Clustering aspects and the shell model

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Abstract: In this talk I shall discuss the clustering aspect and the shell model. I shall first discuss the $\alpha$-cluster aspects based on the shell model calculations. Then I shall discuss the spin zero ground state dominance in the presence of random interactions and a new type of cluster structure for fermions in a single-$j$ shell in the presence of only pairing interaction with the largest multiplicity.
1 INTRODUCTION

As is well known, the clustering phenomenon is universal in nature. The $\alpha$ clustering picture in light nuclei has been studied for many years. In this talk I shall first look back at the history of the $\alpha$ clustering picture in nuclear physics from the view point of the shell model [1]. Next I shall introduce our efforts towards understanding the origin of the spin 0 ground state (0 g.s.) dominance discovered by Johnson, Bertsch and Dean in 1998 using the two-body random ensemble (TBRE) [2]. In the studies of the 0 g.s. dominance we found a new type of cluster which is given by the pairing interaction with the largest multiplicity.

2 THE $\alpha$ CLUSTERING PICTURE BASED ON THE SHELL MODEL

The $^4\text{He}$ ($\alpha$ cluster) has the $(0s)^4$ configuration in the zero-th order approximation which belongs to the [4] symmetry of SU(4). As for the nucleus $^{20}\text{Ne}$, the configuration space for valence nucleons is given by $(1s0d)^4$. The [4] symmetry dominates in the wavefunctions obtained by diagonalizing a shell model Hamiltonian with central two-body interactions and one-body spin-orbit interaction, as shown in Table 1 [1]. This indicates that there is a strong resemblance between the shell model wavefunctions and the $\alpha$ cluster wavefunctions of the nucleus $^{20}\text{Ne}$.

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Table 1: Percentage analysis of wavefunctions into different orbital symmetries.

The matrix element of the Majorana operator $P_M$ between the ground state of $^8\text{Be}$ obtained by using the Cohen-Kurath interaction is $-5.77$. If the SU(4)
symmetry were exact, this value would be $-6$ for the symmetry $[4]$. The overlap between the ground state obtained by the Cohen-Kurath interaction and that by the Majorana interaction with the SU(4) symmetry $[4]$ is 0.97. Therefore, the Cohen-Kurath interaction favors the SU(4) symmetry.

The Elliott SU(3) wavefunctions of the nucleus $^{8}$Be are known to be identical to the Wildermuth $\alpha$-cluster wavefunctions

$$
\Psi(^{8}\text{Be}, (0s)^4(0p)^4[44](40)IM) = \mathcal{N}A \left[ \Phi(\alpha_1)\Phi(\alpha_2)R_{nl}(r_{12})Y^I_M(\theta_{12}\phi_{12}) \right],
$$

where $R_{nl}(r_{12})$ are harmonic oscillator wavefunctions with $2n + l = 4$, and $\mathcal{N}$ is a normalization factor. If we require $R_{nl}$ to satisfy the following equation

$$
H\mathcal{A} \left[ \Phi(\alpha_1)\Phi(\alpha_2)R_{nl}(r_{12})Y^I_M(\theta_{12}\phi_{12}) \right] = E\mathcal{A} \left[ \Phi(\alpha_1)\Phi(\alpha_2)R_{nl}(r_{12})Y^I_M(\theta_{12}\phi_{12}) \right],
$$

$\mathcal{A} \left[ \Phi(\alpha_1)\Phi(\alpha_2)R_{nl}(r_{12})Y^I_M(\theta_{12}\phi_{12}) \right]$ becomes the Resonating Group wavefunctions. We can say that the SU(3) shell model wavefunction provides a good approximation to that of the Generator Coordinate Method (=Resonating Group method). We can see this from the following scenario. For $I = 0$

$$
e^{-(\alpha+\epsilon)\mathbf{r}_{12}^2} = \left(1 - \epsilon r_{12}^2 + \frac{e^2}{2} r_{12}^2 - \frac{e^3}{6} r_{12}^3 + \cdots \right) \\
\sim R_{0s} + \epsilon R_{1s} + \epsilon^2 R_{2s} + \epsilon^3 R_{3s} + \cdots,
$$

one thus obtains

$$
\mathcal{A} \left[ \Phi(\alpha_1)\Phi(\alpha_2)e^{-(\alpha+\epsilon)\mathbf{r}_{12}^2} \right] = \mathcal{A} \left[ \Phi(\alpha_1)\Phi(\alpha_2)(R_{0s} + \epsilon R_{1s} + \epsilon^2 R_{2s} + \epsilon^3 R_{3s} + \cdots) \right].
$$

