1 Introduction

Professor H. Ezawa has worked on vast area related with fundamental physics. His interests and pioneering works are not limited to "elementary-particle-oriented" works but also cover more complex system, such as finite temperature quantum field theory and nonequilibrium theory. It is, therefore, great honor of mine to contribute to this volume by presenting an attempt, information dynamics, to treat various complex systems mathematically and its one of the latest application, a description of recognition process[5] based on the works [7,8]. In this section we give a review on what is the complex system. The discussion leads the introduction of Information Dynamics in natural way.

The complex system has been considered in Santafe research center as follows:

(1) A system is composed of several elements called agents. The size of the system (the number of the elements) is medium.
(2) The agent has intelligence.
(3) Each agent has interaction due to local information. The decision of each agent is determined by not all information but the limited information of the system.

Under a small modification, I define the complex system as follows:
(1) A system is composed of several elements. The scale of the system is often large but not always, in some cases one.

(2) Some elements of the system have special (self) interactions (relations), which produce a dynamics of the system.

(3) The system shows a particular character (not sum of the characters of all elements) due to (2).

**Definition 1** A system having the above three properties is called “complex system”. The ”complexity”of such a complex system is a quantity measuring that complexity, and its change describes the appearance of the particular character of the system.

There exist such measures describing the complexity for a system, for instance, variance, correlation, level - statistics, fluctuation, randomness, multiplicity, entropy, fuzzy, fractal dimension, ergodicity (mixing, flow), bifurcation, localization, computational complexity (Kolmogorov’s or Chaitin’s), catastrophe, dynamical entropy, Lyapunov exponent, etc. These quantities are used case by case and they are often difficult to compute. Moreover, the relations among these are lacking (not clear enough). Therefore it is important to find common property or expression of these quantities. In this paper, we introduce such a common degree to describe the chaotic aspect of quantum dynamical systems. Further we describe the function of barin in the framework of information dynamics [16](ID for short) and we discuss the value of information attached to the brain in terms of the complexity in ID and the chaos degree[19, 20].

## 2 Information Dynamics

There are two aspects for the complexity, that is, the complexity of a state describing the system itself and that of a dynamics causing the change of the system (state). The former complexity is simply called the ”complexity” of the state, and the later is called the ”chaos degree” of the dynamics in this paper. Therefore the examples of the complexity are entropy, fractal dominion, and those of the chaos degree are Lyapunov exponent, dynamical entropy, computational complexity. Let us discuss a common quantity measuring the complexity of a system so that we can easily handle. The complexity of a general quantum state was introduced in the frame of ID [16 9]
and the quantum chaos degree was defined in [10], which we will review in this section.

Information Dynamics is a synthesis of dynamics of state change and complexity of state. More precisely, let \((\mathcal{A}, \mathcal{G}, \alpha(G))\) be an input (or initial) system and \((\overline{\mathcal{A}}, \overline{\mathcal{G}}, \overline{\alpha}(G))\) be an output (or final) system. Here \(\mathcal{A}\) is the set of all objects to be observed and \(\mathcal{G}\) is the set of all means for measurement of \(\mathcal{A}\), \(\alpha(G)\) is a certain evolution of system. Once an input and an output systems are set, the situation of the input system is described by a state, an element of \(\mathcal{G}\), and the change of the state is expressed by a mapping from \(\mathcal{G}\) to \(\mathcal{G}\), called a channel, \(\Lambda^* : \mathcal{G} \rightarrow \mathcal{G}\). Often we have \(\mathcal{A} = \overline{\mathcal{A}}, \mathcal{G} = \overline{\mathcal{G}}, \alpha = \overline{\alpha}\), which is assumed in the sequel. Thus we claim

\[\text{[Giving a mathematical structure to input and output triples}}\]

\[\equiv \text{Having a theory}\]

For instance, when \(\mathcal{A}\) is the set \(M(\Omega)\) of all measurable functions on a measurable space \((\Omega, \mathcal{F})\) and \(\mathcal{G}(\mathcal{A})\) is the set \(P(\Omega)\) of all probability measures on \(\Omega\), we have usual probability theory, by which the classical dynamical system is described. When \(\mathcal{A} = B(\mathcal{H})\), the set of all bounded linear operators on a Hilbert space \(\mathcal{H}\), and \(\mathcal{G}(\mathcal{A}) = \mathcal{G}(\mathcal{H})\), the set of density operators on \(\mathcal{H}\), we have a usual quantum dynamical system. In this paper, we assume that both the input and output triple \((\mathcal{A}, \mathcal{G}, \alpha(G))\) is a \(C^*\)-dynamical system or the usual quantum system as above, and a channel, \(\Lambda^* : \mathcal{G} \rightarrow \mathcal{G}\) is a completely positive map.

