Topologically Alice Strings and Monopoles

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(Dated: April 9, 2004)

Abstract

Symmetry breaking can produce “Alice” strings, which alter scattered charges and carry monopole number and charge when twisted into loops. Alice behavior arises algebraically, when strings obstruct unbroken symmetries — a fragile criterion. We give a topological criterion, compelling Alice behavior or deforming it away. Our criterion, that \( \pi_0(H) \) acts nontrivially on \( \pi_1(H) \), links topologically Alice strings to topological monopoles. We twist topologically Alice loops to form monopoles. We show that Alice strings of condensed matter systems (nematic liquid crystals, \(^3\)He-A, and related non-chiral Bose condensates and amorphous chiral superconductors) are topologically Alice, and support fundamental monopole charge when twisted into loops. Thus they might be observed indirectly, not as strings, but as loop-like point defects. We describe other models, showing Alice strings failing our topological criterion; and twisted Alice loops supporting deposited, but not fundamental, monopole number.
I. INTRODUCTION

Among the defects created when gauged symmetries break down are Alice strings. Alice strings obstruct the global extension of unbroken symmetries, making them multivalued when parallel transported around the string. This algebraic obstruction has two prominent physical consequences. First, it produces nonconservation of associated charges, when Aharonov-Bohm scattered around the string. Second, it induces monopoles, as twisted loops of Alice string.

These Alice features arise due to gauge flux on the string’s core. The gauge flux generates the condensate winding, while acting on asymptotic particles through the Wilson line $U(\varphi)$. This action fixes particles’ Aharonov-Bohm scattering around the string, to one changing both charge and monopole number. Loops of string, which leave charge and monopole number well-defined asymptotically, thus support deposited unlocalizable charge (“Cheshire charge”) and deposited monopole number. Alice loops carry this deposited monopole number by twisting, as we probe further below.

Alice strings in condensed matter systems are global, not gauged, defects. They have no gauge flux to fix their Aharonov-Bohm scattering, and guarantee altered charge and monopole number upon string traversal. However, their Aharonov-Bohm scattering was considered in; with showing that global Alice strings generically share all Alice behaviors. They alter both charge and monopole number on string traversal, with twisted loops supporting both Cheshire charge and deposited monopole number. Furthermore, condensed matter systems offer the most likely prospects for Alice string observation. We show here that known condensed matter Alice strings form twisted loops with fundamental monopole charge, suggesting a second avenue for potential observation: Alice strings might be observed, not as strings, but as looplike point defects, when twisted loops comprise the energetically favored solution of fundamental monopole charge.

The criterion for Alice string formation was first stated algebraically, in terms of the string’s untraced Wilson loop $U(2\pi)$. When $U(2\pi)$ fails to commute with an unbroken symmetry $h$, the symmetry cannot be globally extended; when it fails to commute with unbroken generator $T_h$, the associated charge is nonconserved. Thus Alice strings arise when $U(2\pi)$ lies outside the center of the unbroken symmetry group $H$. As noted in, this is an inherently nontopological criterion, as topologically equivalent choices for $U(2\pi)$
can commute with different subgroups of $H$. Thus emergence of Alice behavior appeared a dynamical question. Steps toward topologizing this criterion came in [4]. They noted that when all topologically equivalent choices for $U(2\pi)$ lie outside the center of $H$, Alice strings must form. Equivalently, topologically Alice strings form when the fiber bundle of $H$ parallel transported around the string is nontrivializable. Both criteria, while accurate, seem difficult to apply.

We here establish an easily applied topological criterion which states when Alice strings must form. Consider the symmetry breakdown of Lie group $G \rightarrow H$, taking for $G$ the simply connected cover of the initial Lie symmetry. A topological string then has homotopy $\pi_o(H)$; that is, its flux $U(2\pi)$ lies in a disconnected component of the unbroken symmetry group $H$. Monopoles have homotopy $\pi_1(H)$, describing loops $h(\alpha)$ of different winding in $H$. Our criterion labels strings with flux $U(2\pi)$ "topologically Alice" if they alter the topological charge of monopoles circumnavigating them:

$$h(\alpha) \rightarrow \hat{h}(\alpha) = U(2\pi) \ h(\alpha) \ U^{-1}(2\pi) \ \not\sim \ h(\alpha).$$

This is a topological criterion, corresponding to a nontrivial action of $\pi_o(H)$ on $\pi_1(H)$, where $h_o = U(2\pi) \in \pi_o(H)$ alters the topological winding of loop $h(\alpha) \in \pi_1(H)$:

$$\hat{h}(\alpha) = h_o \ h(\alpha) \ h_o^{-1} \ \not\sim \ h(\alpha). \quad (1.1)$$

