BIANCHI COSMOLOGIES:
A TALE OF TWO TILTED FLUIDS

ALAN A. COLEY & SIGBJØRN HERVIK

Abstract. We use a dynamical systems approach to study Bianchi type VI0 cosmological models containing two tilted $\gamma$-law perfect fluids. The full state space is 11-dimensional, but the existence of a monotonic function simplifies the analysis considerably. We restrict attention to a particular, physically interesting, invariant subspace and find all equilibrium points that are future stable in the full 11-dimensional state space; these are consequently local attractors and serve as late-time asymptotes for an open set of tilted type VI0 models containing two tilted fluids. We find that if one of the fluids has an equation of state parameter $\hat{\gamma} < 6/5$, the stiffest fluid will be dynamically insignificant at late times. For the value $\hat{\gamma} = 6/5$ there is a 2-dimensional bifurcation set, and if both fluids are stiffer than $\hat{\gamma} = 6/5$ both fluids will have extreme tilt asymptotically. We investigate the case in which one fluid is extremely tilting in detail. We also consider the case with one stiff fluid ($\hat{\gamma} = 2$) close to the initial singularity, and find that the chaotic behaviour which occurs in general Bianchi models with $\hat{\gamma} < 2$ is suppressed.

1. Introduction

In [1] the asymptotic late-time behaviour of general tilted Bianchi type VI0 universes with a tilted $\gamma$-law perfect fluid was analysed. All of the future stable equilibrium points for various subclasses of tilted type VI0 models, as well as for the general tilted type VI0 models, were found. In particular, for the particular value of the equation of state parameter, $\gamma = 6/5$, there exists a bifurcation line which signals a transition of stability between a non-tilted equilibrium point to an extremely tilted equilibrium point, confirming the observation in [2], which followed up on the earlier work on tilted fluids [3–9].

In this paper, we will generalize this work and use the dynamical systems approach [10, 11] to analyse the Bianchi type VI0 model containing two (in general both) tilted (and separately conserved) $\gamma$-law perfect fluids [12]. We shall use the formalism employed in [1], and pay particular attention to some of the cases which were not analyzed in detail there. In particular, we shall study the two physically important cases of when the second fluid is a comoving stiff fluid (i.e., $\Gamma = 2$ and $\tilde{V} = 0$ in the notation below), which represents a scalar field close to the singularity, and when the second fluid is of extreme tilt (i.e., $\tilde{V} = 1$). The study of extremely tilted fluids is partly motivated from the brane-world scenario where gravitational waves in the bulk can, through the interaction of the bulk Weyl tensor with the brane, give rise to extremely tilted fluids as seen from a brane point of view.

The Bianchi type VI0 model is not the most general model but is sufficiently general to account for many interesting phenomena. Bianchi type VI0 models with $\hat{\gamma} < 2$ are suppressed.
various sources of matter has been studied. The type VI$_0$ model with a comoving perfect fluid was considered in [13], with a magnetic and a $\gamma$-law perfect fluid in [14], and with a tilted $\gamma$-law perfect fluid in [1]. In all of these studies there has been evidence that the type VI$_0$ model is asymptotically self-similar at late times; the generic solution will approach an equilibrium point in the dynamical system. In the two tilted model the state space is 11-dimensional, and obtaining all of the equilibrium points is an extremely laborious task. However, guided by previous work we concentrate on the equilibrium points in a particularly interesting subspace; we will then show that these equilibrium points are stable in the full 11-dimensional state space.

The paper is organised as follows: Next, in section 2, we write down the equations of motion using the orthonormal frame formalism and discuss the various invariant subspaces. In section 3 we discuss the (relevant) equilibrium points and their stability, and present some general results regarding the asymptotic behaviour of the tilted type VI$_0$ model. In section 4 we consider the case in which one fluid is extremely tilted. In section 5 we consider the case of a stiff fluid. Close to the initial singularity a scalar field is essentially massless and can be modelled by a comoving perfect fluid with a stiff equation of state parameter $\gamma = 2$. This will enable us to consider the initial singular regime in this case. Finally we summarise our results in section 6.

1.1. Preliminaries. In this paper we shall assume the existence of two (in general both tilting) perfect fluids. We shall denote the second fluid using tildes (e.g., $\tilde{\Omega}$, $\tilde{V}$), and use the notation $\Gamma = \tilde{\gamma}$, where by convention we assume that $\Gamma > \gamma$. We shall also assume that each fluid is separately conserved.

The energy-momentum tensor for each tilted perfect fluid is

$$T_{\mu\nu} = (\hat{\rho} + \hat{p})\hat{u}_\mu\hat{u}_\nu + \hat{p}g_{\mu\nu},$$

where $\hat{u}^\alpha = (\cosh \theta, \sinh \theta c_a)$ is the fluid velocity and $\theta$ its rapidity. The spatial vector $c^a$ is chosen to be a unit vector in the tangent space of the surfaces of homogeneity; i.e. $c^ae_a = 1$. We will further assume that the each fluid obeys the barotropic equation of state,

$$\hat{p} = (\hat{\gamma} - 1)\hat{\rho}. $$

In terms of the unit normal vector $\mathbf{u} = e_0$ to the group orbits the energy-momentum tensor takes the imperfect fluid form, where $\rho, p, q_a, \pi_{ab}$ can be explicitly written down. For the Bianchi cosmologies we can always write the line-element in canonical form with respect to the structure constants $C^a_{bc}$ of the simply transitive Bianchi group type under consideration, which depend only on time [15]. For the type VI$_0$ model, $a_c = 0$ and $n_{ab}$ has two non-zero eigenvalues with opposite sign. This implies that we can choose a frame such that the structure constants can be written

$$n_{ab} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \tilde{n} & n \\ 0 & n & \tilde{n} \end{bmatrix}, \quad a_b = 0. $$

Furthermore, the type VI$_0$ has $\tilde{n}^2 < n^2$. The equations of motion can now be written down in terms of expansion-normalised variables.
2. Equation of motion

We define the expansion-normalised variables:

\[ \Sigma_{ab} = \begin{pmatrix} -2\Sigma_+ & \sqrt{3}\Sigma_{12} & \sqrt{3}\Sigma_{13} \\ \sqrt{3}\Sigma_{12} & \Sigma_+ + \sqrt{3}\Sigma_- & \sqrt{3}\Sigma_{23} \\ \sqrt{3}\Sigma_{13} & \sqrt{3}\Sigma_{23} & \Sigma_+ - \sqrt{3}\Sigma_- \end{pmatrix}, \]

\[ N_{22} = N_{33} = \sqrt{3}N, \quad N_{23} = N_{32} = \sqrt{3}N. \]

We also introduce the three-velocity \( V \) by

\[ \sinh \theta = \frac{V}{\sqrt{1 - V^2}}, \quad 0 \leq V < 1. \]

It is also convenient to introduce a parameter \( \lambda \) defined by \( \bar{N} = \lambda N \), which makes it possible to solve for the function \( R_1 \) (where \( R_a \) is the local angular velocity of a Fermi-propagated axis with respect to the triad \( e_a \)) and eliminate it completely from the equations of motion. For the Bianchi type VI\(_0\) model this parameter will be bounded; i.e. \( \lambda^2 < 1 \).