There exist two complexities in ID, which are axiomatically given as follows:

Let \((\mathcal{A}_t, \mathcal{G}_t, \alpha^t(G^t))\) be the total system of both input and output systems; \(\mathcal{A}_t \equiv \mathcal{A} \otimes \mathcal{A}, \mathcal{G}_t \equiv \mathcal{G} \otimes \mathcal{G}, \alpha^t \equiv \alpha \otimes \alpha\) with suitable tensor products \(\otimes\). Further, let \(C(\varphi)\) be the complexity of a state \(\varphi \in \mathcal{G}\) and \(T(\varphi; \Lambda^*)\) be the transmitted complexity associated with the state change \(\varphi \rightarrow \Lambda^*\varphi\). These complexities \(C\) and \(T\) are the quantities satisfying the following conditions:

(i) For any \(\varphi \in \mathcal{G}\),
\[C(\varphi) \geq 0, \ T(\varphi; \Lambda^*) \geq 0.\]

(ii) For any orthogonal bijection \(j : ex\mathcal{G} \rightarrow ex\mathcal{G}\) (the set of all extreme points in \(\mathcal{G}\)),
\[C(j(\varphi)) = C(\varphi),\]
\[T(j(\varphi); \Lambda^*) = T(\varphi; \Lambda^*).\]
(iii) For $\Phi \equiv \varphi \otimes \psi \in \mathcal{G}_t$,

$$C(\Phi) = C(\varphi) + C(\psi).$$

(iv) For any state $\varphi$ and a channel $\Lambda^*$,

$$T(\varphi; \Lambda^*) \leq C(\varphi).$$

(v) For the identity map "id" from $\mathcal{G}$ to $\mathcal{G}$.

$$T(\varphi; id) = C(\varphi).$$

**Definition 2:** Quantum Information Dynamics (QID) is defined by

$$(\mathcal{A}, \mathcal{G}, \alpha(G); \Lambda^*, C(\varphi), T(\varphi; \Lambda^*))$$

and some relations $R$ among them.

There are several examples of the above complexities $C$ and $T$ such as quantum entropy and quantum mutual entropy [14, 18]. Information Dynamics can be applied to the study of chaos in the following sense:

**Definition 3** [14, 20, 9]: $\psi$ is more chaotic than $\varphi$ as seen from the reference system $\mathcal{S}$ if $C(\psi) \geq C(\varphi)$.

When $\varphi$ changes to $\Lambda^* \varphi$, the degree of chaos associated to this state change (dynamics) $\Lambda^*$ is given by

$$D(\varphi; \Lambda^*) = \inf \left\{ \int_{\mathcal{G}} C(\Lambda^* \omega) d\mu; \mu \in M(\varphi) \right\},$$

where $\varphi = \int_{\mathcal{G}} \omega d\mu$ is a maximal extremal decomposition of $\varphi$ and $M(\varphi)$ is the set of such measures. In some cases such that $\Lambda^*$ is linear, this chaos degree $D(\varphi; \Lambda^*)$ can be written as $C(\Lambda^* \varphi) - T(\varphi; \Lambda^*)$.

Since ID has hierarchy (hierarchical structure), it can be applied several open systems. Later we apply ID to Brain Dynamics.
3 Entropic Chaos Degree (ECD)

In the context of information dynamics, a chaos degree associated with a
dynamics in classical systems was introduced in [19]. It has been applied
to several dynamical maps such logistic map, Baker’s transformation and
tinkerbel map with successful explanations of their chaotic characters [12].
This chaos degree has several merits compared with usual measures such as
Lyapunov exponent.

Here we discuss the quantum version of the classical chaos degree, which
is defined by quantum entropies in Section 2, and we call the quantum chaos
degree the entropic quantum chaos degree. In order to contain both clas-
csical and quantum cases, we define the entropic chaos degree (ECD) in C*-al-
gebraic terminology. This setting will not be used in the sequel application,
but for mathematical completeness we first discuss the C*-algebraic setting.

Let \((\mathcal{A}, \mathcal{S})\) be an input C* system and \((\mathcal{A}, \mathcal{S})\) be an output C* system;
namely, \(\mathcal{A}\) is a C* algebra with unit \(I\) and \(\mathcal{S}\) is the set of all states on
\(\mathcal{A}\). We assume \(\mathcal{A} = \mathcal{A}\) for simlicity. For a weak* compact convex subset \(\mathcal{S}\) (called
the reference space) of \(\mathcal{S}\), take a state \(\varphi\) from the set \(\mathcal{S}\) and let

\[ \varphi = \int_{\mathcal{S}} \omega d\mu_{\varphi} \]

be an extremal orthogonal decomposition of \(\varphi\) in \(\mathcal{S}\), which describes the
degree of mixture of \(\varphi\) in the reference space \(\mathcal{S}\) [15, 21]. The measure \(\mu_{\varphi}\)
is not uniquely determined unless \(\mathcal{S}\) is the Schoque simplex, so that the set
of all such measures is denoted by \(M_{\varphi}(\mathcal{S})\). The entropic chaos degree with
respect to \(\varphi \in \mathcal{S}\) and a channel \(\Lambda^*\) is defined by

\[ D_{S}(\varphi; \Lambda^*) \equiv \inf \left\{ \int_{\mathcal{S}} S_{S}(\Lambda^* \varphi) d\mu_{\varphi}; \mu_{\varphi} \in M_{\varphi}(\mathcal{S}) \right\} \] (3.1)

where \(S_{S}(\Lambda^* \varphi)\) is the mixing entropy of a state \(\varphi\) in the reference space \(\mathcal{S}\) [17
[9]. When \(\mathcal{S} = \mathcal{G}\), \(D_{S}(\varphi; \Lambda^*)\) is simply written as \(D(\varphi; \Lambda^*)\). This \(D_{S}(\varphi; \Lambda^*)\)
contains both the classical chaos degree and the quantum one.