This criterion captures physical Alice behavior and is easily applied. It assures the standard Alice constellation of behaviors: multivalued symmetry, charge-violating Aharonov-Bohm scattering, Cheshire charge on Alice loops. In topology lies both power and limitation. As limitation, Alice behavior survives our criterion only when altering generators alters the topology of loops they generate, eliminating many models with algebraic Alice phenomena. Most simply, for $\pi_o(H)$ to act nontrivially on $\pi_1(H)$, $\pi_1(H)$ itself must be nontrivial. Thus, only a theory with topological strings and monopoles — indeed, with monopoles topologically distinct from antimonopoles — can Alice strings meet our criterion. Strings meeting our criterion, however, have Alice behavior which is topologically ordained; while those failing can be destabilized, continuously deformed to remove all Alice behavior. Their Alice behavior survives only if dynamically stabilized; that is, if energetically favored over non-Alice strings of the same winding.

Our topological criterion for Alice strings ensures that twisted loops carry monopole charge, as we see by explicit construction in section III. We note that topological arguments
indicate only that deposited monopole charge can be carried by a twisted Alice loop. Monopole charge, however, is typically not deposited in single fundamental units, leaving open the question of whether twisted Alice loops can carry fundamental monopole charge. Our analysis of this model-dependent question gives the heuristic answer: twisted Alice loops in condensed matter systems typically support fundamental monopole charge; while twisted Alice loops of particle physics models often do not.

Specifically, we address the best condensed matter candidates for Alice strings: the original Alice string of Schwarz, coinciding with the Alice string of liquid crystals and of non-chiral Bose condensates \[1, 8\]; and the Alice string of \(^3\)He-A \[9\], arising anew for unconventional spin-triplet superconductors \[10\]. In both models, strings are topologically Alice, and form twisted Alice loops supporting fundamental monopole charge, even though monopole charge deposits onto string loops in even increments. Thus, for the Alice strings of interest to condensed matter, even the most fundamental singular point defect, or monopole, can take the form of a twisted Alice loop. We display particle physics-motivated models, however, whose twisted Alice loops support only deposited monopole charge, and cannot form fundamental monopoles.

We present our results as follows. In section II we discuss our topological criterion for Alice behavior. We show that Alice strings failing our criterion have topologically unstable Alice features; that is, their Alice behavior can be deformed away. In section III we show, by construction, that twisted Alice strings carry monopole charge. We argue topologically, using our criterion to display a fully twisted Alice loop, carrying monopole number deposited in the monopole scattering \(h(\alpha) \to \tilde{h}(\alpha)\). We then illustrate our criterion and Alice loop twisting to form monopoles, for key models. First, we treat the condensed matter systems: in section IV A the simple Schwarz Alice string, coinciding with the Alice string of liquid crystals and some non-chiral Bose condensates; and in section IV B the Alice string of \(^3\)He-A and spin-triplet superconductors. In both cases, topologically Alice strings twist into loops carrying fundamental monopole charge; in both cases, this stems from the embedding of monopole loops in an \(SO(3)\) symmetry group, an embedding endemic to condensed matter systems. This suggests that for all topologically Alice condensed matter systems, fundamental point defects — monopoles — can take toroidal shape, as twisted Alice loops.

In the particle physics models of section V we illustrate different outcomes for the topology of Alice strings and their twisted loops: Alice behavior which is nontopological; and
topologically Alice loops which support deposited, but not fundamental, monopole charge. Key points include a focus on how algebraically Alice candidates may fail our criterion; and how the half-twisted Alice loops capable of supporting fundamental monopole charge fail to be single-valued. Thus only fully twisted Alice loops arise, supporting only the deposited monopole charge dictated by topology.

We conclude in section VI.

II. A TOPOLOGICAL CRITERION FOR ALICE BEHAVIOR

We take $G$ to be the simply connected cover of the initial symmetry — a connected Lie group — and $H \subset G$ its unbroken subgroup. A topologically stable string has flux $U(2\pi)$ in a disconnected component of $H$, with topology determined by $\pi_o(H)$. Similarly, the topology of the monopole is given by $\pi_1(H)$, describing loops $h(\alpha)$ of different winding in $H$. By taking seriously the change in monopole number in circumnavigating the Alice string, we construct our criterion. Note that, in Aharonov-Bohm scattering around the string, the monopole $h(\alpha)$ is conjugated by the string’s Wilson loop $U(2\pi)$:

$$h(\alpha) \to \tilde{h}(\alpha) = U(2\pi) \ h(\alpha) \ U^{-1}(2\pi) \ .$$

Monopole number changes if $\tilde{h}(\alpha)$ and $h(\alpha)$ are topologically distinct loops. We represent this transformation topologically, as $\pi_o(H)$ acting naturally on $\pi_1(H)$ by conjugation. Topologically Alice strings form if that action is nontrivial: that is, if, for $h_o$ a representative element of $\pi_o(H)$ and $h(\alpha)$ a representative loop in $\pi_1(H)$,

$$\tilde{h}(\alpha) = h_o \ h(\alpha) \ h_o^{-1} \neq h(\alpha) \ . \ (2.1)$$

A string with untraced Wilson loop $U(2\pi) \sim h_o$ meeting this criterion is topologically guaranteed to change monopole number; we dub it “topologically Alice”.

