Geometric phase for an adiabatically evolving open quantum system

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(Dated: June 2, 2004)

We derive an elegant solution for a two-level system evolving adiabatically under the influence of a driving field with a time-dependent phase, which includes open system effects such as dephasing and spontaneous emission. This solution, which is obtained by working in the representation corresponding to the eigenstates of the time-dependent Hermitian Hamiltonian, enables the dynamic and geometric phases of the evolving density matrix to be separated and relatively easily calculated.

PACS numbers:

I. INTRODUCTION

The discovery by Berry [1, 2] that a (non-degenerate) state of a quantum system can acquire a phase of purely geometric origin when the Hamiltonian of the system undergoes a cyclic, adiabatic change has lead to an explosion of interest in this and related geometric phases in quantum mechanics, both from a theoretical perspective, and from the point of view of possible applications, the latter including applications to optics (where the geometric phase was first discovered [3]), NMR and molecular physics, and to quantum computing [4, 5]. Since Berry’s work, and the demonstration that Berry’s phase can be understood as a holonomy associated with the parallel transport of the quantum state [2], there have been numerous proposals for generalizations. The first of these was due to Wilczek and Zee [6] who, by considering a Hamiltonian with non-degenerate eigenstates, established the existence of an intimate connection between Berry’s phase and non-Abelian gauge theories. The restriction to changes occurring adiabatically was relaxed in the work of Aharonov and Anandan [7] while Anandan [8] generalized the geometric phase to the non-adiabatic non-Abelian case. The restriction of cyclicity was removed by Samuel and Bhandari [8] and by Pati [11]. All of this work is concerned with geometric phases of pure states of closed systems and is now standard, though [6] indicated extensions to taking account of quantum measurements and consequent non-unitary evolution. A nice overview, theoretical as well as experimental, is given in [10].

More recently attention has turned to studying geometric phases for mixed states, though there is not yet a standard description for geometric phases associated with mixed states.

As realistic systems always interact with their environment, and as an open system is almost always to be found in a mixed state, open systems are a natural source of problems involving the geometric phases of mixed states. Garrison and Wright [12] were the first to touch on this issue in a phenomenological way, by describing open system evolution in terms of a non-Hermitean Hamiltonian. This was, in fact, a pure state analysis, so it did not, strictly speaking, directly address the problem of geometric phases for a mixed state, but this work raised issues which could potentially have a bearing on the analysis of the mixed state problem. In fact, they did point out that a proper treatment of an open system would require making use of the density operator approach. Nevertheless, they arrived at an interesting result, a complex geometric phase for dissipative evolution. This is a result that has been recently put into doubt by a master equation treatment by Fonseca Romero et al and Aguiar Pinto and Thomaz [13, 14].

The first complete open systems analyses of geometric phase for a mixed state, from two different perspectives, is to be found in the papers of Ellinas et al [15] and Gambieli and Freed [10]. The former worked with the standard master equation for the density operator of a multilevel atom subject to radiative damping and driven by a laser field with a time-dependent phase. What is of interest in their approach is that it entailed introducing eigenmatrices of the Liouville superoperator of the master equation for the damped system. The system Hamiltonian was allowed to vary adiabatically, with the result that a non-degenerate eigenmatrix acquires a geometric phase as well as a dynamic phase. In [16], the effect of the environment was modelled as an external classical stochastic influence which, when averaged, gives rise to the relaxation terms of the master equation for the system. In both cases the effects of any geometric phase was then shown to be present in measurable quantities such as the inversion of a two state system.

Since then, research has been increasing rapidly into the problem of defining a geometric phase for mixed quantum states for both unitary and non-unitary evolution, motivated to a very large extent by the need to understand the effects of decoherence in quantum computational processes that exploit geometric phases as a means of constructing intrinsically fault-tolerant quantum logic gates. This issue has been addressed from two points of view, the first holistic in nature wherein the aim is to identify a geometric phase to be associated with the mixed state itself, and the second, essentially the ap-
approach of 15, which works with the pure state geometric phases of an appropriate set of parallel transported basis states, which then gives rise to geometric phase factors in the off-diagonal elements of the density operator. No geometric phase is explicitly associated with the mixed state itself; instead observable quantities that will exhibit the effects of the geometric phases of the underlying basis states are determined.

The former, holistic approach was first introduced in a formal way by Uhlmann 17, and in a different way, based on phase-sensitive measurements via interferometry, by Sjöqvist et al 18 for unitary evolution of a mixed state, and later for non-unitary evolution 13, 20. The phase defined in this way is not the same in all respects to that proposed by Uhlmann 21.

The latter kind of approach has been used only for open systems, and involves working with the (Markovian) master equation of the open system. The approaches used involve either solving the master equation of the system 13, 15, 17, 19, or employing a quantum trajectory analysis 22, 23, 24 to unravel the dynamics into pure state trajectories, and calculating the geometric phases associated with individual pure state trajectories. Noise of a classical origin, such as stochastic fluctuations of the parameters of the Hamiltonian have also been studied by Chiara and Palma 25. In essence, the common feature is not so much to propose a new definition of geometric phase for a mixed state as to show how the underlying existence of a geometric phase will nevertheless show up in the observed behaviour of an open quantum system. It is this perspective that is adopted in the work to be presented in this paper.

Here we introduce an elegant approach for solving the central master equation which is based on introducing a unitary transformation due to Kato 26 described in the classic text by Messiah 27.

The new picture is defined via a time-dependent unitary transformation $A(t)$, usually referred to as a rotating axis transformation, which is such that the transformed system Hamiltonian has time independent eigenspaces. This method is extended by showing that under the conditions of adiabatic evolution, all the information on geometric phase for a closed loop is contained within $A(t)$, and is regained by transforming back to the original picture. The goals of this approach is its simplicity, since one needs only to calculate the geometric phases for the eigenstates of the Hamiltonian $H(t)$, and the fact that dynamic and geometric phase are separated in a clear way. In fact, perfect separation of geometric and dynamical contributions is obtained provided the Hamiltonian evolution is adiabatic and the coupling to the environment is weak. This approach bears some similarity to that used by Fonseca RomeroFonseca 13 who make use of several unitary transformations to separate the geometric phase from the dynamic phase. However, the rationale for their transformations, and the origin of the geometric phase, is somewhat elusive in their analysis. In contrast to 13, with the transformation introduced here, the parallel transport condition is essential and explains the appearance of the geometric phase.