In usual quantum system including classical discrete system, \(\mathcal{A}\) is the set
\(B(\mathcal{H})\) of all bounded operators on a Hilbert space \(\mathcal{H}\) and \(\mathcal{S}\) is the set \(\mathcal{G}(\mathcal{H})\) of
all density operators on \(\mathcal{H}\), in which an extreme decomposition of \(\rho \in \mathcal{G}(\mathcal{H})\)
is a Schatten decomposition \(\rho = \sum_k p_k E_k\) (i.e., \(\{E_k\}\) are one dimensional
orthogonal projections with \(\sum E_k = I\)), so that the entropic chaos degree is
written as
where the infimum is taken over all possible Schatten decompositions and $S$ is von Neumann entropy. Note that in classical discrete case, the Schatten decomposition is unique $\rho = \sum_k p_k \delta_k$ with the delta measure $\delta_k(j) \equiv \begin{cases} 1 & (k = j) \\ 0 & (k \neq j) \end{cases}$, and the entropic chaos degree is written by

$$D(\varphi; \Lambda^*) = \sum_k p_k S(\Lambda^* \delta_k),$$

(3)

where $\rho$ is the probability distribution of the orbit obtained from a dynamics of a system and the channel $\Lambda^*$ is generated from the dynamics.

We can judge whether the dynamics $F^*$ causes a chaos or not by the value of $D$ as

$$D > 0 \text{ and not constant } \iff \text{chaotic},$$

$$D = \text{constant } \iff \text{weak stable},$$

$$D = 0 \iff \text{stable}.$$

The classical version of this degree was applied to study the chaotic behaviors of several nonlinear dynamics [12, 19]. The quantum entropic chaos degree is applied to the analysis of quantum spin system [10] and quantum Baker’s type transformation [13], and we could measure the chaos of these systems. The information theoretical meaning of this degree was explained in [20].

The ECD can resolve some inconvenient properties of the Lyapunov exponent, another degree of chaos [12, 11]:

1. Lyapunov exponent takes negative value and sometimes $-\infty$, but the ECD is always positive for any $a \geq 0$.

2. It is difficult to compute the Lyapunov exponent for some maps like Tinkerbell map $f$ because it is difficult to compute $f^n$ for large $n$. On the other hand, the ECD of $f$ is easily computed.

3. Generally, the algorithm for the ECD is much easier than that for the Lyapunov exponent.
4 Quantum Information Dynamic Description of Brain

The Information Dynamics can be employed to describe not only several classical and quantum physical physics but also life sciences. We will construct a model describing the function of brain in the context of Quantum Information Dynamics (QID).

We study a possible function of brain, in particular, we try to describe several aspects of the process of recognition. In order to understand the fundamental parts of the recognition process, the quantum teleportation scheme [8] seems to be useful. We consider a channel expression of the teleportation process that serves for a simplified description of the recognition process in brain.

It is the processing speed that we take as a particular character of the brain, so that the high speed of processing in the brain is here supposed to come from the coherent effects of substances in the brain like quantum computer, as was pointed out by Penrose. Having this in our mind, we propose a model of brain describing its function as follows:

The brain system $BS = \mathfrak{X}$ is supposed to be described by a triple $(B(\mathcal{H}), S(\mathcal{H}), \Lambda^*(G))$ on a certain Hilbert space $\mathcal{H}$ where $B(\mathcal{H})$ is the set of all bounded operators on $\mathcal{H}$, $S(\mathcal{H})$ is the set of all density operators and $\Lambda^*(G)$ is a channel giving a state change with a group $G$.

Further we assume the following:

1. $BS$ is described by a quantum state and the brain itself is divided into several parts, each of which corresponds to a Hilbert space so that $\mathcal{H} = \bigoplus_k \mathcal{H}_k$ and $\varphi = \bigoplus_k \varphi_k$, $\varphi_k \in S(\mathcal{H}_k)$. However, in this paper we simply assume that the brain is in one Hilbert space $\mathcal{H}$ because we only consider the basic mechanism of recognition.

2. The function (action) of the brain is described by a channel $\Lambda^* = \bigoplus_k \Lambda_k^*$. Here as in (1) we take only one channel $\Lambda^*$.

3. $BS$ is composed of two parts; information processing part "P" and others "O" (consciousness, memory, recognition) so that $\mathfrak{X} = \mathfrak{X}_P \otimes \mathfrak{X}_O$, $\mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_O$.

Thus in our model the whole brain may be considered as a parallel quantum computer, but we here explain the function of the brain as a quantum computer, more precisely, a quantum communication process with entanglements like in a quantum teleportation process. We will explain the mathe-
mathematical structure of our model.

