is divergence-free $\partial_M F^{(MN)} = 0$.

It was later noticed in [19] that this divergence-free condition has the same effects as if one were fixing a gauge leaving a *residual* symmetry of *restricted* gauge transformations such that the gauge symmetry parameter obeys the Laplace-like equation $\partial_M \partial^M \Lambda = 0$. Such residual (restricted) symmetries are precisely those that leave invariant the divergence-free condition on the symmetric part of F . Residual, restricted symmetries occur, for example, in the light-cone gauge of p-brane actions leaving a residual symmetry of volume-preserving diffeomorphisms and in string theory, in the conformal gauge, leaving a residual symmetry under conformal reparametrizations; i.e. the Virasoro algebras whose symmetry transformations are given by holomorphic and anti-holomorphic reparametrizations of the string world-sheet.

This Laplace-like condition on the gauge parameter is also the one required such that the action (2-7) is invariant under *finite* (restricted) gauge transformations [19] since under such (restricted) finite transformations the Lagrangian (2-7) changes by second-order terms of the form $(\partial_M \partial_N \Lambda)^2$, which are total derivatives if, and only if, the gauge parameter is *restricted* to obey the analog of Laplace equation $\partial_M \partial^M \Lambda = 0$ [19]. Concluding, the effects of adding the symmetric pieces of F to the action (2-7) leads to a *restricted* gauge symmetry which has the same effects as if one were fixing a gauge $\partial_M F^{(MN)} = 0$ [19].

The matter-field minimal coupling is :

$$\int A_M dX^M = \int J_M A^M. \quad (2.8)$$

where one has reabsorbed the coupling constant, the C-space analog of the electric charge, within the expression for the A field itself. Notice that this term (2-8) has the same form as the coupling of p-branes (whose world volume is $p + 1$ -dimensional) to antisymmetric tensor fields of rank $p + 1$.

The open line integral in C-space of the matter-field interaction term in the action (2-8) is taken from the polyparticles proper time interval Σ ranging from $-\infty$ to $+\infty$ and can be recast via the Stokes law solely in terms of the antisymmetric part of the field strength. This requires closing off the integration contour by a semi-circle that starts at $\Sigma = +\infty$, goes all the way to C-space infinity, and comes back to the point $\Sigma = -\infty$. The field strength vanishes along the points of the semi-circle at infinity, and for this reason the net contribution to the contour integral is given by the open-line integral. Therefore, by rewriting the $\int A_M dX^M$ via the Stokes law relation it yields :

$$\begin{aligned} \int A_M dX^M &= \int F_{[MN]} d\Sigma^{[MN]} = \int F_{[MN]} X^M dX^N = \\ &= \int d\Sigma F_{[MN]} X^M (dX^N / d\Sigma). \end{aligned} \quad (2.9)$$

where in order to go from the second term to the third term in the above equation (2-9) we have integrated by parts and then used the Bianchi identity for the antisymmetric component $F_{[MN]}$:

$$\begin{aligned} \partial_L F_{[MN]} + \partial_M F_{[NL]} + \partial_N F_{[LM]} &= \\ \partial_L (\partial_M A_N - \partial_N A_M) + \partial_M (\partial_N A_L - \partial_L A_N) + \partial_N (\partial_L A_M - \partial_M A_L) &= 0. \end{aligned} \quad (2.10)$$

The integration by parts (2-10) permits us to go from a C-space domain integral represented by the Clifford-value hypersurface Σ^{MN} , to a C-space boundary-line integral :

$$\int d\Sigma^{MN} = \frac{1}{2} \int (X^M dX^N - X^N dX^M). \quad (2.11)$$

The pure matter terms in the action are given by the analog of the proper time integral spanned by the motion of a particle in spacetime :

$$\kappa \int d\Sigma = \kappa \int d\Sigma \sqrt{\frac{dX^M}{d\Sigma} \frac{dX_M}{d\Sigma}}. \quad (2.12)$$

where κ is a parameter whose dimensions are $(mass)^{p+1}$ and Σ is the polyparticle proper time in C-space (1-3). The Lorentz force relation in C-space is directly obtained from a variation of

$$\int d\Sigma F_{[MN]} X^M (dX^N/d\Sigma). \quad (2.13)$$

and

$$\kappa \int d\Sigma = \kappa \int \sqrt{dX^M dX_M}. \quad (2.14)$$

with respect to the X^M variables :

$$\kappa \frac{d^2 X_M}{d\Sigma^2} = e F_{[MN]} \frac{dX^N}{d\Sigma}. \quad (2.15)$$