For the expansion-normalised variables the equations of motion are:

\[ \Sigma_+' = (q - 2)\Sigma_+ + 3(\Sigma_{12}^2 + \Sigma_{13}^2) - 2N^2 \]

\[ + \frac{\gamma\Gamma}{2G_+} (-2v_1^2 + v_2^2 + v_3^2) + \frac{\Gamma\tilde{\Omega}}{2G_+} (-2\tilde{\omega}_1^2 + \tilde{\omega}_2^2 + \tilde{\omega}_3^2) \]

\[ \Sigma_-' = (q - 2)\Sigma_- + \sqrt{3}((\Sigma_{12}^2 - \Sigma_{13}^2) - 2\sqrt{3}\lambda\Sigma_- \Sigma_{23}) \]

\[ + \frac{\sqrt{3}\gamma\Omega}{2G_+} (v_2^2 - v_3^2) - \frac{\sqrt{3}\Gamma\tilde{\Omega}}{2G_+} (\tilde{\omega}_2^2 - \tilde{\omega}_3^2) \]

\[ \Sigma_{12}' = (q - 2 - 3\Sigma_+ - \sqrt{3}\Sigma_-) \Sigma_{12} - \sqrt{3}(\lambda\Sigma_- + \Sigma_{23}) \Sigma_{13} \]

\[ + \frac{\sqrt{3}\gamma\Omega}{G_+} v_1 v_2 + \frac{\sqrt{3}\Gamma\tilde{\Omega}}{G_+} \tilde{\omega}_1 \tilde{\omega}_2 \]

\[ \Sigma_{13}' = (q - 2 - 3\Sigma_+ + \sqrt{3}\Sigma_-) \Sigma_{13} - \sqrt{3}(-\lambda\Sigma_- + \Sigma_{23}) \Sigma_{12} \]

\[ + \frac{\sqrt{3}\gamma\Omega}{G_+} v_1 v_3 + \frac{\sqrt{3}\Gamma\tilde{\Omega}}{G_+} \tilde{\omega}_1 \tilde{\omega}_3 \]

\[ \Sigma_{23}' = (q - 2)\Sigma_{23} - 2\sqrt{3}\lambda N^2 + 2\sqrt{3}\lambda\Sigma_{12}^2 + 2\sqrt{3}\Sigma_{12}\Sigma_{13} \]

\[ + \frac{\sqrt{3}\gamma\Omega}{G_+} v_2 v_3 + \frac{\sqrt{3}\Gamma\tilde{\Omega}}{G_+} \tilde{\omega}_2 \tilde{\omega}_3 \]

\[ N' = (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_{23}\lambda) N \]

\[ \lambda' = 2\sqrt{3}\Sigma_{23}(1 - \lambda^2) \]
The equations for the fluids are (similar equations for the second fluid):

\begin{equation}
\Omega' = \frac{\Omega}{G_+} \left\{ 2q - (3\gamma - 2) + [2q(\gamma - 1) - (2 - \gamma) - \gamma S] V^2 \right\}
\end{equation}

\begin{equation}
v_1' = (T + 2\Sigma_+) v_1 - 2\sqrt{3}\Sigma_1 v_3 - 2\sqrt{3}\Sigma_2 v_2
\end{equation}

\begin{equation}
v_2' = \left( T - \Sigma_+ - \sqrt{3}\Sigma_- \right) v_2 - \sqrt{3} (\Sigma_2 + \lambda \Sigma_-) v_3
\end{equation}

\begin{equation}
v_3' = \left( T - \Sigma_+ + \sqrt{3}\Sigma_- \right) v_3 - \sqrt{3} (\Sigma_2 - \lambda \Sigma_-) v_2
\end{equation}

\begin{equation}
V' = \frac{V(1 - V^2)}{1 - (\gamma - 1) V^2} \left\{ (3\gamma - 4) - S \right\},
\end{equation}

where

\begin{align*}
q &= 2\Sigma^2 + \frac{1}{2} (3\gamma - 2) + (2 - \gamma) V^2 \Omega + \frac{1}{2} (3\Gamma - 2) + (2 - \Gamma) \tilde{\Omega}^2, \\
\Sigma^2 &= \Sigma^2_+ + \Sigma^2_- + \Sigma^2_{12} + \Sigma^2_{13} + \Sigma^2_{23}, \\
S &= \Sigma_{a} c^{a} \bar{c}^{\bar{a}}, \quad c^a e_a = 1, \quad \bar{c}^\alpha = V e^\alpha, \\
V^2 &= v_1^2 + v_2^2 + v_3^2, \\
G_+ &= 1 + (\gamma - 1) V^2, \\
T &= \frac{(3\gamma - 4)(1 - V^2) + (2 - \gamma) V^2 S}{1 - (\gamma - 1) V^2}.
\end{align*}

These variables are subject to the constraints

\begin{align}
1 &= \Sigma^2 + N^2 + \Omega + \tilde{\Omega} \\
0 &= 2\Sigma_- N + \frac{\gamma \Omega v_1}{G_+} + \frac{\Gamma \Omega \tilde{\Omega}}{G_+} \\
0 &= - (\Sigma_{12} + \Sigma_{13} \lambda) N + \frac{\gamma \Omega v_2}{G_+} + \frac{\Gamma \Omega \tilde{\Omega}}{G_+} \\
0 &= (\Sigma_{13} + \Sigma_{12} \lambda) N + \frac{\gamma \Omega v_3}{G_+} + \frac{\Gamma \Omega \tilde{\Omega}}{G_+}.
\end{align}

The parameter \( \gamma \) will be assumed to be in the interval \( \gamma \in (0, 2) \) and we can without loss of generality assume that \( \gamma < \Gamma \).