Let $s = \{s^1, s^2, \cdots, s^n\}$ be a given (input) signal (perception) and $\overline{s} = \{\overline{s}^1, \overline{s}^2, \cdots, \overline{s}^n\}$ the output signal. After the signal $s$ enters the brain, each element $s^j$ of $s$ is coded into a proper quantum state $\rho^j \in S(H_P)$, so that the state corresponding to the signal $s$ is $\rho = \otimes_j \rho^j$. This state may be regarded as a state processed by the brain and it is coupled to a state $\rho_O$ stored as a memory (pre-consciousness) in brain. The processing in the brain is expressed by a properly chosen quantum channel $\Lambda^*$ (or $\Lambda^*_P \otimes \Lambda^*_O$). The channel is determined by the form of the network of neurons and some other biochemical actions, and its function is like a (quantum) gate in quantum computer\textsuperscript{[12]}\textsuperscript{[22]}. The outcome state $\overline{\rho}$ contacts with an operator $F$ describing the work as noema of consciousness (Husserl’s noema), after the contact a certain reduction of state is occurred, which may correspond to the noesis (Husserl’s) of consciousness. A part of the reduced state is stored in brain as a memory. The scheme of our model is represented in the following figure.
5 Value of Information in Brain

The complex system responds to the information and has a particular role to choose the information (value of information). Brain selects some information (inputs) from huge flow of information (inputs). It will be important to find a rule or rules of such selection mechanism. In the model of Sec.4, an output signal $s$ (information) is somehow coded into a quantum state $\varphi$, then it runs in brain with a certain processing effect $\Lambda^*$ and a memory stored, and it changes its own figure. Thus we have two standpoints to catch the value of information in brain. Suppose that we have a fixed purpose (intention)
described by an operator $Q$, then one view of the value of information is whether the signal $s$ is important for the purpose $Q$ and the processing $\Lambda^*$ and another is whether the processing $\Lambda^*$ chosen in brain is effective for $s$ and $Q$. From these considerations, that value should be estimated by a function of the state $\varphi \otimes \varphi_O$, a channel $\Lambda^*$ and an operator $Q$, so that one possibility to define a measure $V(\varphi \otimes \varphi_O, \Lambda^*, Q)$ estimating the effect of a signal and a function of brain is as follows:

Define

$$V(\varphi \otimes \varphi_O, \Lambda^*, Q) = \text{tr} \Lambda^* \varphi \otimes \varphi_Q$$

**Definition 4 Value of Information:**

(1) $s = \{s^1, s^2, \ldots, s^n\}$ is more valuable than $s' = \{s^1', s^2', \ldots, s'^n\}$ for $\Lambda^*$ and $Q$ iff

$$V(\varphi \otimes \varphi_O, \Lambda^*, Q) \geq V(\varphi' \otimes \varphi_O, \Lambda^*, Q).$$

(2) $\Lambda^*$ is more valuable than $\Lambda'^*$ for given $s = \{s^1, s^2, \ldots, s^n\}$ and $Q$ iff

$$V(\varphi \otimes \varphi_O, \Lambda^*, Q) \geq V(\varphi \otimes \varphi_O, \Lambda'^*, Q).$$

The details of this estimator is discussed in [6], where there exist some relations between the information of value and the complexity or the chaos degree under properly chosen complexity $C$ and transmitted complexity $T$. For instance, with entropy type complexities and a certain $Q$, we conjecture (partially proved so far)

$$D(\varphi \otimes \varphi_O, \Lambda^*; Q) \leq D(\varphi \otimes \varphi_O, \Lambda'^*; Q) \iff V(\varphi \otimes \varphi_O, \Lambda^*, Q) \geq V(\varphi \otimes \varphi_O, \Lambda'^*, Q)$$

This result is quite natural because the more chaos a processing produces, the less value it has.

### 6 A Speculation of Brain Function

The set of neurons in brain is divided into several parts and each part corresponds to a configuration domain $G$, each point in which has two states,
excited or not. Thus \( G \equiv \cup_k G_k \) with \( G_k \cap G_j = \emptyset \) for any \( k \neq j \). Let assume that the Hilbert space \( \mathcal{H} \) describing the brain is Fock space on the square integrable random variables \( L_2(G, \mu) \) with the counting measure \( \mu \), so that the whole Hilbert space \( \mathcal{H} \) is decomposed as

\[
\mathcal{H} \equiv - (L_2(G, \mu)) = \bigotimes_k - (L_2(G_k, \mu)).
\]

Let \( \{x_1, x_2, \cdots, x_n\} \) describes the (positions of) excited neurons in as certain domain \( G_k \), so that the vector in \( L^2(G_k, \mu) \) corresponding this configuration is denoted by \( \sum_{j=1}^n \delta_{x_j} \) by the delta measure \( \delta_x \) corresponding to \( x \).