Because the Pauli principle forbids $R_{02}$ and $R_{1s}$ components, Eq. (4) is reduced to

$$
\mathcal{A} \left[ \Phi(\alpha_1)\Phi(\alpha_2)e^{-(\alpha+\epsilon)\mathbf{r}_{12}^2} \right] \sim \epsilon^2 \mathcal{A} \left[ \Phi(\alpha_1)\Phi(\alpha_2)(R_{2s} + \epsilon R_{3s} + \cdots) \right].
$$

Absorbing $\epsilon$ into the normalization factor, Eq. (5) is reduced to

$$
\mathcal{A} \left[ \Phi(\alpha_1)\Phi(\alpha_2)R(r_{12}) \right] \sim \mathcal{A} \left[ \Phi(\alpha_1)\Phi(\alpha_2)R_{2s} \right],
$$

3
We start from a 2-\(\alpha\) condensate wavefunction \(\mathcal{N}(A[\Phi(\alpha_1)\Phi(\alpha_2)]e^{-\beta r_1^2})\) where \(\beta = \alpha + \epsilon\), and have

\[
\mathcal{N}A[\Phi(\alpha_1)e^{-\beta r_1^2}\Phi(\alpha_2)e^{-\beta r_2^2}] = \mathcal{N}A[\Phi(\alpha_1)\Phi(\alpha_2)]e^{-\beta r_{12}^2} \Psi_{\text{C.M.}}(\frac{\vec{r}_1 + \vec{r}_2}{2}), \tag{7}
\]

where \(\Psi_{\text{C.M.}}(\frac{\vec{r}_1 + \vec{r}_2}{2})\) is center-of-mass wavefunction of 2 \(\alpha\) in a single spherical orbit 0S.

As shown in Eq. (6), the right hand side of Eq. (7) can be approximated by the SU(3) shell model wavefunction when \(\epsilon\) is small. Thus we have proved that the 2\(\alpha\)-condensate wavefunctions are the same as those of the shell model if \(\epsilon\) is small. However, when \(\epsilon\) is not very small, higher configurations of the shell model space must contribute to modify the shell model wavefunctions. In such cases the \(\alpha\)-cluster model is more efficient than the shell model. One may similarly discuss the 3-\(\alpha\) condensate states.

3 Towards understanding of the 0 g.s. dominance

The ground state spins\(^{\text{parity}}\) of even-even nuclei are always 0\(^+\). We believed that this fact originates from short range attractive interaction. However, the (0 g.s.) was discovered by using the two-body random ensemble (TBRE) [2]. There have been many efforts to understand this interesting and important observation [3, 4].

Here we exemplify our work [4] by the case of four fermions in a single-\(j\) shell. The Hamiltonian for fermions in a single-\(j\) shell is defined as follows

\[
H = \sum_j G_j A^{J+} \cdot A^I \equiv \sum_j \sqrt{2J + 1} G_j \left( A^{J+} \times \bar{A}^I \right)^0,
\]

\[
A^{J+} = \frac{1}{\sqrt{2}} \left( a^+_j \times a^+_j \right)^J, \quad \bar{A}^J = -\frac{1}{\sqrt{2}} (\bar{a}_j \times \bar{a}_j)^J, \quad G_j = \langle j^2 J | V | j^2 J \rangle. \tag{8}
\]

\(G_j\)'s are taken as a set of Gaussian-type random numbers with a width being 1 and an average being 0. This two-body random ensemble is referred to as “TBRE”. The I g.s. probabilities in this paper are obtained by 1000 runs of a TBRE Hamiltonian.
In Ref. [4] we introduced an empirical formula to predict the $P(I)$’s, which are the probabilities of a state with spin $I$ to be the ground state. This formula was found to be valid for both a single-$j$ shell and many-$j$ shells, for both an even value of particle number and an odd value of particle number, and for both fermions and bosons.