Within the approach used here, it is possible to show explicitly how to achieve, under certain circumstances, operational cancellation of the dynamic phase, thereby making the geometric phase accessible in experiments. An example of where this is possible is given in Section III.

This paper is organized as follows. In Section II we present the main ideas. In Section III we look at several examples. In Section IV we summarize our results while in Section V we indicate possible new directions, including generalisations to non-cyclic evolution and non-Abelian holonomies. An analog to the adiabatic theorem is proved for a general Lindblad equation 28 in the Appendix.

II. THE ROTATING AXIS TRANSFORMATION FOR NONDEGENERATE MULTILEVEL SYSTEMS

A. Geometric Phases for a Closed System

For the present we consider the case of a closed system so as to introduce the basic method employed here. Suppose we have a system with Hamiltonian $H(t)$, a function of time due to the dependence of $H$ on parameters whose values can be changed in time. This Hamiltonian will have instantaneous eigenvectors $|n(t)\rangle$ with eigenvalues $E_n(t)$:

$$H(t)|n(t)\rangle = E_n(t)|n(t)\rangle .$$

(1)

For simplicity we assume $E_i(t) \neq E_j(t)$ for $i \neq j$. This restriction will be removed in section IV to obtain non-Abelian holonomies. For an adiabatically slow change in the system parameters, these eigenvectors will also change in such a way as to satisfy the parallel transport condition 2:

$$\langle n(t)|\frac{d}{dt}|n(t)\rangle = 0 .$$

(2)

At this point we introduce a unitary operator $A(t)$ via the equation

$$A(t)|n(0)\rangle = |n(t)\rangle .$$

(3)

This completely defines $A(t)$. Note that because of the path dependence of the parallel transported eigenstates $|n(t)\rangle$, the operator $A(t)$ has a non-integrable nature, and, as we see later, will contain the information on the geometric phase.

This unitary operator can now be used to remove the time dependence of the eigenstates of the Hamiltonian. Thus, if we define

$$H' = A(t)HA$$

(4)
we note that the eigenvectors of \( H^A \) are now just \( |n(0)\rangle \) and hence are time independent. The transformed Schrödinger equation is then

\[
H^A|\psi^A\rangle = i\hbar \left( A^\dagger \dot{A}|\psi^A\rangle + \frac{d}{dt} |\psi^A\rangle \right) \tag{5}
\]

where \( |\psi^A\rangle = A^\dagger |\psi\rangle \). If \( |n(t)\rangle \) is parallel transported, then, to the lowest-order adiabatic approximation, one can neglect the terms containing \( A^\dagger(t)\dot{A}(t) \) so that the transformed Schrödinger equation becomes \(28\)

\[
H^A|\psi^A\rangle = i\hbar \frac{d}{dt} |\psi^A\rangle. \tag{6}
\]

The solution of Eq. (6) contains no geometric contribution — it gives the dynamic contribution to the phase of any adiabatically evolving state. So we have extracted the dynamics from the geometric influence of the time-varying Hamiltonian. The geometric contribution is entirely contained within the operator \( A \). To obtain this information we have to transform back to the original picture in terms of the states \( |\psi\rangle \):

\[
|\psi\rangle = A|\psi^A\rangle. \tag{7}
\]

If the Hamiltonian undergoes a closed loop in time \( T \), i.e. \( H(T) = H(0) \), then the parallel transported eigenstates \( |n(T)\rangle \) return to the initial eigenstates \( |n(0)\rangle \) up to the geometric phase. Hence we have \( A(T) = \text{diag}(e^{i\varphi_1}, \ldots, e^{i\varphi_N}) \) where \( \varphi_n \) is the geometric phase associated with the eigenstate \( |n(T)\rangle \). This result can be readily generalized if the system is in a mixed state \( \rho \). Introducing the notation

\[
\rho^A(t) = A^\dagger(t)\rho(t)A(t) \tag{8}
\]

we obtain

\[
\rho(T) = A(T)\rho^A(T)A^\dagger(T) = \text{diag}(e^{i\varphi_1}, \ldots, e^{i\varphi_N})\rho^A(T) \times \text{diag}(e^{-i\varphi_1}, \ldots, e^{-i\varphi_N}). \tag{9}
\]

This holonomic transformation multiplies the off-diagonal elements of the density operator \( \rho^{A}_{\alpha\beta} \) by a phase \( e^{i(\varphi_\alpha - \varphi_\beta)} \), which is the difference of the geometric phases of the eigenstates of the Hamiltonian \( H(T) \).

### B. Open System and Master Equation

Systems that are coupled to a reservoir (or environment) can usually be described by a reduced density operator that evolves according to a master equation, which, in many cases, can be written in the Lindblad form \(28\)

\[
\dot{\rho}(t) = \frac{1}{i\hbar}[H(t), \rho(t)] + \frac{1}{2} \sum_{\alpha=1}^{k} \mathcal{L}_{\Gamma_\alpha}[\rho(t)] \tag{10}
\]

for

\[
\mathcal{L}_{\Gamma}[\rho] = 2\Gamma[\rho^\dagger - \rho] - \rho \Gamma^\dagger \Gamma, \tag{11}
\]

and where \( \rho(t) \) is the density operator for the system of interest, \( \dot{\rho}(t) \) is its derivative with respect to time, \( H(t) \) is the system Hamiltonian and the dissipation operators \( \Gamma_\alpha \) arise due to the presence of the reservoir. To obtain a geometric phase we once again consider an adiabatically changing Hamiltonian \( H(t) \), with \( H(T) = H(0) \), as in the preceding section. Upon introducing the operator \( A(t) \) defined in Eq. (6) we can transform the master equation with the unitary operator \( A^\dagger(t) \), which leads us to the new master equation

\[
\dot{\rho}^A(t) = \frac{1}{i\hbar}[H^A(t), \rho^A(t)] + \rho^A(t)A^\dagger(t)\dot{A}(t) - A^\dagger(t)\dot{A}(t)^\dagger \rho^A(t) + \frac{1}{2} \sum_{\alpha=1}^{k} \mathcal{L}_{\Gamma^A_\alpha}[\rho^A(t)]. \tag{12}
\]