Eqs \( (2.18), (2.20) \) essentially define \( \tilde{v}_i \), and so the governing equations are a twelve-dimensional set subject to a single constraint (eq. \( (2.17) \)); hence the dynamical system is effectively eleven-dimensional.

2.1. **The state space.** The constraint \( (2.17) \) implies that \( \Sigma_1, \Sigma_{12}, \Sigma_{13}, \Sigma_{23}, N, \Omega \) and \( \Omega \) are all bounded. Combining the equations for \( N \) and \( \tilde{N} \) we get the equations

\begin{equation}
(N \pm \tilde{N})' = \left( q + 2\Sigma_+ \pm 2\sqrt{3}\Sigma_{23} \right) (N \pm \tilde{N}).
\end{equation}
Thus if the initial data have $\bar{N}^2 < N^2$, then this will hold for all times\(^1\). Hence, for type VI\(_0\), $\bar{N}$ will be bounded as well. The invariant subspaces $N \pm \bar{N} = 0$ correspond to Bianchi type II universes. We also require $0 \leq V, \tilde{V} \leq 1$ for physical reasons. This implies the bounds

$$\Sigma_2^2 + \Sigma_+^2 + \Sigma_{12}^2 + \Sigma_{13}^2 + \Sigma_{23}^2 + \Sigma_{22}^2 + \Sigma_{23}^2 + N^2 \leq 1$$

$$\bar{N}^2 \leq N^2.$$ 

We have, therefore, the 15 bounded variables $\Sigma_{\pm}$, $\Sigma_{12}$, $\Sigma_{13}$, $\Sigma_{23}$, $N$, $\bar{N}$, $v_i$, $\tilde{v}_i$, $\Omega$ and $\bar{\Omega}$. However, due to 4 constraints, these are not all independent. Hence, the state space can be considered a subspace of a compact region in $\mathbb{R}^{11}$. There are also some discrete symmetries which are intrinsic to the type VI\(_0\) geometry [1].

2.2. Invariant subspaces. Some of the physically interesting invariant subspaces are as follows (we will also assume that the boundaries are included):

1. $T(VI_0)$: The full state space of tilted type VI\(_0\).
2. $F(VI_0)$: A Bianchi tilted type VI\(_0\) with two tilt-degrees of freedom; $\Sigma_{12} = \Sigma_{23} = \bar{N} = 0$ (or $\Sigma_{13} = \Sigma_{23} = \bar{N} = 0$), $v_2 = 0$.
3. $T_2(VI_0)$: A Bianchi tilted type VI\(_0\) with two tilt degrees of freedom; $\Sigma_{12} = \Sigma_{13} = \Sigma_{23} = 0$.
4. $T_1(VI_0)$: Tilted Bianchi type VI\(_0\) with one tilt degree of freedom; $\Sigma_{12} = \Sigma_{13} = 0$, $v_2 = v_3 = 0$.
5. $B(VI_0)$: Non-tilted Bianchi type VI\(_0\); $\Sigma_{\pm} = \Sigma_{12} = \Sigma_{13} = V = \tilde{V} = 0$.
6. $T_\pm(II)$: The tilted type II boundary; $N = \pm \bar{N}$.
7. $B(II)$: Non-tilted type II; $N^2 = \bar{N}^2$ and $V = \tilde{V} = 0$.
8. $B(I)$: Bianchi type I universes; $N = \bar{N} = V = \Omega = \tilde{V} = 0$.
9. $\partial V B(I)$: The Bianchi type I vacuum boundary; $N = \bar{N} = V = \Omega = \tilde{V} = \bar{\Omega} = 0$.

2.3. A monotonic function. Fortunately, the above system of equations possess a monotonic function with makes it possible to determine an important result regarding the asymptotic behaviour of the two-fluid model. We define the two functions:

$$\beta = \frac{(1 - V^2)^{(2 - \gamma)}}{1 + (\gamma - 1)V^2}, \quad \bar{\beta} = \frac{(1 - \tilde{V}^2)^{(2 - \Gamma)}}{1 + (\Gamma - 1)\tilde{V}^2}.$$ 

Then the function,

$$\chi = \frac{\beta \Omega - \bar{\beta} \bar{\Omega}}{\beta \Omega + \bar{\beta} \bar{\Omega}},$$ 

where $\chi$ is bounded and satisfies $-1 \leq \chi \leq 1$, satisfies the differential equation

$$\chi' = \frac{3}{2}(\Gamma - \gamma)(1 - \chi^2).$$

Hence, assuming $\Gamma > \gamma$, then $\chi' \geq 0$ and $\chi$ is monotonically increasing. From this we can deduce that (for $\chi^2 \neq 1$)

1. $\lim_{\tau \to -\infty} \chi = -1$; i.e., at early times $\beta \Omega \to 0$,
2. $\lim_{\tau \to +\infty} \chi = +1$; i.e., at late times $\bar{\beta} \bar{\Omega} \to 0$.

\(^1\)This can also be seen by using $\lambda$ and eq. (2.10): $\bar{N}^2 < N^2 \Leftrightarrow \lambda^2 < 1.$
At early times, we have that $\beta \Omega \to 0$. If $\beta \neq 0$ (i.e., not extreme tilt), it follows that $\Omega \to 0$ and the first (less stiff) fluid is dynamically negligible at early times. Indeed, if neither fluids have extreme tilt asymptotically, then $\lim_{r \to -\infty} \Omega = 0$, and $\lim_{r \to +\infty} \bar{\Omega} = 0$. Hence, asymptotically the dynamical behaviour is that of a single fluid cosmology. It is therefore of interest to study the behaviour for when (at least) one of the fluids have extreme tilt and under what circumstances this can occur asymptotically. Note that in the case of a comoving stiff perfect fluid ($\bar{\beta} = 1$ and $\Gamma = 2$), the future asymptotic behaviour is governed by the first fluid (and is given in [1]), and the behaviour close to the initial singularity need not be oscillatory.

The behaviour of the monotonic function $\chi$ combined with the earlier work [1] also gives us a hint of where to find the relevant equilibrium points. In [1] we noticed that all the locally future stable equilibrium points occurred in the invariant subspace $F(VI_0)$. The monotonic function $\chi$ tells us that the late-time behaviour is almost entirely dictated by the less stiff fluid. Hence, one would expect that all the locally future stable equilibrium points for the two-fluid model also lie in this invariant subspace. Since the total state space is 11-dimensional, this observation simplifies our analysis considerably. In the following we shall therefore only consider equilibrium points in $F(VI_0)$, but the stability analysis will be performed in the full 11-dimensional state space.