Our procedure to predict the $P(I)$’s is as follows. Let us set only one of the $G_J$’s equal to $-1$ and the others to zero, and find the spin $I$ of the ground state. We repeat this process for all two-body interactions $G_J$. We can find how many times the ground state has angular momentum $I$. This number is denoted as $N_I$. The values of $N_I$ for four nucleons in a single-$j$ shell can be easily counted from Table 2 when $j$ is not very large.

Using the $N_I$, we can predict the probability that the ground state has angular momentum $I$ as $P(I) = N_I/N$, where $N$ is the number of independent two-body matrix elements ($N = j + 1/2$ for fermions in a single-$j$ shell). A nice agreement between our predicted $P(0)$’s and those obtained by diagonalizing a TBRE Hamiltonian is shown in Fig. 1. Comparison for more complicated cases can be found in [4].

For the case of fermions in a single-$j$ shell, one easily notices that the $P(I_{\text{max}})$ is sizable. The reason is very simple: $N_{I_{\text{max}}} \equiv 1$, which comes from the $G_{J_{\text{max}}}$ term. Therefore, we predict that the $P(I_{\text{max}}) = 1/N$, where $N = j + 1/2$ for a single-$j$ shell. Fig. 2 shows a comparison between the predicted $P(I_{\text{max}})$’s by $1/N$ and those by

### Table 2: The angular momenta $I$’s which give the lowest eigenvalues for 4 fermions in a single-$j$ shell, when $G_J = -1$ and all other parameters are 0.

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<th>$2j$</th>
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5
the TBRE Hamiltonians. We see a remarkable agreement.

The above empirical formula also provides us with other very important insights. It presents a guideline to tell which interactions are essential to produce a sizable \( P(I) \) in a many-body system. For example, \( P(0) \) for \( j = 31/2 \) is given essentially by the two-body matrix elements with \( J = 0, 6, 8, 12, \) and 22. The \( P(0) \) would be close to zero without these five terms. This disproves a popular idea that the angular momentum 0 ground state (0 g.s.) dominance may be independent of two-body interactions.

The 0 g.s. dominance has not been understood yet from a more sophisticated level. Further works are therefore warranted.

4 A NEW TYPE OF CLUSTER

When one examines the eigenvalues and wavefunctions for \( G_{J_{\text{max}}} = -1 \) and other \( G_{J'} = 0 \) for a Hamiltonian of Eq. (8), one finds a very interesting phenomenon: the states can be classified by clusters. Here we discuss systems up to four particles \((n = 4)\) with \( j = 31/2\). Systems with more particles exhibit a similar behavior.

For \( n = 2 \), there are only two cases: \( I = 30 \) which has \( E = -1 \) and \( I \neq 30 \) which have \( E = 0 \).

For \( n = 3 \), it was proved \([5]\) that the nonzero eigenvalue of each \( I \) states for \( G_{J_{\text{max}}} = -1 \) and other \( G_{J'} = 0 \) is given by the configuration \( |\Psi_{n=3}^{E \neq 0}\rangle = \frac{1}{\sqrt{N_{J_{\text{max}}}^I}} (a_j^\dagger \times A^{J=J_{\text{max}}} I) |0\rangle \),

where \( N_{J_{\text{max}}}^I \) is the normalization factor. Namely, for each \( I \) there is one and only one state which has non-zero eigenvalue. One can prove that this value is very close to \(-1\) unless \( I \sim I_{\text{max}}^{(3)} \) (The \( I_{\text{max}}^{(n)} \) refers to the maximum angular momentum of \( n \) particles in a single-\( j \) shell). Therefore, the first particle \( a_j^\dagger \) in \( |\Psi_{n=3}^{E \neq 0}\rangle \) behaves like a “spectator”, and the \( A^{J_{\text{max}} \dagger} \) behaves like a two-particle “cluster”. Eigenvalues for those \( J \neq J_{\text{max}} = 2j - 1 \) in \( |\Psi_{n=3}^E\rangle \) are zero.

Now let us come to the case of \( n = 4 \). In \([6]\) it was proved that the eigenvalues \( E \) of \( n = 4 \) are asymptotically 0, -1 or -2 for small \( I \), and that the states with \( E \sim 0, -1 \) or \(-2\) are constructed by zero, one or two pairs with spin \( J = J_{\text{max}} \).
Besides these “integer” eigenvalues, “non-zero” eigenvalues arise as $I$ is larger than $2j - 9$. The values of these “non-integer” eigenvalues are very close to those of $n = 3$ states with $I^{(3)} \sim I_{\text{max}}^{(3)}$. The configurations for these states can be approximately given by one cluster consisting of three particles with $I^{(3)} \sim I_{\text{max}}^{(3)}$ and one spectator. We shall explain this in more details below.