Note that \( \Gamma^A_\alpha \equiv A^\dagger \Gamma_\alpha A \) is in general time-dependent even when \( \Gamma_\alpha \) is not. As shown in the Appendix, the terms containing \( A^\dagger(t)\dot{A}(t) \) can be neglected, as in the case of a unitary evolution but with the additional requirement that the damping is weak. This establishes the parallel transport condition Eq. (4) for weakly damped systems in an adiabatic evolution. We then get

\[
\dot{\rho}^A(t) = \frac{1}{i\hbar}[H^A(t), \rho^A(t)] + \frac{1}{2} \sum_{\alpha=1}^{k} \mathcal{L}_{\Gamma^A_\alpha}[\rho^A(t)]. \tag{13}
\]

The solution of Eq. (13) contains no geometric contributions. To regain \( \rho(T) \) we have to transform back to the original picture as in the unitary case, Eq. (6).

\[
\rho(T) = A(T)\rho^A(T)A^\dagger(T) = \text{diag}(e^{i\varphi_1}, \ldots, e^{i\varphi_N})\rho^A(T) \times \text{diag}(e^{-i\varphi_1}, \ldots, e^{-i\varphi_N}). \tag{14}
\]

with \( \varphi_n \) being the geometric phase for the state \( |n\rangle \).

### III. EXAMPLES FOR TWO-LEVEL SYSTEMS

#### A. Optical Resonance with Spontaneous Emission

We consider a two level atom in a classical resonant laser field. In the rotating-wave approximation the Hamiltonian for this system is

\[
H = \hbar \begin{pmatrix} \Delta & \Omega e^{-i\phi} \\ \Omega e^{i\phi} & -\Delta \end{pmatrix} \tag{15}
\]

The detuning \( \Delta \), the coupling strength \( \Omega \) and the phase \( \phi \) are properties of the laser. To induce a geometric phase we change the phase \( \phi(t) \) of the laser field slowly in comparison to \( E/h = (\Omega^2 + \frac{1}{2} \Delta^2)^{\frac{1}{2}} \), which is the absolute
value of the eigenenergies of the Hamiltonian divided by $\hbar$. The eigenvalue equation is

$$H(t)| + (t)\rangle = E| + (t)\rangle$$

$$H(t)| - (t)\rangle = -E| - (t)\rangle$$

with

$$| + (t)\rangle = e^{-i\phi(t)}\sin^2 \frac{\theta}{2}\cos \frac{\theta}{2}|e\rangle + e^{i\phi(t)}\cos^2 \frac{\theta}{2}\sin \frac{\theta}{2}|g\rangle$$

$$| - (t)\rangle = -e^{-i\phi(t)}\cos^2 \frac{\theta}{2}\sin \frac{\theta}{2}|e\rangle + e^{i\phi(t)}\sin^2 \frac{\theta}{2}\cos \frac{\theta}{2}|g\rangle$$

and

$$\sin \frac{\theta}{2} = \sqrt{\frac{E - \frac{1}{2}\hbar \Delta}{2E}}$$

$$\cos \frac{\theta}{2} = \sqrt{\frac{E + \frac{1}{2}\hbar \Delta}{2E}}$$

$|e\rangle$ and $|g\rangle$ denotes the excited state and the ground state of the two level atom, respectively. Note that $| + \rangle$ and $| - \rangle$ satisfy the parallel transport condition as required in Section 11.

Furthermore we want to include spontaneous emission as a source of dissipation. In the weak coupling limit the master equation is known to be

$$\dot{\rho}(t) = \frac{1}{i\hbar}[H(t),\rho(t)] + \frac{1}{2}\mathcal{L}_{\Gamma}[\rho(t)].$$

for

$$\Gamma = \sqrt{\lambda}\sigma_-. = \sqrt{\lambda} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$  

Here $\lambda$ denotes the spontaneous emission rate. The task here is to solve Eq. (22) in the adiabatic and weak damping limit. As in Section 11 we define the operator $A(t)$ by $A(t)| \pm (0)\rangle = | \pm (t)\rangle$. After the transformation of Eq. (22) with $A^\dagger(t)$, the Hamiltonian is not diagonal. Hence we carry out another transformation with an operator $B^\dagger$ that is defined by

$$B^\dagger| + (0)\rangle = |e\rangle$$

$$B^\dagger| - (0)\rangle = |g\rangle.$$  

As $B^\dagger$ is time-independent, we obtain no term $B^\dagger B$ in the master equation. We can carry out both transformations together with the operator $C^\dagger(t) = B^\dagger A^\dagger(t)$, which turns out to be

$$C(t) = \begin{pmatrix} e^{-i\phi(t)}\sin^2 \frac{\theta}{2}\cos \frac{\theta}{2} & -e^{-i\phi(t)}\cos^2 \frac{\theta}{2}\sin \frac{\theta}{2} \\ e^{i\phi(t)}\cos^2 \frac{\theta}{2}\sin \frac{\theta}{2} & e^{i\phi(t)}\sin^2 \frac{\theta}{2}\cos \frac{\theta}{2} \end{pmatrix}.$$  

The master equation for $\rho^C(t) = C^\dagger(t)\rho(t)C(t)$ is, from Eq. (13) and Eq. (22),

$$\ddot{\rho}^C(t) = \frac{1}{i\hbar}[H^C,\rho^C(t)] + \frac{1}{2}\mathcal{L}_{\Gamma^C(t)}[\rho^C(t)].$$

for

$$H^C = C^\dagger(t)H(t)C(t) = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$$

and

$$\Gamma^C(t) = C^\dagger(t)\Gamma C(t) = \begin{pmatrix} \cos \frac{\theta}{2}\sin \frac{\theta}{2} & \sin^2 \frac{\theta}{2}e^{-i\phi(t)}
\cos \frac{\theta}{2} & -\cos \frac{\theta}{2}\sin \frac{\theta}{2} \end{pmatrix} f(\phi).$$