3. Late-time behaviour

The system of equations possesses a wealth of equilibrium points. In Appendix 12 we have listed all of the physically interesting ones in the invariant subspace $F(VI_0)$.

By inspection of the equilibrium points and their eigenvalues, we see that they all play a role in the late-time behaviour for various values of the equation of state parameters. Based on the analysis in the Appendix, the equilibrium points that are locally stable to the future are summarized in Table 11; these represent the possible future asymptotic states of the models. It should be emphasised that some of these equilibrium points have zero eigenvalues so when we refer to stability we mean in the technical sense; e.g., there is 1-dimensional equilibrium set which is normally hyperbolic.

There is another thing worth mentioning regarding the second fluid. If $\bar{\Omega} \to 0$, then the second fluid will decouple and become dynamically insignificant. For these cases the tilt velocity $\bar{V}$ of the equilibrium points indicates the tilt velocity close to the equilibrium points. Only away from (but arbitrary close to) $\bar{\Omega} = 0$ is $\bar{V}$ a meaningful quantity.

Note that for $\gamma = 6/5$ there is a 2-dimensional bifurcation set and for $\gamma \geq 6/5$ both of the fluids are dynamically significant at late times. If $\gamma < 6/5$, on the other hand, the stiffest fluid becomes dynamically insignificant and the late time asymptotic behaviour of the model is entirely dominated by the less stiff fluid.

4. An extreme tilted perfect fluid

In this case we assume that the second fluid is of extreme tilt; i.e.,

$$\bar{V}^2 = 1.$$  

This case was not explicitly studied in [1].
By a careful analysis of the equilibrium points in the invariant set $F(VI_0)$ we can prove the following:

**Theorem 4.1.** When $\tilde{V}^2 = 1$, all of the equilibrium points of the resulting dynamical system of physical interest (in the invariant set $F(VI_0)$) necessarily have the property that (i) $V^2 = 0$ (i.e., the first fluid is comoving), (ii) $V^2 = 1$ (i.e., the first fluid is also of extreme tilt), or (iii) $(v_1, v_2, v_3) = -\omega (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$ (i.e., the two fluids are aligned).

This makes the resulting analysis of the equilibrium points and their stability relatively straightforward (and in many cases the results can be inferred from the analysis of the single fluid case [1]).

For the extremely tilted invariant subspace, $\tilde{V} = 1$, the equation for $\tilde{\Omega}$ simplifies to (with $\Gamma < 2$)

$$\tilde{\Omega}' = (2q - 2 - \tilde{S})\tilde{\Omega}.$$  

(4.2)

The extremely tilted fluid loses its dependence on the parameter $\Gamma$, and dynamically it behaves similarly to an anisotropic radiation fluid with $\Gamma = 4/3$.\(^2\)

Close to the initial singularity we typically have that $q \approx 2$ (separated by downward spikes in the chaotic case). Hence, since $|\tilde{S}| \leq 2\Sigma$, we see that $\tilde{\Omega}$ is decaying towards zero as we approach the initial singularity. Thus for $\Gamma < 2$, the extremely tilted fluid will become negligible close to the initial singularity (in the sense that $\tilde{\Omega} \to 0$).

At late times we have that $\dot{\tilde{\Omega}} = 0$. But since $\ddot{\tilde{\Omega}} = 0$, the future evolution is more complicated; we cannot conclude anything about the late-time behaviour from the monotonic function alone. However, some general results can be obtained from the existence of equilibrium points and their local stability and using Theorem 4.1.\(^2\)

From Table II we see that when the second fluid is extreme, then the future attractors are: $I(E)(I)$, for $0 < \gamma < 2/3$ and $4/3 < \Gamma$, $C(E)(VI_0)$ for $2/3 < \gamma < 6/5$ and $4/3 - (3\gamma - 2)/12 < \Gamma$, and $E(VI_0)$ for $6/5 < \gamma$ and $\gamma < \Gamma$. But both $I(E)(I)$ and $C(E)(VI_0)$ have $\tilde{\Omega} = 0$ (and $\Omega \neq 0$), so that for $\gamma < 6/5$ the second (extreme) fluid is dynamically negligible at late times. Only when $\gamma > 6/5$ can the second

\(^2\)Compare eq.(4.2) with a non-tilted radiation fluid with anisotropic stresses for which the evolution equation takes the form $\Omega_r' = (2q - 2 - \Sigma_{ab}\Pi^{ab})\Omega_r$ where $\Pi^{ab}$ is the expansion-normalised anisotropic stress tensor.
fluid be non-negligible, and in this case the future attractor is $E(V I_0)$, where both fluids are of extreme tilt.

In principle, we should study the special case in which both fluids are of extreme tilt separately. However, the main results follow directly from the arguments above. Both extreme tilting fluids will be negligible dynamically at early times for appropriate values of the parameters and the initial singularity will be oscillatory in general. And at late times the local future attractor will be $E(V I_0)$ for $6/5 < \gamma < \Gamma$.

Thus we have the following: In the Bianchi type VI model the extremely tilted fluid becomes dynamically insignificant ($\tilde{\Omega} \to 0$) in the future if the non-extremely tilted fluid has $\gamma < \frac{6}{5}$. Note that this includes, for example, the dust case (for which $\gamma = 1$).

We have identified the extreme tilting fluid to be the second fluid. In the above we assumed that $\gamma < \Gamma$. The case $\gamma > \Gamma$ is not obtained by simply interchanging the two fluids in the case described above (due to the asymmetry in the choice of the extreme fluid), and care should be taken in describing the asymptotic dynamics in this case. However, this can be done using Theorem 4.1 and the local stability of the equilibrium points (see below).

4.1. Null fluid; Relation to branes. In brane-world cosmology matter fields and gauge interactions are confined to a four-dimensional brane moving in a higher-dimensional “bulk” spacetime. The five-dimensional Weyl tensor in the generalized Randall-Sundrum-type models [16] is felt on the brane via its projection, $E_{\mu\nu}$. In general, in the 4-dimensional picture the conservation equations do not determine all of the independent components of $E_{\mu\nu}$ on the brane. In these models the gravitational field can also propagate in the extra dimensions. For example, at sufficiently high energies particle interactions can produce 5D gravitons which are emitted into the bulk. In some applications these gravitational waves may be approximated as of type N [17].

