Auxiliary fields and the flux tube model

Fabien Buisseret and Claude Semay

Service de Physique Générale et de Physique des Particules Élémentaires,
Groupe de Physique Nucléaire Théorique, Université de Mons-Hainaut,
Place du Parc 20, B-7000 Mons, Belgium

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Abstract

It is possible to eliminate exactly all the auxiliary fields (einbein fields) appearing in the rotating string Hamiltonian to obtain the classical equations of motion of the relativistic flux tube model. A clear interpretation can then be done for the characteristic variables of the rotating string model.

It has been recently shown that the solutions of the Nambu-Goto Lagrangian for two quarks attached with a straight string are given by the solutions of the relativistic flux tube model \[1\]. To get rid of the square roots appearing in this Lagrangian, a Hamiltonian for a rotating string containing auxiliary fields (also known as einbein fields) was previously written from this Lagrangian \[2, 3\]. It has been shown that, in the case of two equal quark masses, the equations of motion of the rotating string reduce exactly to the classical equations of motion of the relativistic flux tube model, provided all auxiliary fields are eliminated correctly \[4\]. In this work, we generalize this result to the case of two quarks with different masses, and we provide a clear interpretation for the characteristic variables of the rotating string model.

Partial results about the elimination of the auxiliary fields for the rotating string Hamiltonian have been obtained in Ref. \[2\], but we perform here the complete calculation. Moreover, we show that some approximations which seems to be used in Ref. \[2\] are not necessary.

The Lagrangian for a system of two (spinless) quarks with masses \(m_1\) and \(m_2\) attached with a string can be written \[3\]

\[
\mathcal{L}(\tau) = -m_1 \sqrt{x_1^2} - m_2 \sqrt{x_2^2} - a \int_0^1 d\beta \sqrt{(\dot{w}w')^2 - \dot{w}^2 w'^2},
\]

(1)

where \(a\) is the string tension. \(x_1(\tau)\) and \(x_2(\tau)\) are the world lines of the two quarks, which depend on the common proper time \(\tau_1 = \tau_2 = \tau\). The four-vector \(w(\tau, \beta)\) is the string coordinate, parameterized by \(\beta \in [0, 1]\). The notations \(\dot{x} = dx/d\tau\) and \(w' = dw/d\beta\) are used. The introduction of auxiliary fields to get rid of the square roots leads to the following Lagrangian

\[
\mathcal{L} = -\frac{1}{2} \left\{ \frac{m_1^2}{\mu_1} + \mu_1 \dot{x}_1^2 + \frac{m_2^2}{\mu_2} + \mu_2 \dot{x}_2^2 \right. \\
+ \left. \int_0^1 d\beta \left[ \dot{\nu}^2 - \frac{\eta^2}{\nu^2} w'^2 - 2\nu(\dot{w}w') + \eta^2 w'^2 \right] \right\}.
\]

(2)

As we will see later, the field \(\mu_i(\tau)\) can be considered as a constituent mass for the quark \(i\), while the field \(\nu(\tau, \beta)\) plays the role of a string energy density. The field \(\eta(\tau, \beta)\) is the easiest field to eliminate. In the following, we will adopt the straight-line ansatz for the minimal string

\[
w(\tau, \beta) = \beta x_1(\tau) + (1 - \beta) x_2(\tau).
\]

(3)
We define a set of relative and center of mass coordinates

\[ r(\tau) = x_1(\tau) - x_2(\tau), \quad (4a) \]
\[ R(\tau) = \zeta(\tau)x_1(\tau) + (1 - \zeta(\tau))x_2(\tau), \quad (4b) \]

The parameter \( \zeta(\tau) \) will be determined below. With these new coordinates, the condition (3) is written (we will now drop the notations of the various dependencies of the fields on \( \tau \) and \( \beta \))

\[ w = R + (\beta - \zeta)r, \quad (5) \]

and the Lagrangian (2) becomes

\[
\mathcal{L} = -\frac{1}{2} \left[ \frac{m_1^2}{\mu_1} + \frac{m_2^2}{\mu_2} + a_1 \dot{R}^2 + 2a_2 \dot{R}\dot{r} - 2(c_1 + \dot{\zeta}a_1)\dot{R}r - 2(c_2 + \dot{\zeta}a_2)\dot{r}r \\
+ a_3r^2 + (a_4 + 2\dot{\zeta}c_1 + \dot{\zeta}^2a_1)r^2 \right]. \quad (6)
\]