We show in the Appendix that only the absolute value of the off-diagonal elements of the dissipation operator $\Gamma^C(t)$ contributes in the solution of Eq. (26). Using only the absolute values of these elements does not simplify the calculation, but multiplying them with a time independent factor $e^{i\phi}$ and averaging over all $0 < \beta < 2\pi$ does. Doing so, we first rewrite Eq. (26)

$$\rho^C(t) = \frac{1}{i\hbar}[H^C,\rho^C(t)] + \frac{1}{4\pi} \int_{0}^{2\pi} \mathcal{L}_{\Gamma^C}^\beta[\rho^C(t)] d\beta.$$  

In Eq. (29) we substitute the Lindblad operator $\Gamma^C$ with the new Lindblad operators $\Gamma^\beta$ with

$$\Gamma^\beta = \sqrt{\lambda}\begin{pmatrix} \cos \frac{\theta}{2}\sin \frac{\theta}{2} & \sin^2 \frac{\theta}{2}e^{-i\beta} \\ \cos \frac{\theta}{2}\sin \frac{\theta}{2} & -\cos \frac{\theta}{2}\sin \frac{\theta}{2} \end{pmatrix} f(\beta), \quad 0 \leq \beta < 2\pi.$$  

as explained previously. Finally we get a new master equation, which is equivalent to Eq. (26) in the adiabatic and weak damping limit:

$$\ddot{\rho}^C(t) = \frac{1}{i\hbar}[H^C,\rho^C(t)] + \frac{1}{4\pi} \int_{0}^{2\pi} \mathcal{L}_{\Gamma^\beta}^\beta[\rho^C(t)] d\beta.$$  

Eq. (31) may seem to be more complicated than Eq. (26), but if we evaluate the term behind the integral in Eq. (31) we realize that many contributions are cancelled out by the integration. If we write

$$\rho^C(t) = \begin{pmatrix} a & b \\ b^* & 1 - a \end{pmatrix}$$

then Eq. (31) reduces to the two independent equations

$$\dot{a}(t) = \lambda \left[ \sin^4 \frac{\theta}{2} - a \left( \sin^4 \frac{\theta}{2} + \cos^4 \frac{\theta}{2} \right) \right]$$

$$\dot{b}(t) = -\frac{2i}{\hbar}Eb - \lambda b \left( \sin^2 \frac{\theta}{2}\cos^2 \frac{\theta}{2} + \frac{1}{2} \right)$$

with the solutions

$$a(t) = \begin{pmatrix} a(0) - \frac{\sin^4 \frac{\theta}{2}}{\sin^4 \frac{\theta}{2} + \cos^4 \frac{\theta}{2}} \end{pmatrix} e^{-\lambda t(\sin^4 \frac{\theta}{2} + \cos^4 \frac{\theta}{2})}$$

$$+ \frac{\sin^4 \frac{\theta}{2}}{\sin^4 \frac{\theta}{2} + \cos^4 \frac{\theta}{2}} e^{-\lambda \left( \sin^2 \frac{\theta}{2}\cos^2 \frac{\theta}{2} + \frac{1}{2} \right)}$$

$$b(t) = b(0) e^{-2i\phi(t)} e^{-\lambda t\left( \sin^2 \frac{\theta}{2}\cos^2 \frac{\theta}{2} + \frac{1}{2} \right)}.$$
As the last step we need to evaluate \( \rho(t) = C(t) \rho^C(t) C^\dagger(t) \). As the inversion provides an operational quantity for inferring the geometric phase by measuring the relative proportion of ground vs excited states, and because the terms become rather long, we only write the inversion \( w(t) \), which is

\[
\begin{align*}
\mathcal{W}(t) &= \rho_{11} - \rho_{22} = (2a(t) - 1) \cos \theta \\
&\quad - 2 \sin \theta \Re \left( b(t) e^{i \phi(t) \cos \theta} \right) \tag{37}
\end{align*}
\]

To compare this result with that found by Ellinas et al

\[
\mathcal{W}(T) = 2p \left( \frac{(\Delta \hbar)}{2 E} \right)^2 \frac{2Kp + 1}{2Kp} e^{-\kappa \lambda T} - \frac{1}{2Kp} \cos \left( \frac{2ET}{\hbar} - 2 \pi \frac{\hbar \Delta}{2E} \right) e^{-G \lambda T} \left( \frac{\Omega \hbar}{E} \right)^2 \tag{39}
\]

which is the same as that derived by \[12\]. The dynamic and geometric phases are found in the cosine term in this expression: the difference of the dynamic phases of the eigenstates of the Hamiltonian is given by \( 2ET/\hbar \), and the difference of the geometric phases of these eigenstates (for \( \phi(T) = 2\pi \)) given by \( 2\pi \hbar \Delta/2E \). This term is diminished by a damping factor \( \exp(-G \lambda T) \) which can influence the observability of the geometric phase effect on the inversion. The issues of time scales to observe the effect of the geometric phase have been discussed in \[15\]. However, for the present, we wish to point out that the result above has been derived here by use of a simple transformation into a rotating frame. This is to be contrasted with the much more complicated approach of \[15\], based on calculating the eigenmatrices of the Liouvillian.