Let us assume that the 5-dimensional bulk is algebraically special and of type N. Then there exists a null frame such that the 5-dimensional Weyl tensor is given by [18]:

$$C_{abcd} = 4C_{1i1j}l_1^a n_i^b n_j^c.$$

Defining $n^c$ to be the normal vector on the brane, the non-local stresses on the brane can be written as

$$E_{ab} = C_{1i1j} \left[ l_1^a (n_i^b n^c) - m_b^i (l_a n^c) \right] \left[ l_1^b (m_i^d n^c) - m_b^i (l_a n^c) \right],$$

where $E_{ab} n^b = E^a = 0$. Using the projection operator on the brane and defining $\hat{\ell}^a$ as the projection of the null vector $l_a$ onto the brane, we consider the physically interesting case in which $\hat{\ell}^a \hat{\ell}_a = 0$, so that $\hat{\ell}_a$ is null. The four-dimensional projected Weyl tensor on the brane can then be written

$$E_{\mu\nu} = - \left( \frac{\tilde{\kappa}}{\kappa} \right)^4 \hat{\ell}_\mu \hat{\ell}_\nu.$$

This is equivalent to the energy-momentum tensor of a null fluid, which is formally equivalent to the energy-momentum tensor of an extreme tilted perfect fluid. Using a covariant decomposition of $E_{\mu\nu}$, the non-local energy terms are given by:

$$U = \epsilon (\hat{\ell}_\nu u^\nu)^2, \quad Q_{\mu} = \epsilon (\hat{\ell}_\nu u^\nu) \hat{\ell}_\mu, \quad P_{\mu\nu} = \epsilon (\hat{\ell}_\mu \hat{\ell}_\nu).$$
The equations on the brane now close and the dynamical behaviour can be analysed. We can investigate the effect this type N bulk may have on the cosmological evolution of the brane. Since dynamically $\Gamma \sim 4/3$ in the isotropic case, for $\gamma > \Gamma = 4/3$ we need to reverse the arguments given in the previous subsection. First, we note that if $\gamma > 4/3$, then the null fluid is not dynamically important at early times; that is, the effects of the projected Weyl tensor will not affect the dynamical behaviour close to the initial singularity. This supports the result that an isotropic singularity is a local stable past attractor, and hence constitute the most likely initial conditions in a classical brane-world [19]. Also, if $\gamma < 4/3$, the extreme tilting fluid is dynamically negligible to the future.

Therefore, the null brane fluid is not dynamically important asymptotically at early and late times for all values of $\gamma$ of physical import, supporting the phenomenological analysis presented in [17].

5. Stiff perfect fluid

In this section we consider the case that the second fluid is a stiff comoving fluid. Close to the initial singularity a scalar field is essentially massless and can be modelled by a comoving perfect fluid with a stiff equation of state parameter $\Gamma = 2$. The stiff fluid was not explicitly studied in [1]. This will enable us to consider the initial singular regime as $\tau \to -\infty$ in this case. Unlike for non-stiff fluids with $\Gamma < 2$, for which the initial singular regime possesses an oscillatory, and very likely a chaotic, behaviour [6, 10, 20], in this case there exists a global past attractor [21].

In the Appendix we have proven Theorem A.2: For $\gamma < \Gamma = 2$, and $\tilde{\Omega} > 0$ we have that

$$\lim_{\tau \to -\infty} \tilde{V} = 0, \quad \lim_{\tau \to -\infty} \left(\tilde{\Sigma}^2 + \tilde{\Omega}\right) = 1. \quad (5.1)$$

The overbar denotes an appropriate mean value (see eq.(A.3)). Hence, the stiff fluid will be asymptotically co-moving at early times. Note also that $\Sigma^2 + \tilde{\Omega} = 1$ implies $\Omega = N = 0$. This means that the second fluid $\gamma < \Gamma$ will be dynamically insignificant at early times.

For the stiff fluid case, $\Gamma = 2$, there exist equilibrium points given by (usually referred to as Jacobs disc):

$$\Sigma_+^2 + \Sigma_{23}^2 + \tilde{\Omega} = 1, \quad \lambda = 1,$$

$$V = \tilde{V} = N = \Sigma_+ = \Sigma_{12} = \Sigma_{13} = \Omega = 0. \quad (5.2)$$

The equilibrium points can thus be considered as the solid disc $\Sigma_+^2 + \Sigma_{23}^2 \leq 1$ in the $(\Sigma_+, \Sigma_{23})$-plane. The centre of the disc represents the FRW model with a stiff fluid and the unit circle correspond to vacuum Kasner solutions.

By linearising the system of equations with respect to the Jacobs disc, we get the following eigenvalues from the shear, $N$, $\lambda$ and $\Omega$ equations:

$$\lambda_{1,2} = 0, \quad \lambda_3 = -2\sqrt{3}\Sigma_{23}, \quad \lambda_{4,5} = -3\Sigma_+ \pm \sqrt{3}\Sigma_{23}$$

$$\lambda_6 = 2(1 + \Sigma_+ + \sqrt{3}\Sigma_{23}), \quad \lambda_7 = -4\sqrt{3}\Sigma_{23}, \quad \lambda_8 = 3(2 - \gamma). \quad (5.3)$$

In the brane scenario the Friedmann equation contains quadratic matter terms $\rho^2$ (compared to the general-relativistic $\rho$ dependence), so that close to the initial singularity we can effectively replace $\gamma \rightarrow 2\gamma$. This result is consequently consistent with [19].
Figure 1. Jacobs Disc. Equilibrium points inside the triangular region are stable to the past.

The remaining (independent) eigenvalues come from the velocities $\tilde{v}_i$ which, according to Theorem A.2, have $\tilde{v}_i \to 0$. Hence, there exists a triangular region inside the Jacobs disc in which the equilibrium points are locally stable into the past (see Fig.1). This indicates that solutions including a stiff fluid are not generically chaotic as we approach the initial singularity. From the existence of the monotonic function $\chi$, we know that the less stiff fluid has $\beta \Omega \to 0$ as we approach the initial singularity. From the eq. (5.1) we see that $\tilde{\Omega} > 0$ implies $\Omega \to 0$ at early times; i.e. we expect that the fluid with $\gamma < 2$ will decouple from the equations of motion and the initial singularity is almost entirely determined by the stiffest fluid. In this case the chaotic behaviour, conjectured to exist for $\Gamma < 2$, can be avoided.

Therefore to conclude: If one of the fluids is stiff ($\Gamma = 2$), there exist local past attractors corresponding to non-tilted solutions with a stiff perfect fluid. Furthermore, the less stiff fluid will be dynamically insignificant at early times.