When we consider only one domain, for simplicity, denoted by the same \( G \) and it is decomposed into the processing part \( G^P \) and other part \( G^O \) including the effect of consciousness as in the previous section, our Hilbert space of the brain is \( \mathcal{H} \equiv - (L_2(G, \mu)) = \Gamma (L_2(G^P, \mu)) \otimes \Gamma (L_2(G^O, \mu)). \) Along the above settings we may explain some functions of brain in the terminologies of Fock space and quantum teleportation\([7, 8]\), on which we are working now \([6]\).

In the sequel, we will explain the first trial explaining the brain function, in particular the memory change due to recognition, based on the quantum teleportation scheme done in \([3]\).

Let us assume the Hilbert space \( \mathcal{H}_O \) is composed of two parts, before and after recognition. For notational simplicity, we denote the Hilbert spaces by \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 \) where \( \mathcal{H}_1 \) represents the processing part, \( \mathcal{H}_2 \) the memory before recognition and \( \mathcal{H}_3 = \mathcal{H}_2 \) the memory after recognition. Throughout this paper we will have in mind this interpretation of the Hilbert spaces \( \mathcal{H}_j \) \((j = 1, 2, 3)\). However, this is just an illustration of what we are going to do, and the teleportation scheme may be applied to very different situations.

We are mainly interested in the changes of the memory after the process of recognition. For that reason we consider channels from the set of states on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) into \( \mathcal{H}_3 \). Main object to be measured causing the recognition is here assumed to be a self-adjoint operator

\[
F = \sum_{k,l=1}^n z_{k,l} F_{k,l}
\]

on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) where the operators \( F_{k,l} \) are orthogonal projections (alternatively, we may take \( F_{k,l} \) as an operator valued measure). The channel \( \Lambda_{k,l} \) describes the state of the memory after the process of recognition if the outcome of
the measurement according to $F$ was $z_{k,l}$ and is given by

$$\Lambda_{k,l}(\rho \otimes \gamma) := \frac{\text{Tr}_{1,2}(F_{k,l} \otimes 1)(\rho \otimes J \gamma J^*) (F_{k,l} \otimes 1)}{\text{Tr}_{1,2,3}(F_{k,l} \otimes 1)(\rho \otimes J \gamma J^*) (F_{k,l} \otimes 1)}$$

where $\rho$ and $\gamma$ (denoted $\rho_0$ above) are the state of the processing part and of the memory before recognition and $J$ an isometry extending from $H_2$ to $H_2 \otimes H_3$ and $1$ denotes the identical operator. The value $\text{Tr}_{1,2,3}(F_{k,l} \otimes 1)(\rho \otimes J \gamma J^*) (F_{k,l} \otimes 1)$ represents the probability to measure the value $z_{k,l}$. So, obviously, we have to assume that this probability is greater than 0. The state $\Lambda_{k,l}(\rho \otimes \gamma)$ gives the state of the memory after the process of recognition. The elements of a basis $(b_k)_{k=1}^{n}$ of $H_j$ are interpreted as elementary signals.

In this first attempt to our model described above, there appear still a lot of effects being non-realistic for the process of recognition. Some examples (cf. the last subsection) show that with this model one can describe extreme cases such as storing the full information or total loss of memory, but - as mentioned above - that is still far from being a realistic description.

In the paper [5], we restrict ourselves to finite dimensional Hilbert spaces. Moreover, we assume equal dimension of the Hilbert spaces $H_j$ ($j = 1, 2, 3$). It seems that infinite dimensional schemes will lead to more realistic models. However, this is just a first attempt to describe the brain function. Moreover, for finite dimensional Hilbert spaces the mathematical model becomes more transparent and one can obtain easily a general idea of the model. To indicate obvious generalizations to more general situations and especially to infinite dimensional Hilbert spaces we sometimes use notions and notations from the general functional analysis [3, 4].

6.1 Basic Notions

Let $H_1, H_2, H_3$ be Hilbert spaces with equal finite dimension:

$$\dim H_j = n, \ (j \in \{1, 2, 3\}).$$

First we will represent these Hilbert spaces in a way that it seems to be convenient for our considerations. Each of the spaces $H_1, H_2, H_3$ can be identified with the space $\mathbb{C}^n$ of $n$-dimensional complex vectors. The space $\mathbb{C}^n$ again may be identified with the space $\{f : G \rightarrow \mathbb{C}\}$ of all complex-valued function on $G := \{1, \ldots, n\}$. The scalar product then is given by

$$\langle f, g \rangle := \sum_{k=1}^{n} \overline{f(k)}g(k) = \int \overline{f(k)}g(k)\mu(dk)$$

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where \( \mu \) is the counting measure on \( G \), i.e. \( \mu = \sum_{k=1}^{n} \delta_k \) with \( \delta_k \) denoting the Dirac measure in \( k \). So, each of the spaces \( H_j \) can be written formally as an \( L_2 \)-space:

\[
H_j = L_2(G, \mu) := L_2(G), \quad (j \in \{1, 2, 3\}).
\]

For the tensor product one obtains

\[
f \otimes g(k, l) = f(k)g(l) \quad (f, g \in L_2(G), k, l \in G),
\]

and we have

\[
H_1 \otimes H_2 = L_2(G \times G, \mu \times \mu) = H_2 \otimes H_3.
\]

We will abbreviate this tensor product by \( L_2(G^2, \mu^2) \) or just by \( L_2(G^2) \).