The following notations are used

\[
a_1 = \mu_1 + \mu_2 + \int_0^1 d\beta \nu, \quad (7a) \\
a_2 = \mu_1 - \zeta(\mu_1 + \mu_2) + \int_0^1 d\beta (\beta - \zeta) \nu, \quad (7b) \\
a_3 = \mu_1(1 - \zeta)^2 + \mu_2\zeta^2 + \int_0^1 d\beta (\beta - \zeta)^2 \nu, \quad (7c) \\
a_4 = \int_0^1 d\beta \left( \eta^2 \nu - \frac{a_1^2}{\nu} \right), \quad (7d) \\
c_1 = \int_0^1 d\beta \eta \nu, \quad (7e) \\
c_2 = \int_0^1 d\beta (\beta - \zeta) \eta \nu. \quad (7f)
\]

We can extract three useful relations

\[
\int_0^1 d\beta \nu = a_1 - \mu_1 - \mu_2, \quad (8a) \\
\int_0^1 d\beta \beta \nu = a_2 + \zeta a_1 - \mu_1, \quad (8b) \\
\int_0^1 d\beta \beta^2 \nu = a_3 + \zeta^2 a_1 + 2\zeta a_2 - \mu_1. \quad (8c)
\]

In the following, the contravariant components of a four-vector \( v \) will be noted \((v^0, \vec{v})\) and \( v^2 = (v^0)^2 - \vec{v}^2 \). The use of a common proper time implies that \( r^0 = \tau^0 = 0 \) and that
\[ \dot{\vec{R}}^0 = 1. \] Consequently, the Lagrangian can be recast into

\[
\mathcal{L} = \frac{1}{2} \left[ \frac{m_1^2}{\mu_1} + \frac{m_2^2}{\mu_2} + a_1 - a_1 \ddot{\vec{R}}^2 - 2a_2 \dot{\vec{R}} \vec{r} + 2(c_1 + \dot{\zeta} a_1) \ddot{\vec{r}} + 2(c_2 + \dot{\zeta} a_2) \dot{\vec{r}} \vec{r} - a_3 \vec{r}^2 - (a_4 + 2\dot{\zeta} c_1 + \zeta^2 a_1) \vec{r}^2 \right].
\] (9)

As we will work in the center of mass frame, the total vector momentum \( \vec{P}_{cm} \) of the system must vanish

\[
\vec{P}_{cm} = -\frac{\partial \mathcal{L}}{\partial \dot{\vec{R}}} = \vec{0},
\] (10)

which implies that

\[
\dot{\vec{R}} = \frac{(c_1 + \dot{\zeta} a_1) \vec{r} - a_2 \vec{r}}{a_1}.
\] (11)

Moreover, the relative vector momentum \( \vec{p} \) is given by

\[
\vec{p} = -\frac{\partial \mathcal{L}}{\partial \vec{r}} = -a_2 \dot{\vec{R}} + (c_2 + \dot{\zeta} a_2) \vec{r} - a_3 \vec{r}.
\] (12)

Thus, we impose \( a_2 = 0 \) in order that \( \vec{p} \) does not depend on the motion of the center of mass. This leads to the following value for the parameter \( \zeta \)

\[
\zeta = \frac{\mu_1 + \int_0^1 d\beta \beta \nu}{\mu_1 + \mu_2 + \int_0^1 d\beta \nu}.
\] (13)

This quantity, which determines the center of mass coordinate, contains contributions not only from the quarks but also from the string. From this expression, it is clear that fields \( \mu_1 \) and \( \mu_2 \) play the role of dynamical masses for the quarks, while the field \( \nu \) can be interpreted as a dynamical string energy density. The substitution of \( \dot{\vec{R}} \) by the formula (11) and the replacement of \( a_2 \) by zero give the following Lagrangian

\[
\mathcal{L} = -\frac{1}{2} \left[ \frac{m_1^2}{\mu_1} + \frac{m_2^2}{\mu_2} + a_1 + \frac{1}{a_1} \left( (c_1^2 - a_4 a_1) \vec{r}^2 - a_3 a_1 \vec{r}^2 + 2a_1 c_2 \vec{r} \vec{r} \right) \right].
\] (14)

It is worth noting that all terms with \( \dot{\zeta} \) have disappeared. It is thus not necessary to assume \( \zeta \) independent of \( \tau \) as it seems to be the case in Ref. [2].