6. Discussion

We have studied Bianchi type VI$_0$ universe models containing two tilted $\gamma$-law perfect fluids using dynamical systems techniques. The state space is 11-dimensional. We utilized the existence of a monotonic function, analysed the equilibrium points in a particular important invariant subspace, derived Theorem 4.1, and subsequently found equilibrium points that are future stable in the full 11-dimensional state space and are consequently local attractors; the solutions, which are summarized in Table 1, serve as late-time asymptotes for an open set of tilted type VI$_0$ models containing two tilted perfect fluids.

We then studied the case in which one fluid is extremely tilting in more detail (this analysis, plus the analysis of the stiff fluid case, completed the analysis of [1]). We determined possible future behaviour and discussed when the extreme fluid is negligible asymptotically.

We then applied these results to a problem in brane-world cosmology. The four-dimensional equations on the brane are modified by the effect of the projection of the five-dimensional Weyl tensor on the brane, $E_{\mu\nu}$, and when the bulk gravitational field can be approximated as type N gravitational waves close to the brane, this implies that the form of $E_{\mu\nu}$ is that of a null fluid, which is formally dynamically equivalent to a perfect fluid with extreme tilt. Therefore, we can model the
dynamical effects of $\mathcal{E}_{\mu\nu}$ in brane cosmology by assuming that the second fluid is of extreme tilt. We found that the effects of $\mathcal{E}_{\mu\nu}$ are not dynamically important (at least in this class of models) in the asymptotic regime close to the singularity (as expected from the work of [17]) nor to the future at late times for physically important values for the fluid parameters.

Finally, we studied the important case of a stiff fluid. We found that in this case there is a local past attractor (5.2), so that unlike previous analysis (for models with $\gamma < 2$), there are no chaotic oscillations as the initial singularity is approached.

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Appendix A. Some simple proofs

Lemma A.1. For $0 < \gamma < \Gamma \leq 2$, we have

$$q \leq 2, \text{ and } q = 2 \Rightarrow \Sigma^2 + \tilde{\Omega} = 1. \tag{A.1}$$

Proof. We can use the constraint equation and write

$$q = 2 - 2N^2 - f(\gamma, V)\Omega - f(\Gamma, \tilde{V})\tilde{\Omega}$$

where

$$f(\gamma, V) = \frac{1}{2} \left[ \frac{3(2 - \gamma) + (5\gamma - 6)V^2}{1 + (\gamma - 1)V^2} \right]. \tag{A.2}$$

Note that for $0 < \gamma \leq 2, 0 \leq V \leq 1$, the function $f$ obeys $f(\gamma, V) \geq 0$, and equality holds if and only if $\gamma = 2, V = 0$. From this we can see that eq. (A.1) follows automatically. \qed

As the initial singularity is approached the solutions typically become oscillatory and chaotic. In order to control this oscillatory behaviour it is useful to introduce an appropriate mean value, $\overline{A}$, defined by

$$\overline{A}(\tau) \equiv \frac{1}{\tau_0 - \tau} \int_{\tau}^{\tau_0} A(t)dt, \quad \tau_0 > \tau. \tag{A.3}$$

This definition of a mean value is very useful since it basically measures the integrated effect of a certain quantity. The possible sharp spikes appearing in the curvature variables as the solution oscillates are therefore integrated out. As $\tau \to -\infty$, these spikes can therefore be ignored as an integrated effect.

In the case of a stiff fluid, $\Gamma = 2$, the following theorem is useful:

Theorem A.2. For $\gamma < \Gamma = 2$, and $\tilde{\Omega} > 0$ we have that

$$\lim_{\tau \to -\infty} \tilde{V} = 0, \quad \lim_{\tau \to -\infty} \left( \Sigma^2 + \tilde{\Omega} \right) = 1. \tag{A.4}$$

Proof. For a stiff fluid the equations for $\tilde{V}$ and $\tilde{\Omega}$ are:

$$\tilde{V}' = (2 - \tilde{S})\tilde{V},$$

$$\tilde{\Omega}' = 2 \left[ (q - 2) + (2 - \tilde{S})\frac{\tilde{V}^2}{1 + V^2} \right] \tilde{\Omega}. \tag{A.4}$$
As shown above, \( q \) is bounded by \( q \leq 2 \). Therefore, there exists an \( \epsilon(\tau) \geq 0 \) such that \( q = 2 - \epsilon \). Integrating the latter equation in \( A.4 \) (after dividing by \( \bar{\Omega} \) and replacing \( (2 - \bar{S})V \) with \( \bar{V}' \)), we obtain \( (\tau < \tau_0) \)
\[
\ln \bar{\Omega} \bigg|_{\tau_0}^{\tau} = -2 \int_{\tau_0}^{\tau} \epsilon d\tau + \ln(1 + \bar{V}^2) \bigg|_{\tau_0}^{\tau}.
\]
Since \( \ln(1 + \bar{V}^2) \) is bounded and \( \epsilon \geq 0 \), the right-hand side is bounded from above by \( \ln 2 \). The left-hand side, on the other hand, is bounded from below by \( \ln \bar{\Omega}(\tau_0) \) (which is negative). Hence, both sides must be bounded. More precisely, there must exist a \( \delta > 0 \) such that \( \bar{\Omega} \geq \delta > 0 \) for \( \tau < \tau_0 \). Moreover, boundedness implies that \( \int_{\tau_0}^{\tau} \epsilon d\tau \) is bounded, and hence,
\[
\bar{\epsilon} = \frac{1}{\tau_0 - \tau} \int_{\tau_0}^{\tau} \epsilon d\tau \to 0 \quad \text{for} \quad \tau \to -\infty,
\]
which leads to \( \bar{S} \to 2 \) at early times. Using eq. \( A.1 \) we thus have \( \lim_{\tau \to -\infty} (\Sigma^2 + \bar{\Omega}) = 1 \).

Furthermore, we note that \( \bar{S} \) obeys the bound \( |\bar{S}| \leq 2\Sigma \), which for \( 0 < \bar{\Omega} \leq 1 \) implies that \( \bar{V} \) is strictly monotonically increasing. Since \( \bar{V} \) is bounded, either \( \bar{V} \to 0 \) or \( \bar{S} \to 2 \). The latter case can only happen if \( \Sigma \to 1 \), which implies that \( \bar{\Omega} \to 0 \). However, this case is excluded since \( \bar{\Omega} \geq \delta > 0 \) for \( \tau < \tau_0 \). The theorem now follows.

\[\square\]

**Appendix B. Equilibrium Points**

The system of equations possesses a wealth of equilibrium points. Here, we will describe some of the interesting ones which are important for the late-time behaviour.