Fig. 3(a)-(c) plots the distribution of all non-zero eigenvalues for $n = 3$ and 4. From Fig. 3, one sees that these eigenvalues are converged at a few values but with exceptions. The “converged” values are very close to those of eigenvalues of $n = 3$.

Table 3: The lowest eigenvalue $\mathcal{E}_I$ (columns “SM” and “coupled”) for $I \geq 28$ states of four fermions in a $j = 31/2$ shell and its overlap (last column) between the wavefunction obtained by the shell model calculation and that obtained by coupling $a_j^\dagger$ to the $I_{\text{max}}^{(3)}$ state. The column “(SM)” is obtained by the shell model diagonalization, and the “$F_I$” is the matrix element of $H_{I_{\text{max}}}$ in the configuration of coupling $a_j^\dagger$ to the $I_{\text{max}}^{(3)}$ state. Italic font is used for the overlaps which are not close to 1.

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<th>$F_I$ (coupled)</th>
<th>overlap</th>
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For $j = 31/2$ and $n = 4$ the total number of states is 790. The number of states with non-zero eigenvalues is 380. Within a precision $10^{-2}$, 308 states which
eigenvalues are located at the eigenvalues of \( n = 3 \), and 21 states have eigenvalues at \(-2\). It is noted that most of exceptions of eigenvalues of \( n = 4 \) can be nicely given by a three-particle cluster coupled with a single-\( j \) particle in a high precision. In this example only four states with \( I = 48 \), two states with \( I = 46 \), two states with \( I = 44 \) are not given well by the pictures of a three-particle cluster coupled with a single-\( j \) particle or two pairs with spin \( J_{\text{max}} \). These states behave as four-body clusters in systems with \( n > 4 \).

Let us exemplify this by the peak near \( 2.25 \). \( E_{I_{\text{max}}(n=3)} = -\frac{267}{118} = -2.26271186440678 \) for \( j = 31/2 \). The smallest \( I \) which can be coupled by three particles with \( I = I_{\text{max}}(n=3) \) and a single-\( j \) spectator is \( I_{\text{max}}(3) - j = 2j - 3 \) and here 28. The lowest eigenvalue \( E_I \) of \( I = 28 \) obtained by a shell model diagonalization for \( n = 4 \) is \(-2.26271186440689\), which is very close to \( E_{I_{\text{max}}(n=3)} \). Some of the \( E_I \) with \( I \) between 28 to 56 are listed in Table 3.

We have calculated all overlaps between states of \( n = 4 \) with energies focused in the peaks and those of simple configurations obtained by coupling a non-zero energy cluster with \( I_{\text{max}}(3) \sim I_{\text{max}}(3) \) of three fermions and a single-\( j \) particles, which shows similar situation as Table 1. Therefore, we conclude that those “non-integer” eigenvalues of \( n = 4 \) with \( H = H_{J_{\text{max}}} \) are given in a high precision by three-particle cluster (nonzero energy) coupled with a single-\( j \) particle.

5 SUMMARY

I first discussed the \( \alpha \)-cluster picture for a few typical nuclei from the view point of the shell model. Then I discussed an empirical approach of predicting the \( I \) g.s. probabilities in the presence of random interactions. A new type of clustering phenomenon was discussed. We are now trying to find applications of this clustering phenomenon.

I thank Drs. Y. M. Zhao and N. Yoshinaga for their collaborations in my work.
Figure 1     A comparison between our predicted $P(0)$’s and those obtained by 1000 runs of a TBRE Hamiltonian.

Figure 2     A comparison between the predicted $P(J_{\text{max}})$ ($\sim 1/(j + \frac{1}{2})$) and those obtained by 1000 runs of a TBRE Hamiltonian.

Figure 3     Distribution of all non-zero eigenvalues for $n = 4$. It is seen that these eigenvalues of $n = 4$ are converged at eigenvalues of $n = 3$ but with very few exceptions.
References