By \( \mathcal{B}(H) \) we denote the space of all bounded linear operators on a Hilbert space \( H \). In \( \mathcal{B}(L_2(G)) \) the operator of multiplication by a function \( g \in L_2(G) \) is given by

\[
(O_g f)(k) = g(k)f(k) \quad (f \in L_2(G), k \in G).
\]

Observe that for all \( f, g \in L_2(G) \) one has

\[
O_f g = O_g f, \quad O_f^* = O_f
\]

and for \( f \in L_2(G) \) with \( f(k) \neq 0 \) for all \( k \in G \) it holds \( O_f^{-1} = O_{1/f} \).

The function \( 1, 1(k) = 1 \) for all \( k \in G \), obviously belongs to \( L_2(G) \) and \( 1 = O_1 \) is the identity in \( \mathcal{B}(L_2(G)) \).

Consequently, an operator of multiplication \( O_f \) is unitary if and only if \( |f(k)| = 1 \) for all \( k \in G \).

Further, we will use the mapping \( J \) from \( L_2(G) \) into \( L_2(G^2) \) given by

\[
(Jf)(k, l) = f(k)\delta_{k,l} \quad (f \in L_2(G), k, l \in G)
\]

where \( \delta_{k,l} \) denotes the Kronecker symbol. It is immediate to see that \( J \) is an isometry. For the adjoint \( J^* : L_2(G^2) \rightarrow L_2(G) \) we obtain

\[
(J^* \Phi)(k) = \Phi(k, k) \quad (\Phi \in L_2(G^2), k \in G).
\]

Observe that \( G \) equipped with the operation \( \oplus : G \times G \rightarrow G \), \( k \oplus l := (k + l) \mod n \) is a group. The operation inverse to \( \oplus \) we denote by \( \ominus \). Let us remark that \( k \ominus l = k - l \) in the case \( k > l \) and \( k \ominus l = k - l + n \) if \( k \leq l \). We conclude that for all \( k \in G \) the operator \( U_k \in \mathcal{B}(L_2(G)) \) given by

\[
(U_k f)(m) := f(k \oplus m) \quad (f \in L_2(G))
\]
is unitary.

Now, let \((b_k)_{k=1}^n\) be an orthonormal basis in \(L_2(G)\), and denote by \((B_k)_{k=1}^n\) the sequence of multiplication operators corresponding to the elements of this basis, i.e. \(B_k := \mathcal{O}_{b_k}, \ k \in G\). Then for \(k, l \in G\) we put

\[
\xi_{k,l} := (B_k \otimes U_l)J1.
\]

One can show that the sequence \((\xi_{k,l})_{k,l \in G}\) is an orthonormal basis in \(L_2(G^2)\). And we denote by \(F_{i,j} \in B(L_2(G^2))\) the projection onto \(\xi_{i,j}\), i.e.

\[
F_{i,j} := |\xi_{i,j}\rangle \langle \xi_{i,j}| = \langle \xi_{i,j}, \cdot \rangle \xi_{i,j}.
\]

6.2 Channels

**Definition 5** Let \(\gamma\) be a state on \(H_2 = L_2(G)\) (i.e. \(\gamma\) is a positive trace-class operator with \(\text{Tr}(\gamma) = 1\)). The state \(e(\gamma)\) on \(L_2(G^2) = H_2 \otimes H_3\) given by

\[
e(\gamma) = J\gamma J^*\]

where \(J\) is the isometry given by (4) we call the entangled state corresponding to \(\gamma\).

Now, let \(\rho\) and \(\gamma\) be states on \(H_1\) resp. \(H_2\), the state \(e(\gamma)\) (usually denoted by \(\sigma\)) will be a state on \(H_2 \otimes H_3\). Remember that we assumed \(H_1 = H_2 = H_3 = L_2(G)\). The numbering only indicates the meaning of the states (we recall that \(H_1\) represents the processing part, \(H_2\) the memory before and \(H_3\) the memory after the recognition process.) Then \(\rho \otimes e(\gamma)\) is a state on \(H_1 \otimes H_2 \otimes H_3\) and we observe immediately

\[
\rho \otimes e(\gamma) = (1 \otimes J)(\rho \otimes \gamma)(1 \otimes J^*)
\]

In subsection 6.3 we calculate explicitly the trace of

\[
(F_{i,j} \otimes 1)(\rho \otimes e(\gamma))(F_{i,j} \otimes 1) = (F_{i,j} \otimes 1)(1 \otimes J)(\rho \otimes \gamma)(1 \otimes J^*)(F_{i,j} \otimes 1).
\]

The following proposition will be very useful for this.