**B.1. Non-tilted + Non-tilted.**

**B.1.1.** \( I(I) \), \( FRW: \) \( \Sigma^2 = N = \lambda = 0 = V = \bar{V} = \bar{\Omega} = 0, \) \( \Omega = 1, \) \( q = \frac{1}{2}(3\gamma - 2), \) \( 0 < \gamma, \) \( \Gamma < 2. \)

Eigenvalues:
\[
\lambda_{1,2,3,4,5} = -\frac{3}{2}(2 - \gamma), \quad \lambda_{6,7} = \frac{1}{2}(3\gamma - 2), \quad \lambda_8 = -3(\Gamma - \gamma), \quad \lambda_{9,10,11} = (3\Gamma - 4).
\]

**B.1.2.** \( C(VI_0) \), \( Collins \) \( VI_0: \) \( \Sigma_- = \Sigma_{12} = \Sigma_{13} = \Sigma_{23} = \lambda = V = \bar{V} = \bar{\Omega} = 0, \)
\( \Sigma_+ = -\frac{3}{4}(3\gamma - 2), \) \( N^2 = \frac{1}{16}(3\gamma - 2)(2 - \gamma), \) \( \Omega = \frac{1}{2}(2 - \gamma), \) \( q = \frac{1}{2}(3\gamma - 2), \) \( 2/3 < \gamma < 2, \) \( 0 < \Gamma < 2. \)

Eigenvalues:
\[
\lambda_{1,2} = -\frac{3}{4}(2 - \gamma) \left(1 \pm \sqrt{5\gamma - 6}\right), \quad \lambda_{3,4} = -\frac{3}{4}(2 - \gamma) \left(1 \pm \sqrt{10 - 13\gamma}\right),
\]
\[
\lambda_5 = -\frac{3}{2}(2 - \gamma), \quad \lambda_{6,7} = \frac{3}{4}(5\gamma - 6), \quad \lambda_8 = -3(\Gamma - \gamma),
\]
\[
\lambda_9 = (3\Gamma - 4) + \frac{1}{2}(3\gamma - 2), \quad \lambda_{10,11} = (3\Gamma - 4) - \frac{1}{4}(3\gamma - 2)
\]

**B.2. Non-tilted + Intermediately tilted.**
B.2.1. $I_T(I)$, FRW with a tilted fluid: $\Sigma^2 = N = \lambda = 0 = V = 0$, $\Omega = 1$, $q = \frac{1}{3}(3\gamma - 2)$. $0 < \gamma < 2$.

For the second fluid:

$$\Gamma = \frac{4}{3}, \tilde{\Omega} = 0, \tilde{v}_1 = 0, \tilde{v}_2 = \tilde{v}_3 = \frac{V}{\sqrt{2}}.$$  

Eigenvalues:

$$\lambda_{1,2,3,4,5} = -\frac{3}{2}(2 - \gamma), \lambda_{6,7} = \frac{1}{2}(3\gamma - 2), \lambda_8 = (3\gamma - 4), \lambda_{9,10,11} = 0.$$  

B.2.2. $C_T(V I_0)$, Collins type VI$_0$ with a tilted fluid: $\Sigma_- = \Sigma_{12} = \Sigma_{13} = \Sigma_{23} = \lambda = V = 0$, $\Sigma_+ = -\frac{1}{4}(3\gamma - 2)$, $N^2 = \frac{3}{16}(3\gamma - 2)(2 - \gamma)$, $\Omega = \frac{3}{4}(2 - \gamma)$, $q = \frac{1}{2}(3\gamma - 2)$, $2/3 < \gamma < 2$.

For the second fluid:

$$\Gamma = \frac{4}{3} - \frac{1}{12}(3\gamma - 2), \tilde{\Omega} = 0, \tilde{v}_1 = 0, \tilde{v}_2 = \tilde{v}_3 = \frac{V}{\sqrt{2}}.$$  

Eigenvalues:

$$\lambda_{1,2,3,4,5,6,7} \text{ as for } C(V I_0), \lambda_8 = \frac{3\tilde{G}_-}{4\tilde{G}_+}(5\gamma - 6), \lambda_9 = 0,$$

$$\lambda_{10,11} = -\frac{3}{2}(3\gamma - 2) \left( 1 \pm \sqrt{1 - 8(2 - \gamma)V^2/(3\gamma - 2)} \right) \quad (B.1)$$


B.3.1. $I_E(I)$, FRW with an extremely tilted fluid: $\Sigma^2 = N = \lambda = 0 = V = 0$, $\Omega = 1$, $q = \frac{1}{2}(3\gamma - 2)$. $0 < \gamma < 2$.

For the second fluid:

$$0 < \Gamma < 2, \tilde{\Omega} = 0, \tilde{v}_1 = 0, \tilde{v}_2 = \tilde{v}_3 = \frac{1}{\sqrt{2}}.$$  

Eigenvalues:

$$\lambda_{1,2,3,4,5} = -\frac{3}{2}(2 - \gamma), \lambda_{6,7} = \frac{1}{2}(3\gamma - 2),$$

$$\lambda_8 = (3\gamma - 4), \lambda_9 = -\frac{2(3\Gamma - 4)}{2 - \Gamma}, \lambda_{10,11} = 0.$$  

The zero eigenvalues correspond to the fact that the equilibrium point is a member of a two-dimensional set of equilibrium points.

B.3.2. $C_E(V I_0)$, Collins type VI$_0$ with an extremely tilted fluid: $\Sigma_- = \Sigma_{12} = \Sigma_{13} = \Sigma_{23} = \lambda = V = 0$, $\Sigma_+ = -\frac{1}{4}(3\gamma - 2)$, $N^2 = \frac{3}{16}(3\gamma - 2)(2 - \gamma)$, $\Omega = \frac{3}{4}(2 - \gamma)$, $q = \frac{1}{2}(3\gamma - 2)$. $2/3 < \gamma < 2$.