### B. Optical Resonance with Dephasing

As in the previous subsection we treat a two level atom driven by a resonant electromagnetic field. This time we assume the damping is due to dephasing that occurs as a consequence of phase changing collisions, which changes the relative phase between the excited state and the ground state of the atom (in contrast to strong collisions that change populations of eigenstates). Since the phase change can vary for each collision we have to consider a one dimensional manifold of dissipation operators

\[
\Gamma_\alpha = \sqrt{\lambda(\alpha)} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}, \quad -\pi < \alpha < \pi, \tag{40}
\]

where \( \lambda(\alpha) \) is the dephasing rate density. Hence we get the master equation

\[
\dot{\rho}(t) = \frac{1}{i\hbar} \left[ H(t), \rho(t) \right] + \int_{-\pi}^{\pi} \left( \Gamma_\alpha(t) \rho(t) \Gamma_\alpha^\dagger(t) - \lambda(\alpha) \rho(t) \right) \, d\alpha \tag{41}
\]

with the Hamiltonian \( H(t) \) from the previous subsection with a slowly changing phase \( \phi(t) \) again. Thus, we get the same parallel transported eigenstates of the Hamiltonian and we can start by carrying out the same transformation as in the previous subsection. In the adiabatic and weak damping limit we obtain the transformed master equation

\[
\dot{\rho}^C(t) = \frac{1}{i\hbar} \left[ H^C, \rho^C(t) \right] + \int_{-\pi}^{\pi} \left( \Gamma_\alpha^C(t) \rho^C(t) \Gamma_\alpha^C(t) - \lambda(\alpha) \rho^C(t) \right) \, d\alpha \tag{42}
\]

for

\[
H^C = C^\dagger(t) H(t) C(t) = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \tag{43}
\]

and

\[
\begin{align*}
\Gamma_\alpha^C(t) &= C^\dagger(t) \Gamma_\alpha(t) C(t) = \sqrt{\lambda(\alpha)} \begin{pmatrix} \cos^2 \theta & \cos \alpha \sin^2 \theta \\ i \sin \theta \sin^2 \theta & \cos \alpha \sin^2 \theta \end{pmatrix} \left( \frac{i \sin \phi(t) \cos \theta + \frac{2}{\pi} \sin \frac{\phi(t)}{\pi}}{\sin^2 \frac{\phi(t)}{\pi} + \cos^2 \theta} + e^{i \alpha} \right) \\
&\quad \begin{pmatrix} i \sin \theta \sin^2 \theta & e^{i \phi(t) \cos \theta + \frac{2}{\pi}} \\ \sin^2 \frac{\phi(t)}{\pi} & \cos^2 \theta \end{pmatrix} \tag{44}
\end{align*}
\]

Furthermore we define

\[
K = \frac{2\Omega^2 + \Delta^2}{4\Omega^2 + \Delta^2} \quad \text{and} \quad G = \frac{6\Omega^2 + \Delta^2}{8\Omega^2 + 2\Delta^2}. \tag{38}
\]

If we furthermore consider the inversion at a time \( T \) at the end of the cyclic evolution we finally get for the inversion:

\[
\mathcal{W}(T) = 2p \left( \frac{(\Delta \hbar)}{2 E} \right)^2 \frac{2Kp + 1}{2Kp} e^{-\kappa \lambda T} - \frac{1}{2Kp} \cos \left( \frac{2ET}{\hbar} - 2 \pi \frac{\hbar \Delta}{2E} \right) e^{-G \lambda T} \left( \frac{\Omega \hbar}{E} \right)^2 \tag{39}
\]
Now everything is much the same as in the previous subsection. Finally we find for the components $a$ and $b$ of the density operator $\rho^C(t)$ the decoupled differential equations

\begin{align}
\dot{a} &= -4f a \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} + 2f \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} \\
\dot{b} &= i \left( -\frac{2E}{\hbar} + gb \left( \sin^4 \frac{\theta}{2} - \cos^4 \frac{\theta}{2} \right) \right) - fb \left( \sin^4 \frac{\theta}{2} + \cos^4 \frac{\theta}{2} \right). \tag{45}
\end{align}

where

\[ f = \int_{-\pi}^{\pi} \lambda(\alpha) (1 - \cos \alpha) \, d\alpha \]

and

\[ g = \int_{-\pi}^{\pi} \lambda(\alpha) \sin \alpha \, d\alpha \]

are properties of the model describing the damping collisions. The solutions of these equations are

\[ a(t) = (a(0) - \frac{1}{2}) e^{-4f T \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} + \frac{1}{2}}, \tag{49} \]

\[ b(t) = b(0) e^{(\frac{-2E}{\hbar} + gb(\sin^4 \frac{\theta}{2} - \cos^4 \frac{\theta}{2}))T} e^{-(\sin^4 \frac{\theta}{2} + \cos^4 \frac{\theta}{2})fT}. \tag{50} \]

To calculate the inversion $w(t)$ we can take Eq. (39) and substitute $a(t)$ and $b(t)$ with Eq. (49) and Eq. (50), respectively. Using Eq. (20) and Eq. (21) as well as the previous definition of $K$ we finally find at time $T$:

\[ w(T) = 2p \left( \frac{\hbar \Delta}{2E} \right)^2 e^{-(\frac{4\pi}{\hbar})^2 fT + \frac{1}{2} \left( \frac{\Omega h}{E} \right)^2 e^{-KfT}} \times \cos \left( \left( \frac{\hbar \Delta}{2E} - \frac{2E}{\hbar} \right) T + 2\pi \frac{\hbar \Delta}{2E} \right). \tag{51} \]

This result is similar to that found in the case of spontaneous emission in Eq. (39). There appears in the cosine term in Eq. (41) the difference of the dynamic phases of the eigenstates of the Hamiltonian, as given by $2ET/\hbar$, and the difference of the geometric phases of these eigenstates (for $\phi(T) = 2\pi$), given by $2\pi \hbar \Delta / 2E$. This term is also diminished by a damping factor $\exp(-KfT)$ which influences the observability of the geometric phase effect on the inversion. Both this term and an additional contribution of a shift in the Rabi frequency by $\frac{gA}{2E}$ arise through the presence of dephasing, though the latter will only appear if the dephasing rate density $\lambda(\alpha)$ is not symmetric.