The Lagrangian (14) contains auxiliary fields through the variables \( a_i \) and \( c_i \). The field \( \eta \) can be first eliminated by the condition \( \delta \mathcal{L}/\delta \eta = 0 \) [2]. This gives the following expression

\[
2(\vec{r} \vec{r}) \delta c_2 + \frac{\vec{r}^2}{a_1} [2c_1 \delta c_1 - a_1 \delta a_4] = 0.
\] (15)
With relations (8), this equation can be recast into the form
\[ c_1 = \frac{\dot{r} \cdot r}{r^2} a_1 \left( \zeta - \frac{\mu_1}{\mu_1 + \mu_2} \right). \]  
(16)

The use of Eq. (7a), with the condition \( a_2 = 0 \), gives the solution
\[ \eta_0 = \frac{\dot{r} \cdot r}{r^2} \left( \beta - \frac{\mu_1}{\mu_1 + \mu_2} \right). \]  
(17)

Knowing the extremal value of the auxiliary field \( \eta_0 \), the coefficients \( c_1, c_2, \) and \( a_4 \) can be computed. We thus obtain
\[ c_2 = \frac{\dot{r} \cdot r}{r^2} (a_3 - \mu), \]  
(18a)
\[ c_1^2 - a_4 a_1 = a_1 \left[ \int_0^1 d\beta \frac{a^2}{\nu} - (a_3 - \mu) \frac{(\dot{r} \cdot r)^2}{r^4} \right], \]  
(18b)
where \( \mu = \mu_1 \mu_2 / (\mu_1 + \mu_2) \).

It is now interesting to separate the longitudinal and transverse components of \( \dot{r} \) with respect to \( r \)
\[ \dot{r}^2 = \dot{r}_\parallel^2 + \dot{r}_\perp^2 = \frac{(\dot{r} \cdot r)^2}{r^2} + \frac{(\dot{r} \times r)^2}{r^2}, \]  
(19)
to obtain
\[ \mathcal{L} = -\frac{1}{2} \left[ \frac{p^2_\parallel}{\mu_1} + \frac{p^2_\perp}{\mu_2} + a_1 + a^2 \dot{r}^2 \int_0^1 \frac{d\beta}{\nu} - a_3 \dot{r}_\perp^2 - \mu \dot{r}_\parallel^2 \right]. \]  
(20)

The canonical transformation \( H = -\dot{r}_\parallel p_\parallel - \dot{r}_\perp p_\perp - \mathcal{L} \), with
\[ \begin{align*}
\dot{p}_\parallel &= -\frac{\partial \mathcal{L}}{\partial \dot{r}_\parallel} = -\mu \dot{r}_\parallel, \\
\dot{p}_\perp &= -\frac{\partial \mathcal{L}}{\partial \dot{r}_\perp} = -a_3 \dot{r}_\perp,
\end{align*} \]  
(21a)
(21b)
lead to the following Hamiltonian (now, \( r^2 \) always means \( \dot{r}^2 \))
\[ H = \frac{1}{2} \left[ \frac{p^2_\parallel}{\mu_1} + \frac{p^2_\perp}{\mu_2} + \mu_1 + \mu_2 + a^2 \dot{r}^2 \int_0^1 \frac{d\beta}{\nu} + \int_0^1 d\beta \nu + \frac{L^2}{a_3 \dot{r}^2} \right], \]  
(22)
where the orbital angular momentum is given by the usual relation \( L^2 = \overline{p}_\perp \dot{r}^2 \). Let us note that the string does not contribute to the longitudinal component of the momentum \( 2 \).