For the second fluid:

$$0 < \Gamma < 2, \tilde{\Omega} = 0, \tilde{v}_1 = 0, \tilde{v}_2 = \tilde{v}_3 = \frac{1}{\sqrt{2}}.$$
Eigenvalues:
\[ \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \] as for \( C(\mathcal{V} I_0) \), \( \lambda_8 = \frac{-3(2 - \Gamma)}{4\Gamma} (5\gamma - 6) \), \( \lambda_9 = \frac{-3}{2} (3\gamma - 2) \left( \pm \sqrt{\frac{11\gamma - 18}{3\gamma - 2}} \right) \), \( \lambda_{10,11} = \frac{-2}{\Gamma} \left[ 3\Gamma - 4 + \frac{1}{4} (3\gamma - 2) \right] \).

B.4. Intermediately tilted + Extremely tilted.

B.4.1. \( S(\mathcal{V} I_0) \), a \( \gamma = 6/5 \) bifurcation:
\[ \gamma = \frac{6}{5}, \quad 0 < \Gamma < 2, \quad q = \frac{4}{5}, \quad \Sigma_+ = -\frac{2}{5}, \quad \Sigma_+ = \Sigma_{23} = 0, \quad \Sigma_{13} = -\Sigma_{12} \]
(\text{B.2})
\[ v_1 = \tilde{v}_1 = 0, \quad v_2 = v_3 = \frac{V}{\sqrt{2}}, \quad 0 \leq V < 1, \quad \tilde{v}_2 = \tilde{v}_3 = \frac{1}{\sqrt{2}}, \]
\[ \Omega = \frac{[12 - 25(1 + \lambda)N^2] (5 + V^2)}{50(1 - V^2)}, \quad \tilde{\Omega} = \frac{25N^2(2V^2 + 3\lambda + 1) - 6(5V^2 + 1)}{25(1 - V^2)}, \]
(\text{B.3})
\[ \Sigma_{12} = \frac{1}{10} \sqrt{25N^2(1 - \lambda)} - 6, \quad N = \sqrt{B \pm 3(V + 1)(1 - \lambda)\sqrt{\sigma}} \]
where
\[ \begin{align*}
A &= \frac{1 + \lambda}{\lambda^2 - 4\lambda + 7} \left\{ \left[ \left( \lambda^2 - 4\lambda + 7 \right) V + (1 + \lambda)(\lambda - 5) \right]^2 - 18(1 - \lambda)^3 \right\} \\
\sigma &= \frac{\lambda^2 - 202\lambda + 121}{\lambda^2 - 202\lambda + 121} V + \lambda^2 + 98\lambda - 111^2 - 432\lambda(\lambda + 5)(\lambda + 1) \\
B &= \frac{3 \left\{ \left[ (\lambda^2 - 2\lambda - 39) V + \lambda^2 + 28\lambda + 15 \right]^2 - 36(3 + 8\lambda + 23\lambda^2 + 2\lambda^3) \right\}}{\lambda^2 - 2\lambda - 39}.
\end{align*} \]
(\text{B.4})

In addition we have to make sure that the energy densities are positive and that \( \Sigma_{12} \) is real. This puts bounds on \( N \):
\[ \begin{align*}
\Omega \geq 0 & \quad \Rightarrow \quad N^2 \leq \frac{12}{25(1 + \lambda)} \\
\tilde{\Omega} \geq 0 & \quad \Rightarrow \quad N^2 \geq \frac{6(5V^2 + 1)}{25(2V^2 + 3\lambda + 1)} \\
\Sigma_{12}^2 \geq 0 & \quad \Rightarrow \quad N^2 \geq \frac{6}{25(1 - \lambda)} \\
\end{align*} \]
(\text{B.5})

The bifurcation \( S(\mathcal{V} I_0) \) is thus parameterized by \( \lambda \) and \( V \) provided that the bounds hold.

The region is bounded by the following lines:
(1) \( \tilde{\Omega} = 0 \): Given by the line bifurcation \( \mathcal{L}(\mathcal{V} I_0) \) for a single tilted fluid (see [1]).
(2) \( \tilde{\Omega} = 0 \):
\[ N^2 = \frac{12}{25}, \quad \lambda = 0. \]
(3) \( V = 0 \):
\[ N^2 = \frac{9 + 42\lambda - 3\lambda^2 \pm 3(1 - \lambda)\sqrt{\lambda^2 - 34\lambda + 1}}{25(1 + \lambda)(\lambda^2 + 14\lambda + 1)}, \quad 0 \leq \lambda \leq 17 - 12\sqrt{2} \]
(4) $V = 1$: 

$$N^2 = \frac{12}{25}, \quad \lambda = 0, \quad \Omega + \tilde{\Omega} = \frac{6}{25}.$$ 

Eigenvalues: Using a combination of numerical and analytical work the eigenvalues seem to be:

$$\lambda_{1,2} = 0, \quad \text{Re}(\lambda_{3,4,5,6}) = -\frac{3}{5}, \quad \lambda_7 = \frac{6(5\Gamma - 6)}{5(2 - \Gamma)},$$ 

$$\lambda_8 + \lambda_9 = \lambda_{10} + \lambda_{11} = -\frac{6}{5}, \quad \text{Re}(\lambda_{8,9,10,11}) < 0.$$ 

B.5. Extremely tilted + Extremely tilted.

B.5.1. $\mathcal{E}(V_{I0})$: Extremely tilted type $V_{I0}$. $V = \tilde{V} = 1$, $v_1 = \tilde{v}_1 = 0$, $v_2 = v_3 = \tilde{v}_2 = \tilde{v}_3 = \frac{1}{2}\sqrt{2}$, $\Sigma_+ = \Sigma_{23} = \tilde{N} = 0$, $\Sigma_+ = -\frac{2}{5}$, $\Sigma_{12} = -\Sigma_{13} = \frac{\sqrt{6}}{10}$, $N = \frac{2\sqrt{3}}{5}$, $\Omega + \tilde{\Omega} = \frac{6}{25}$, $q = \frac{4}{5}$.

Eigenvalues:

$$\lambda_1 = -\frac{6(5\gamma - 6)}{5(2 - \gamma)}, \quad \lambda_{2,3} = -\frac{3}{5} \left(1 \pm i\sqrt{19}\right), \quad \lambda_{4,5} = -\frac{3}{5} \left(1 \pm i\sqrt{11}\right),$$ 

$$\lambda_{6,7} = -\frac{3}{5} \left(1 \pm i\sqrt{14 + 5\sqrt{2}}\right), \quad \lambda_8 = 0, \quad \lambda_9 = -\frac{6(5\Gamma - 6)}{5(2 - \Gamma)},$$ 

$$\lambda_{10,11} = -\frac{3}{5} \left(1 \pm i\sqrt{11 + 4\sqrt{2}}\right).$$

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