where we have re-introduced the C-space charge e back into the Lorentz force equation in C-space. A variation of the terms (2-7, 2-8) of the action w.r.t the A_M field furnishes the following equation of motion for the A fields :

$$\partial_M F^{(MN)} + \partial_M F^{[MN]} = J^N. \quad (2.16)$$

Gauge invariance will impose further restrictions on the symmetric part $F^{(MN)}$. The gauge variation of the symmetric part $F_{(MN)}$ is not zero but equals :

$$\delta F_{(MN)} = \partial_M \partial_N \Lambda \neq 0 \quad (2.17)$$

since the gauge variation of the fields is given by :

$$\delta A_M = \partial_M \Lambda. \quad (2.18)$$

where Λ is a C-space *scalar*-valued parameter. One can notice that the gauge variation of $F_{[MN]} = 0$ since

$$[\partial_M, \partial_N] \Lambda = 0. \quad (2.19)$$

As we said above, the action (2-7) would be gauge invariant if, and only if, one sets the following condition :

$$\partial_M F^{(MN)} = 0. \quad (2.20)$$

which leads to a *restricted* gauge-invariance of the moste general action (2-7) compared to the ordinary unrestricted gauge-invaraince of the Maxwell action involving only the antisymmetric components $F_{[MN]}$.

If this condition (2-20) is imposed then the *restricted* gauge variation of (2-7) yields :

$$\begin{aligned} \delta \int [\mathcal{D}X] F_{MN} F^{MN} &= \delta \int [\mathcal{D}X] (F_{(MN)} F^{(MN)} + F_{[MN]} F^{[MN]}) = \\ &= 2 \int [\mathcal{D}X] F^{(MN)} \partial_M \partial_N \Lambda. \end{aligned} \quad (2.21)$$

since the gauge variation of $F^{[MN]} = 0$. If we integrate by parts the last term of (2-21) we get for the gauge variation of the Maxwell term :

$$2 \int [\mathcal{D}X] \partial_M [F^{(MN)} \partial_N \Lambda] - 2 \int [\mathcal{D}X] (\partial_N \Lambda) (\partial_M F^{(MN)}). \quad (2.22)$$

If there are no boundary terms and/or if the fields vanish fast enough at infinity the total derivative terms are zero and the first terms of the gauge variation is zero. Hence, the C-space generalized Maxwell action (2-7) is invariant under the (restricted) gauge transformations provided the symmetric component of the field strength is divergence-free : $\partial_M F^{(MN)} = 0$. This condition is also compatible with the field equations of the gauge fields. After breaking the field equations into a symmetric and antisymmetric component yields after implementing the condition (2-20) :

$$\partial_M F^{(MN)} = 0. \quad (2.24)$$

and

$$\partial_M F^{[MN]} = J^N. \quad (2.25)$$

Finally, by taking derivatives on both sides of the last equation with respect to the X^N coordinate , one obtains due to the symmetry condition of $\partial_M \partial_N$ versus the antisymmetry of $F^{[MN]}$ that :

$$\partial_N \partial_M F^{[MN]} = 0 = \partial_N J^N = 0. \quad (2.26)$$

which is precisely the continuity equation for the current.

Therefore we have found that a restricted gauge invariance of the action (2-7) demands that $\partial_M F^{(MN)} = 0$, which is indeed compatible with the *continuity* equation for the current. The continuity equation is essential to ensure that the matter-field coupling term of the action $\int A_M dX^M = \int J_M A^M$ is also gauge invariant, which can be readily verified after an integration by parts and setting the boundary terms to zero :

$$\delta \int [\mathcal{D}X] J^M A_M = \int [\mathcal{D}X] J^M \partial_M \Lambda = - \int [\mathcal{D}X] (\partial_M J^M) \Lambda = 0. \quad (2.27)$$

Gauge invariance also ensures the conservation of the energy-momentum (via Noethers theorem) defined in terms of the Lagrangian density variation as [13] :

$$T^{MN} = (\partial^M A_K) \left(\frac{\delta \mathcal{L}}{\delta (\partial_N A_K)} \right) - G^{MN} \mathcal{L}. \quad (2.28)$$

where G^{MN} is the C-space metric (in this case flat) . The canonical conjugate momenta for the field A is :

$$\pi_{NM} = \partial_N A_M. \quad (2.29)$$

It remains to be seen if this construction of C-space generalized Maxwell Electrodynamics of p-forms can be generalized to the Nonabelian case when we replace ordinary derivatives by gauge-covariant ones :

$$F = dA \rightarrow F = DA = (dA + A \bullet A). \quad (2.30)$$

For example, one could define the graded-symmetric product $E_M \bullet E_N$ based on the graded commutator of Superalgebras:

$$[A, B] = AB - (-1)^{s_A s_B} BA. \quad (2.31)$$

s_A, s_B is the grade of A and B respectively. For bosons the grade is even and for fermions is odd. In this fashion the graded commutator captures both the anti-commutator of two fermions and the commutator of two bosons in one stroke. Using (2-31) one may extend this definition to :

$$E_M \bullet E_N = E_M E_N - (-1)^{s_M s_N} E_N E_M. \quad (2.32)$$

s_M, s_N is the grade of E_M and E_N respectively. Even or odd depending on the grade of the basis elements.