We will now proceed to the elimination of the auxiliary field \( \nu \) by demanding that \( \delta H/\delta \nu = 0 \) \( 2 \). This leads to the following expression
\[ \int_0^1 d\beta \left( 1 - \frac{a^2 \dot{r}^2}{\nu^2} \right) \delta \nu - \frac{L^2}{a_3 \dot{r}^2} \delta a_3 = 0. \]  
(23)
Using the definition (7c) and the condition $a_2 = 0$ (13), we obtain

$$\delta a_3 = -2a_2 \delta \zeta + \int_0^1 d\beta (\beta - \zeta)^2 \delta \nu = \int_0^1 d\beta (\beta - \zeta)^2 \delta \nu. \quad (24)$$

So, we have to solve

$$\int_0^1 d\beta \delta \nu \left( 1 - \frac{a^2 r^2}{\nu^2} - \frac{L^2}{a^2 r^2} (\beta - \zeta)^2 \right) = 0. \quad (25)$$

In order to simplify the notation, we introduce the variable $y$

$$y^2 = \frac{L^2}{a^2 r^2}. \quad (26)$$

The extremal field $\nu_0$ solution of Eq. (25) is

$$\nu_0 = \frac{ar}{\sqrt{1 - y^2 (\beta - \zeta)^2}}. \quad (27)$$

It is now possible to compute the various coefficients present in the Hamiltonian (22). The following integrals are given by analytical formulas

$$\int_0^1 d\beta \nu = \frac{ar}{y} [\arcsin s]^y_1, \quad (28)$$

$$\int_0^1 d\beta \frac{\nu}{\nu} = \frac{1}{2ary} \left[ s\sqrt{1 - s^2} + \arcsin s \right]^y_1, \quad (29)$$

$$\int_0^1 d\beta (\beta - \zeta) \nu = -\frac{ar}{y^2} \left[ 1 - s^2 \right]^y_1, \quad (30)$$

$$\int_0^1 d\beta (\beta - \zeta)^2 \nu = \frac{ar}{2y^3} \left[ -s\sqrt{1 - s^2} + \arcsin s \right]^y_1, \quad (31)$$

with the notations

$$y_1 = (1 - \zeta)y \quad \text{and} \quad y_2 = \zeta y. \quad (32)$$

Using the constraint $a_2 y = 0$ (13), the relation $a_3 y = L/r$ (26), and the Hamiltonian (22), we obtain a set of three coupled equations for the rotating string

$$0 = \mu_1 y_1 - \mu_2 y_2 - \frac{ar}{y} \left( \sqrt{1 - y_1^2} - \sqrt{1 - y_2^2} \right), \quad (33a)$$

$$\frac{L}{r} = \frac{1}{y} (\mu_1 y_1^2 + \mu_2 y_2^2) + \frac{ar}{y^2} (F(y_1) + F(y_2)), \quad (33b)$$

$$H = \frac{1}{2} \left[ \frac{p^2}{\mu_1} + \frac{m^2}{\mu_1} + \frac{p^2}{\mu_2} + \frac{m^2}{\mu_2} + \mu_1 (1 + y_1^2) + \mu_2 (1 + y_2^2) \right]$$

$$+ \frac{ar}{y} (\arcsin y_1 + \arcsin y_2), \quad (33c)$$

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with

$$F(y_i) = \frac{1}{2} \left[ \arcsin y_i - y_i \sqrt{1 - y_i^2} \right]. \quad (33d)$$

The elimination of the remaining auxiliary fields $\mu_1$ and $\mu_2$ is obtained using the constraints $\delta H/\delta \mu_i = 0$. As in Ref. [5], we also ask $\delta (L/r)/\delta \mu_i = 0$. So we can write

$$2\partial_{\mu_i} H - 2y \partial_{\mu_i} (L/r) = 0. \quad (34)$$

These equations lead to the following relations

$$- \frac{p_r^2 + m_1^2}{\mu_1^2} + 1 - y_1^2 + 2A\partial_{\mu_1} y_1 + 2B\partial_{\mu_1} y_2 = 0, \quad (35a)$$

$$- \frac{p_r^2 + m_2^2}{\mu_2^2} + 1 - y_2^2 + 2B\partial_{\mu_2} y_1 + 2A\partial_{\mu_2} y_2 = 0, \quad (35b)$$

with

$$A = -\mu_1 y_1 + \frac{ar}{y} \sqrt{1 - y_1^2} + \frac{1}{y} (\mu_1 y_1^2 + \mu_2 y_2^2) - \frac{ar}{y^2} \left( y_1 \sqrt{1 - y_1^2} + y_2 \sqrt{1 - y_2^2} \right), \quad (36a)$$

$$B = -\mu_2 y_2 + \frac{ar}{y} \sqrt{1 - y_2^2} + \frac{1}{y} (\mu_1 y_1^2 + \mu_2 y_2^2) - \frac{ar}{y^2} \left( y_1 \sqrt{1 - y_1^2} + y_2 \sqrt{1 - y_2^2} \right). \quad (36b)$$