**Proposition 6** Let \((g_k)_{k=1}^n\) and \((h_k)_{k=1}^n\) be orthonormal systems in \(L_2(G)\) and \(\rho\) and \(\gamma\) states on \(L_2(G)\) having the following representations:

\[
\rho = \sum_{k=1}^n \alpha_k |g_k><g_k|, \quad \gamma = \sum_{k=1}^n \beta_k |h_k><h_k|,
\]

\[
\alpha_k \geq 0, \beta_k \geq 0, \sum_{k=1}^n \alpha_k = \sum_{k=1}^n \beta_k = 1.
\]
Then for all $i, j \in G$

$$(F_{i,j} \otimes 1)(\rho \otimes e(\gamma))(F_{i,j} \otimes 1) = F_{i,j} \otimes \sum_{k,l=1}^{n} \alpha_k \beta_l |G_{i,j} g_k \otimes h_l > < G_{i,j} g_k \otimes h_l| \quad (12)$$

where $G_{i,j}$ is given by, for $i, j \in G$

$$G_{i,j} := J^*(U_j \otimes 1)(B^*_i \otimes 1) = J^*(U_j B^*_i \otimes 1) \quad (13)$$

Denote by $\mathcal{T}$ the set of all positive trace-class operators on $L^2(G)$ including the null operator $0$,

$$0(f) = 0 \quad (f \in L^2(G)).$$

We fix an operator $\tau \in \mathcal{T}$ having the representation

$$\tau = \sum_{k=1}^{n} \gamma_k |h_k > < h_k| \quad (14)$$

with $(\gamma_k)_{k \in G} \subseteq [0, \infty)$ and $(h_k)_{k \in G}$ being an orthonormal basis in $L^2(G)$.

The linear mapping $K_\tau : \mathcal{T} \rightarrow \mathcal{T}$ given by

$$K_\tau(\rho) := \sum_{k=1}^{n} \gamma_k O_{h_k} \rho O^*_{h_k} \quad (\rho \in \mathcal{T}) \quad (15)$$

depends only on the operator $\tau$ but not on its special representation.

**Definition 7** Denote by $\mathcal{S}$ the set of all states on $L^2(G)$ and for $\tau \in \mathcal{T}$ by $\mathcal{S}_\tau$ the set of all states $\rho$ from $\mathcal{S}$ with the property that $\text{Tr}K_\tau(\rho)$ is positive:

$$\mathcal{S}_\tau := \{ \rho \in \mathcal{S} : \text{Tr}K_\tau(\rho) > 0 \}. \quad (16)$$

For $\tau \in \mathcal{T}$ the mapping $\hat{K}_\tau : \mathcal{S}_\tau \rightarrow \mathcal{S}$ given by

$$\hat{K}_\tau(\rho) := \frac{1}{\text{Tr}K_\tau(\rho)} K_\tau(\rho) \quad (\rho \in \mathcal{S}_\tau) \quad (17)$$

is called the channel corresponding to $\tau$. The channel corresponding to $\tau$ is called unitary if there exists an unitary operator $U$ on $L^2(G)$ such that $\hat{K}_\tau(\rho) = U \rho U^*$. 

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Observe that the channel $\hat{K}$ is in general nonlinear.

Let us make some remarks on the physical meaning of the channels $K_\tau$ and $\hat{K}_\tau$. The channels $K_\tau$ are mixtures of linear channels of the type

$$K^h(\rho) := O_h \rho O_h^* \quad (\rho \in)$$

with $h \in L_2(G)$, $\|h\| = 1$. Let us consider the more general case

$$\|h\| > 0, \ |h(k)| \leq 1 \quad (k \in G).$$

We define an operator $t_h : L_2(G) \longrightarrow L_2(\{1, 2\} \times G)$ by setting for all $f \in L_2(G)$ and $k \in G$

$$(t_h f)(l, k) = \begin{cases} h(k)f(k) & \text{for } l = 1 \\ \sqrt{1 - |h(k)|^2}f(k) & \text{for } l = 2. \end{cases}$$

The operator $t_h$ is an isometry from $L_2(G)$ to $L_2(\{1, 2\} \times G) \cong L_2(\{1, 2\}) \otimes L_2(G)$. Indeed,

$$\|t_h f\|^2 = \sum_{l=1}^{2} \sum_{k=1}^{n} |t_h f(l, k)|^2 = \sum_{k=1}^{n} (|h(k)|^2 + 1 - |h(k)|^2)|f(k)|^2 = \|f\|^2.$$

Consequently, the mapping $E_h : \mathcal{B}(L_2(\{1, 2\} \times G)) \longrightarrow \mathcal{B}(L_2(G))$ given by

$$E_h(B) := t_h^* B t_h$$

is completely positive and identity preserving. The channel $E_h^*(\rho) = t_h \rho t_h^*$ is the corresponding linear channel from the set of states on $L_2(G)$ into the set of states on $L_2(\{1, 2\} \times G)$. The space $L_2(\{1, 2\} \times G)$ has an orthogonal decomposition into $L_2(\{1\} \times G)$ and $L_2(\{2\} \times G)$ both being trivially isomorphic to $L_2(G)$. Performing a measurement according to the projection onto $L_2(\{1\} \times G) \cong L_2(G)$ given the state $E_h^*(\rho)$ one obtains the state $\hat{K}^h(\rho)$. A measurement according to the projection onto $L_2(\{2\} \times G) \cong L_2(G)$ leads to the state $\hat{K}\sqrt{1-|h|^2}(\rho)$. 16
6.3 The State of the Memory after Recognition