### C. Spin in Magnetic Field with Dephasing

As another example we consider the simple model of a spin-$\frac{1}{2}$ particle in a magnetic field with constant field strength, which demonstrates how to remove the dynamic phase in a standard model. To induce a geometric phase we change the direction of the magnetic field slowly in comparison to $E/\hbar$. As a source of decoherence we consider dephasing which is defined by

\[ \Gamma_\alpha(t)|e(t)\rangle = \sqrt{\lambda(\alpha)}|e(t)\rangle \]

\[ \Gamma_\alpha(t)|g(t)\rangle = \sqrt{\lambda(\alpha)}e^{i\alpha}|g(t)\rangle. \tag{53} \]

$|g(t)\rangle$ and $|e(t)\rangle$ are the parallel transported eigenstates of the Hamiltonian with spin parallel and antiparallel to the magnetic field, respectively. Further $\lambda(\alpha)$ is the dephasing rate density. Note that dephasing does not change the energy of the spin-system. It is important to distinguish between this model and the two level atom in the previous subsection. Here the Lindblad operators are defined in the basis of the time-dependent eigenstates of the Hamiltonian whereas before the Lindblad operators have been defined in the basis of the excited and the ground state of the two level atom which are independent of the properties of the applied laserfield and hence not the eigenstates of the Hamiltonian. Such dephasing operators could be realized by random fluctuations of the field strength of the applied magnetic field. Since the Hamiltonian changes in time, the dephasing operators have to be time-dependent, too. As in Section II we define the operator

\[ A(t)|e(0\rangle = |e(t)\rangle \]

\[ A(t)|g(0\rangle = |g(t)\rangle \]

and find, from Eq. (13), for $\rho^A = A^d(t)\rho(t)A(t)$

\[ \dot{\rho}^A(t) = \frac{1}{i\hbar} \left[ \left( \frac{E}{0} \quad 0 \right), \rho^A(t) \right] \]

\[ + \frac{1}{2} \int_{-\pi}^{\pi} \left[ 2 \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \rho^A(t) \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} - 2\rho^A(t) \right] \times \lambda(\alpha) \, d\alpha \]

\[ = \left( \begin{array}{cc} 0 & \frac{-2iE}{\hbar} \rho^A_{21}(t) \\ \frac{2iE}{\hbar} \rho^A_{12}(t) & 0 \end{array} \right) \]

\[ + \int_{-\pi}^{\pi} \left( e^{i\alpha} - 1 \right) \rho^A_{12}(t) \left( e^{-i\alpha} - 1 \right) \rho^A_{11}(t) \times \lambda(\alpha) \, d\alpha. \tag{55} \]

Here $\rho^A_{ij}(t)$ denotes the components of $\rho^A(t)$. The solution of Eq. (55) is

\[ \rho^A_{11}(t) = \rho^A_{11}(0) \]

\[ \rho^A_{12}(t) = \rho^A_{12}(0) e^{-i\Delta \frac{4\pi}{\hbar} + g} e^{-ft} \tag{57} \]
with \( f \) and \( g \) defined in Eqs. [17] and [18], respectively. As the last step we need to calculate \( \rho(t) = A(t) \rho^A(t) A(t) \). If the evolution of the Hamiltonian is cyclic we have \( A(t) = \text{diag}(e^{i\varphi}, e^{-i\varphi}) \) where \( \varphi \) is the geometric phase for \( |e\rangle \). The geometric phase is half of the solid angle enclosed by the path which \( |e(t)\rangle \) drives on the Bloch sphere [11]. This is equivalent to half of the solid angle enclosed by the path determined by the direction of the magnetic field. Hence we finally get for the components of the density operator after the Hamiltonian undergoes a closed loop

\[
\rho_{11}(T) = \rho_{11}(0) \\
\rho_{12}(T) = \rho_{12}(0) e^{-i\frac{2\varphi T}{g} + gT - 2\varphi} e^{-TF} 
\]

(58)

In the latter equation we can see a phase change due to the energy difference of the system, an additional phase change due to the dephasing and the geometric phase. Furthermore we see how the absolute value of the off-diagonal element of the density operator decreases exponentially in time because of the dephasing.

Our task now is to remove the dynamic phase. We do a \( \sigma_z \)-transformation in our system and then in the time interval \([T, 2T]\) we drive the direction of the magnetic field around the same loop as before but backwards: \( \hat{B}(T + t) = \hat{B}(T - t) \). The components of the density operator \( \rho'(T) \) after the \( \sigma_z \)-transformation are

\[
\rho'_{11}(T) = \rho_{22}(T) = 1 - \rho_{11}(0) \\
\rho'_{12}(T) = \rho_{12}(T) = \rho_{12}(0) e^{i\frac{2\varphi T}{g} + gT - 2\varphi} e^{-TF} 
\]

(59)

When we drive the magnetic field backwards, then the parallel transported eigenstates are \( |e(T + t)\rangle = |e(T - t)\rangle \) and \( |g(T + t)\rangle = |g(T - t)\rangle \).

Now we define the operator

\[
A'(T + t) = \rho'_{11}(T) = 1 - \rho_{11}(0) \\
A'(T + t) = \rho'_{12}(T) = \rho_{12}(T) = \rho_{12}(0) e^{-i\frac{2\varphi T}{g} + gT - 2\varphi} e^{-TF} 
\]

(60)

which parallel transports the eigenstates of the Hamiltonian \( H(T + t) \). Again we transform the density operator \( \rho'(T + t) = A'(T + t) \rho'(T + t) A'(T + t) \) and find for the components of \( \rho'(T + t) \) Eq. [59]

\[
\rho'_{11}(2T) = \rho_{11}(T) = 1 - \rho_{11}(0) \\
\rho'_{12}(2T) = \rho_{12}(T) = \rho_{12}(0) e^{-i\frac{2\varphi T}{g} + gT - 2\varphi} e^{-TF} 
\]

(61)

where in the last step Eq. [59] is used. Now we need to find \( \rho'(2T) = A'(2T) \rho'(2T) A'(2T) \). From Eq. [59], and because now we drive the loop backwards and hence get the same geometric phase up to a sign, it follows that

\[
A'(2T) = A'(T) \text{diag}(e^{-i\varphi}, e^{+i\varphi}) \\
A'(2T) = \text{diag}(e^{i\varphi}, e^{-i\varphi}) 
\]

After another \( \sigma_z \)-transformation we finally get the density operator

\[
\rho(2T) = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) e^{-i\varphi - 2 TF} \\ \rho_{12}(0) e^{-i\varphi - 2 TF} & 1 - \rho_{11}(0) \end{pmatrix}. 
\]

(62)

Hence we see that not only the dynamic phase of the Hamiltonian is removed, but also the phase shift through dephasing. What stays is twice the difference of the geometric phases of the ground state and the excited state. This geometric effect appears in the off-diagonal components of the density operator and is damped out exponentially in time through dephasing.