One may generalize Maxwells theory to that of Born-Infeld nonlinear Electrodynamics in C-spaces based on our construction presented in this letter. Finally, the extension from a flat C-space to a *curved* C-space also warrants further investigation in order to couple C-space Yang-Mills theory to C-space gravity (a higher derivative gravity with torsion [11]) . This will be left for a future publication.

Acknowledgements

We are indebted to Matej Pavsic for numerous discussions which led to this work and to Frank (Tony) Smith and M. Bowers for their invaluable help.

References

1-D. Hestenes, "Spacetime Algebra" Gordon and Breach, New York, 1996.

- D. Hestenes and G. Sobczyk, "Clifford Algebra to Geometric Calculus" D. Reidel Publishing Company, Dordrecht, 1984.
- 2-I. R. Porteous, Clifford Algebras and the Classical Groups , Cambridge University Press, 1995.
- 3-W. Baylis, Electrodynamics , A Modern Geometric Approach , Boston, Birkhauser, 1999.
- G. Trayling and W. Baylis, A Geometric Approach for the Standard Model hep-th/0103137.
- 4-B. Jancewicz, Multivectors and Clifford Algebra in Electrodynamics World Scientific, Singapore 1989.
- 5- "Clifford Algebras and their applications in Mathematical Physics" Vol 1: Algebras and Physics. eds by R. Ablamowicz, B. Fauser. Vol 2: Clifford analysis. eds by J. Ryan, W. Sprosig Birkhauser, Boston 2000.
- 6-P. Lounesto, "Clifford Algebras and Spinors". Cambridge University Press. 1997.
- 7-C. Castro, Chaos, Solitons and Fractals **11** (2000) 1663. Foundations of Physics **30** (2000) 1301. Chaos, Solitons and Fractals **11** (2000) 1721. Chaos, Solitons and Fractals **12** (2001) 1585. The Programs of the Extended Relativity in C-spaces, towards physical foundations of String Theory Advance NATO Workshop on the Nature of Time, Geometry and the Physics of Perception. Kluwer Publishers 2003.
- 8-W. Pezzaglia, "Physical Applications of a Generalized Geometric Calculus" gr-qc/9710027. Dimensionally Democratic calculus and Principles of Polydimensional Physics gr-qc/9912025. Classification of Multivector Theories and Modifications of the Postulates of Physics gr-qc/9306006.
- 9-M.Pavsic: "The Landscape of Theoretical Physics": A Global View Kluwer, Dordrecht 1993. Foundations of Physics **31** (2001) 1185. Phys. Let **A 242** (1998) 187. Nuovo Cimento **A 110** (1997) 369. Clifford Space as the Arena for Physics gr-qc/0211085, to appear in Foundations of Physics (2003).
- 10- S. Ansoldi, A. Aurilia, C. Castro, E. Spallucci, Phys. Rev. **D 64** 026003 (2001)
- 11-C. Castro, M. Pavsic : Phys. Letts **B 539** (2002) 133.
- 12-C. Castro, Jour. of Entropy **3** (2001) 12-26.
- 13-M. Defaria-Rosa, E. Recami and W. Rodriguez : Phys. Letts **B 173** (1986) 233.
- 14-T. Ivezic, Foundations of Physics Lett **15** (2002) 27. Invariant Relativistic Electrodynamics , Clifford Algebra Approach hep-th/0207250.
- 15-H. Pfeiffer : Higher Gauge Theoy and Nonabelian generalization of 2-form Electrodynamics hep-th/0304074. J. Baez : Higher Yang-Mills theory hep-th/0206130.
- 16- L. Nottale : Fractal Spacetime and Microphysics, towards the Theory of Scale Relativity World Scientific 1992.
- 17- P. Ramond, Exceptional Groups and Physics hep-th/0301050.
- 18- D. Hestenes and R. Ziegler : Projective Geometry with Clifford Algebra , Acta Applicandae Mathematicae **23** (1991) 25-63.
- 19-C. Castro, M.Pavsic, The Extended Relativity Theory in Clifford Spaces to appear in the Int.Jour.Mod.Phys. ■
A.