It is easy to see that $B - A = a_2 y = 0$. Moreover, using definitions (32), it can be shown that $A = -\zeta a_2 y = 0$. Finally, $A = B = 0$ and the extremal values of the auxiliary fields $\mu_i$ are given by

$$\mu_i = \sqrt{\frac{p_r^2 + m_i^2}{1 - y_i^2}}. \quad (37)$$

If we interpret the variables $y_i$ as the perpendicular speed of the quark $v_{i\perp}$, we can write $\mu_i = W_{ir} \gamma_i$ with $W_{ir} = \sqrt{p_r^2 + m_i^2}$ and $\gamma_i = 1/\sqrt{1 - v_{i\perp}^2}$, in the usual relativistic flux tube notations [3]. The set of equations (33) can then be recast into the form

$$P_\perp = 0 = W_{1r} \gamma_{1\perp} v_{1\perp} - W_{2r} \gamma_{2\perp} v_{2\perp} + \frac{ar}{v_{1\perp} + v_{2\perp}} \left( \sqrt{1 - v_{1\perp}^2} - \sqrt{1 - v_{2\perp}^2} \right), \quad (38a)$$

$$\frac{L}{r} = \frac{W_{1r} \gamma_{1\perp} v_{1\perp}^2}{v_{1\perp} + v_{2\perp}} + W_{2r} \frac{\gamma_{2\perp} v_{2\perp}^2}{v_{1\perp} + v_{2\perp}} + \frac{ar}{(v_{1\perp} + v_{2\perp})^2} (F(v_{1\perp}) + F(v_{2\perp})), \quad (38b)$$

$$H = \gamma_{1\perp} W_{1r} + \gamma_{2\perp} W_{2r} + \frac{ar}{v_{1\perp} + v_{2\perp}} (\arcsin v_{1\perp} + \arcsin v_{2\perp}), \quad (38c)$$

with

$$F(v_{i\perp}) = \frac{1}{2} \left[ \arcsin v_{i\perp} - v_{i\perp} \sqrt{1 - v_{i\perp}^2} \right]. \quad (38d)$$

These equations are exactly the asymmetric relativistic flux tube equations of Ref. [5]. Let us note that, in Ref. [2], conditions about the extremal values of the auxiliary fields $\mu_i$ and
\( \nu \) are given. It seems that they are obtained assuming that \( \zeta \) is independent of these fields. We see here that the calculation can be completed without this assumption.

The equations obtained are the classical ones. For practical calculations, they must be quantized. This can be done in three steps:

1. to replace \( L \) by \( \sqrt{\ell(\ell + 1)} \), where \( \ell \) is the orbital angular momentum quantum number;
2. to replace \( p_r \) by the operator \(-\frac{1}{r} \frac{\partial^2}{\partial r^2} r\);
3. to symmetrize the various non-commuting operators.

The contribution of a potential \( V \), simulating effects not taken into account by the rotating string, can be considered in the starting Lagrangian. In this case, the calculation shows that it is simply added to the Hamiltonian in Eq. Such an interaction must depend on the module of the relative separation between quarks. With the equal time approximation, we have simply \( V = V(\vec{r}^2) \).

It is shown here that the rotating string model with the straight-line ansatz reduces exactly to the relativistic flux tube model, when all auxiliary fields are correctly eliminated. This result generalizes the work of Ref. to the case of two different quark masses. Again a clear interpretation can be done for the various characteristic fields of the rotating string Hamiltonian.

Despite these considerations, the rotating string equations are nevertheless interesting to consider, because analytical approximated solutions and quite precise numerical solutions (using the WKB approximation) can be obtained for these equations. These solutions are good approximations of the solutions of the genuine relativistic flux tube equations.

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