### IV. GENERALIZATIONS

#### A. Non-Cyclic Evolution

To consider a non-cyclic evolution we first outline Pati’s analysis [11]. If \( H(T) = H(0) \), Pati compared the phase of the parallel transported eigenstate of the Hamiltonian at time \( t = T \), \( |n(T)\rangle \) with the phase of the eigenstate at time \( t = 0 \), \( |n(0)\rangle \). If the Hamiltonian does not undergo a closed loop, i.e. \( H(T) \neq H(0) \), then \( |n(T)\rangle \) is not \( |n(0)\rangle \) up to a geometric phase. Comparing the phases of states which differ not only by a phase is not straightforward. Pati introduced a reference section \( \tilde{n}(t) \) which is supported by eigenstates of the Hamiltonian \( H(t) \). The phase of \( \tilde{n}(t) \) is fixed by the requirement to make \( |n(t)\rangle \) in phase with \( |n(0)\rangle \) as defined by means of the work of [3], i.e. \( \langle n(0)|n(t)\rangle = 0 \). Then \( |n(T)\rangle \) and \( |n(0)\rangle \) differ only by a phase and this phase is defined to be the generalization of the geometric phase to non-cyclic evolution.

We use this idea and generalize it to open systems. First one has to calculate the density operator \( \rho^A(T) \) in the rotating axis representation as in Section II. Instead of transforming back to the original picture we transform to the picture given by the reference section introduced in [11]. We define the operator \( \tilde{A}(T) \) by

\[
|\tilde{n}(T)\rangle = \tilde{A}(T)|n(0)\rangle 
\]

(63)

The density operator in this new picture is

\[
\rho^\tilde{A}(T) = \tilde{A}(T) A(T) \rho^A(T) A^\dagger(T) \tilde{A}(T) = \text{diag}(e^{i\varphi_1}, \ldots, e^{i\varphi_N}) \rho^A(T) \text{diag}(e^{-i\varphi_1}, \ldots, e^{-i\varphi_N}) 
\]

(64)

and \( \tilde{A}(T) A(T) = \text{diag}(e^{i\varphi_1}, \ldots, e^{i\varphi_N}) \) is the generalized holonomy transformation with respect to the reference section \( |n(T)\rangle \).

#### B. Non-Abelian Holonomies

Again we consider the master equation Eq. [10] with an adiabatically changing Hamiltonian. For simplicity we restrict to a cyclic Hamiltonian \( H(T) = H(0) \).

Until now we assumed the eigenenergies of the Hamiltonian to be non-degenerate. However, if the eigenvalues
are degenerate we expect to get non-Abelian holonomies as the generalization of the geometric phase. In this case we have the eigenvalue equation

$$H(t)|n_m(t)\rangle = E_n(t)|n_m(t)\rangle$$  \hspace{1cm} (65)$$

in which $n = 1, \cdots , N$; $m = 1, \cdots , M_n$ and $M_n$ is the degree of degeneracy of the subspace of the Hamiltonian with energy $E_n$. The $M_n$ are required to be constant in time, i.e., we do not allow any level crossings of the Hamiltonian $H(t)$. The $|n_m(t)\rangle$ are now assumed to satisfy the modified parallel transport condition $\hspace{1cm}$

$$\langle n_m(t)|\frac{d}{dt}|n_{m'}(t)\rangle = 0 \quad \forall \ m, m' = 1, \cdots , M_n. \hspace{1cm} (66)$$

Now we can define the operator $A(t)$ by

$$A(t)|n_m(0)\rangle = |n_m(t)\rangle$$  \hspace{1cm} (67)$$

and with its help we transform the master equation in the rotating axis representation to remove the time dependence of the eigenspaces of the Hamiltonian analogous to Section 11. Doing this we get the new master equation Eq. (12) for $\rho^A(t) = A(t)\rho(t)A(t)^\dagger$ as before. Again it can be shown that the terms $\rho^A(t)A(t)^\dagger A(t)$ and $A(t)^\dagger A(t)\rho^A(t)$ can be neglected if the $|n_m(t)\rangle$ satisfy the modified parallel transport condition Eq. (66) in the adiabatic and weak damping limit. This justifies the condition Eq. (66). Since the proof for this is much the same as for non-degenerate Hamiltonians as done in the Appendix, we do not carry out the proof in this case. Now we have to solve (16) which represents the dynamics. Finally we have to transform back to the original picture:

$$\rho(T) = A(T)^\dagger \rho^A(T)^\dagger A(T).$$  \hspace{1cm} (68)$$

We obtain the non-Abelian holonomy $A(T) \in U(M_1) \otimes \cdots \otimes U(M_N)$ similar to the way obtained the geometric phase for non-degenerate systems.

V. CONCLUSIONS AND FUTURE DIRECTIONS

We have introduced the rotating axis transformation, in which parallel transport of the eigenstates of the Hamiltonian plays an important role, to study the geometric phase for an adiabatically evolving multilevel system. This transformation was shown to be particularly useful in simplifying the calculation of open-system evolution, described by a master equation of the Lindblad form, as it allows an easy separation of dynamic and geometric phases. These advantages were illustrated by applying it to optical resonance with spontaneous emission, where we obtain known results but more easily. The method was then used to quickly and easily study the effects of the geometric phase in a number of new problems.

In one application we show explicitly how to remove the dynamic phase.