Let us recall that for states $\rho, \gamma$ on $L^2(G)$ and $i, j \in G$

$$(F_i \otimes 1)(\rho \otimes e(\gamma))(F_i \otimes 1)$$

is a linear operator from $L^2(G^3)$ into $L^2(G^2)$, and that (cf. [3]) it is equal

$$(F_i \otimes 1)(1 \otimes J)(\rho \otimes \gamma)(1 \otimes J^*)(F_i \otimes 1).$$

In the following we consider the family of channels $(\Lambda_{i,j})_{i,j \in G}$ from the set of product states $\rho \otimes \gamma$ on $H_1 \otimes H_2$ into the states on $H_3$ given by

$$\Lambda_{i,j}(\rho \otimes \gamma) := \frac{\text{Tr}_{1,2}(F_{i,j} \otimes 1)(\rho \otimes e(\gamma))(F_{i,j} \otimes 1)}{\text{Tr}_{1,2,3}(F_{i,j} \otimes 1)(\rho \otimes e(\gamma))(F_{i,j} \otimes 1)}$$

where $\text{Tr}_{1,2}$ resp. $\text{Tr}_{1,2,3}$ denotes the partial trace with respect to the first two components resp. the full trace with respect to all three spaces. In the sequel we always will assume that

$$\text{Tr}_{1,2,3}(F_{i,j} \otimes 1)(\rho \otimes e(\gamma))(F_{i,j} \otimes 1) > 0. \quad (22)$$

Let $\rho$ and $\gamma$ are given as in Proposition 6. Since $(\xi_{i,j})_{i,j \in G}$ is an orthonormal basis in $L^2(G^2)$ we get from Proposition 6

$$\text{Tr}_{1,2}(F_{i,j} \otimes 1)(\rho \otimes e(\gamma))(F_{i,j} \otimes 1) = \sum_{k,l=1}^n \alpha_k \beta_l \langle G_{i,j}g_k \otimes h_l, \cdot \rangle G_{i,j}g_k \otimes h_l. \quad (23)$$

Summarizing, we get the following representation of $\Lambda_{i,j}$:

**Proposition 8** Let $\rho$ and $\gamma$ be given as in Proposition 6. Further, assume (22). Then

$$\Lambda_{i,j}(\rho \otimes \gamma) = \frac{\sum_{k,l=1}^n \alpha_k \beta_l \langle G_{i,j}g_k \otimes h_l, \cdot \rangle G_{i,j}g_k \otimes h_l}{\sum_{k,l=1}^n \alpha_k \beta_l \|G_{i,j}g_k \otimes h_l\|^2} \quad (24)$$

where for $\Phi \in L_2(G^2)$

$$\|G_{i,j} \Phi\|^2 = \sum_{m=1}^n |b_i|^2(m \oplus j) |\Phi(m \oplus j, m)|^2. \quad (25)$$
Fortunately, we can find expressions for the state $\Lambda_{i,j}(\rho \otimes \gamma)$ of the memory after the recognition process being in many cases simpler. We can express the teleportation channel $\Lambda_{i,j}$ with the help of the channels $K_\tau$ we introduced in the previous subsection.

**Proposition 9** Let $i, j \in G$ and let $\rho$ be a state from $S_{\|b_i\rangle\langle b_i\|}$ (cf. (15) and Definition 7). Further, let $\gamma$ be a state from $\mathcal{S}$ such that

$$U_j K_{\|b_i\rangle\langle b_i\|}(\rho) U_j^* \in \mathcal{S}_\gamma.$$  \hfill (26)

Then

$$\Lambda_{i,j}(\rho \otimes \gamma) = \hat{K}_\gamma \circ K_j \circ \hat{K}_{\|b_i\rangle\langle b_i\|}(\rho)$$  \hfill (27)

where $K_j$ denotes the unitary channel given by $K_j(\rho) = U_j \rho U_j^*$. 

**Remark 10** All proofs of this paper can be seen in [5].

**Concluding remarks:** We touched the problem of finding simplified models for the recognition process. We were interested in how the input signal arriving at the brain is entangled (connected) to the memory already stored and the consciousness that existed in the brain, and how a part of the signal will be finally stored as a memory. It is clear that this simple model is just for illustration and can not serve for describing realistic aspects of recognition. Choosing a more complex basis one obtains expressions depending heavily on the states $\rho$ and $\gamma$. Though the above presented model is only a first attempt it shows that there are possibilities to model the process of recognition. To get closer to realistic models we will try to refine the above models by

- passing over to infinite Hilbert spaces,
- replacing pure states by coherent states on the Fock space,
- making more complex measurements than simple one-dimensional projections $F_{i,j}$,
- replacing the trivial entanglement $J$ by a more complex one based on beam splitting procedures, and finally
- examing whether some symmetry breaking as in [2] will occur in the process of recognition and storing memory.
References