Although, in our applications, we concentrated on Abelian holonomies for nondegenerate systems, the generalization to non-Abelian holonomies for degenerate Hamiltonians and to non-cyclic evolution is straightforward.

Acknowledgments

BCS acknowledges financial support from an Australian DEST IAP grant to support participation in the European Fifth Framework project QUPRODIS and from Alberta’s informatics Circle of Research Excellence (iCORE) as well as valuable discussions with S. Ghose and K.-P. Marzlin.

APPENDIX A: NEGLECTING $A^\dagger(t)\dot{A}(t)$ IN THE ADIABATIC APPROXIMATION

Here we prove that the terms containing $A^\dagger(t)\dot{A}(t)$ in Eq. (12) can be neglected in the adiabatic approximation. The proof will be analogous to the proof of the adiabatic theorem given in [27]. For simplicity we assume $H(t)$ in Eq. (12) to be diagonal which can always be achieved by a proper time independent transformation and hence is no restriction. We start by transforming Eq. (12) in the interaction picture. We define

$$\rho^H(t) = e^{-i\int_0^t H^A(t) dt} \rho^A(t) e^{i\int_0^t H^A(t) dt}$$

and get by use of Eq. (12) the master equation in the interaction picture

$$\dot{\rho}^H = \rho^H (A^\dagger \dot{A})^H - (A^\dagger \dot{A})^H \rho^H + \frac{1}{2} \sum_{\alpha=1}^k \mathcal{L}^H_{\alpha} [\rho^H]$$  \hspace{1cm} (A1)$$

The formal solution of this equation is

$$\rho^H(t) = \rho^H(0) + \int_0^t \left( \rho^H (A^\dagger \dot{A})^H - (A^\dagger \dot{A})^H \rho^H + \frac{1}{2} \sum_{\alpha=1}^k \mathcal{L}^H_{\alpha} [\rho^H] \right) (s) ds.$$  \hspace{1cm} (A2)$$

Within the integral, there are some contributions of the form of a product of a slowly varying function and a fast oscillating function. These contributions are known to become small when the frequency of the oscillating function increases in comparison with the time derivative of the slowly varying function. To see this we make use...
of the result, following [27],
\[ \int_0^t f(s)e^{i\omega s} \, ds = \frac{1}{i\omega} \left[ f(s)e^{i\omega s} \right]_0^t - \int_0^t f'(s)e^{i\omega s} \, ds \]
\[ \lim_{t \to \infty} 0 \] (A3)

To make use of this we write Eq. (A2) in components

\[ \rho_{ij}^H(t) = \rho_{ij}^H(0) + \int_0^t \left( \rho_{ik}^H (A^\dagger A)^H_{kj} - (A^\dagger A)^H_{ik} \rho_{kj}^H \right) \]
\[ + \frac{i}{\omega} \sum_{\alpha=1}^k 2\Gamma_{\alpha ik} \rho_{kl}^H \Gamma_{\alpha jl} - \Gamma_{\alpha ik} \Gamma_{\alpha jl} \rho_{ij}^H \]
\[ - \rho_{ik}^H \Gamma_{\alpha kl} \Gamma_{\alpha lj}^* (s) \, ds. \] (A4)

Furthermore we can rewrite Eq. (A4) as

\[ \rho_{ij}^H(t) = \rho_{ij}^H(0) + \int_0^t \left( \frac{i}{\omega} \sum_{\alpha=1}^k 2\Gamma_{\alpha ik} \rho_{kl}^H \Gamma_{\alpha jl}^* \right) \]
\[ - \Gamma_{\alpha ik} \Gamma_{\alpha kl} \rho_{ij}^H + \Gamma_{\alpha ik} \Gamma_{\alpha kl}^* \rho_{ij}^H \]
\[ - \Gamma_{\alpha ik} \Gamma_{\alpha kl}^* \rho_{ij}^H + \Gamma_{\alpha kl} \rho_{ij}^H \]
\[ - \rho_{ik}^H \Gamma_{\alpha kl} \Gamma_{\alpha lj}^* (s) \, ds. \] (A5)

The star denotes complex conjugation. The functions \( \Gamma_{\alpha il}^* \) are oscillating with frequency \( \omega_{ik} - \omega_{ij} \). Hence, we can achieve a significant simplification if the differences of all eigenfrequencies, \( \omega_{ik} - \omega_{ij} \), are vanishing (are big in comparison with \( \tilde{H} \) and \( \Gamma_{\alpha} \)) which is always the case if we consider a two-level system. Then the condition \( \omega_{ik} - \omega_{ij} = 0 \) implies \( i = k, j = l \) or \( i = j, k = l \) and hence only corresponding parts will contribute to the integral:

\[ \rho_{ij}^H(t) = \rho_{ij}^H(0) + \int_0^t \left( \frac{i}{\omega} \sum_{\alpha=1}^k 2\delta_{ij} \Gamma_{\alpha ik} \rho_{kl}^H \Gamma_{\alpha ik}^* \right) \]
\[ - 2\delta_{ij} \Gamma_{\alpha ik}^* \rho_{kl}^H \Gamma_{\alpha ik} + 2\Gamma_{\alpha il}^* \rho_{ij}^H \Gamma_{\alpha il} \]
\[ - \Gamma_{\alpha ik} \Gamma_{\alpha kl} \rho_{ij}^H + \Gamma_{\alpha kl} \rho_{ij}^H \]
\[ - \rho_{ik}^H \Gamma_{\alpha kl} \Gamma_{\alpha lj}^* (s) \, ds. \] (A6)

Here we see that only the absolute value of the off-diagonal elements of \( \Gamma_{\alpha}^* \) and hence \( \Gamma_{\alpha} \) contribute in Eq. (12).

We can further note that if we set \( i = j \), we find that the diagonal elements of the density operator are coupled only to diagonal elements, whereas for \( i \neq j \), we find that \( \rho_{ij}^H \) is coupled only to itself.

[23] A. Carollo, I. Fuentes-Guridi, M. Franca Santos, and