This criterion captures physical Alice behavior, is easily applied, and is topological. Its result, for any chosen $h_o$ and $h(\alpha)$, remains invariant under deformations of either flux $U(2\pi) = h_o$ or monopole loop $h(\alpha)$. By construction, strings are topologically Alice if monopoles change topologically in traversing them. Of course, monopoles $h(\alpha)$ change because of algebraic Alice behavior: loop $h(\alpha) = e^{i\alpha T_h}$ alters only if its generator $T_h$ alters; that is, if $U(2\pi)$ fails to commute with generator $T_h$. This algebraic noncommutation creates the standard Alice constellation of behaviors: multivalued symmetry, charge-violating
Aharonov-Bohm scattering, Cheshire charge on Alice loops. It is captured by our criterion only when altering generators alters the topology of the loops they generate. This misses some Alice phenomena — particularly in models with poorly distinguished loops, when \( \pi_1(H) = 0 \) (and all loops are trivial) or \( \pi_1(H) = \mathbb{Z}_2 \) (and all nontrivial loops, including a loop and its inverse, are identified). Such models possess either no, or only \( \mathbb{Z}_2 \), topological monopoles. We claim that Alice behavior in these models is not robust topologically; that is, continuous deformation of such strings removes their Alice behavior. In such cases, persistence of Alice behavior can arise only from dynamical arguments, favoring Alice strings over non-Alice strings of the same winding. Dynamically stabilized features remain interesting — for example, nontopological defects including embedded, semilocal, or electroweak strings.\[11\] However, we seek here for Alice behavior the more robust motivation of topological imperative.

Consider an Alice string with Wilson loop \( U(2\pi) = h_o \) which fails our topological criterion. This occurs if, for a nontrivial loop \( h(\alpha) \in H \) describing a monopole, the parallel transported monopole is homotopic to the original; that is,

\[
\tilde{h}(\alpha) = h_o \ h(\alpha) \ h_o^{-1} \sim h(\alpha) .
\]  

(2.2)

This occurs only if there exists some continuous map \( f(x) \) deforming \( \tilde{h}(\alpha) \) to \( h(\alpha) \); that is

\[
f(x) : \tilde{h}(\alpha) \rightarrow \begin{cases} 
\tilde{h}(\alpha) & \text{when } x = 0 \\
h(\alpha) & \text{when } x = 1 
\end{cases}.
\]

Note that the map \( f(x) \) relates nontrivial loops in \( H \), with basepoint \( \alpha = 0 \) fixed at the group identity. We write it as a continuously varying group element \( f(x) \) acting on \( \tilde{h}(\alpha) \) by conjugation,

\[
f(x) : \tilde{h}(\alpha) \rightarrow f(x) \tilde{h}(\alpha) f(x)^{-1} .
\]

Without loss of generality we take \( f(0) = \mathbb{1} \).

Now consider the continuous map \( h'_o(x) = f(x) \ h_o \), where \( f(x) \) acts on \( h_o \) by left multiplication. This interpolates between the Alice string’s Wilson loop \( h_o \), and group element \( h'_o(1) \) in the same disconnected component of \( H \). \( h'_o(1) \) thus defines a topologically equivalent string, with Wilson loop \( U(2\pi) = h'_o(1) \). In circumnavigating this deformed string, the original monopole \( h(\alpha) \) is unchanged: it goes to

\[
\tilde{h}'(\alpha) = h'_o(1) \ h(\alpha) \ h'_o^{-1}(1) = f(1) \ h_o \ h(\alpha) \ h_o^{-1} \ f(1)^{-1}
\]

\[
= f(1) \tilde{h}(\alpha) \ f(1)^{-1} = h(\alpha) ,
\]
by construction of $f(x)$. Thus the monopole loop $h(\alpha)$ remains identical on circumnavigating the string. Choosing as our nontrivial loop $h(\alpha) = e^{i\alpha T_h}$, the loop (at each value of $\alpha$) remains unchanged only if the generator $T_h$ remains unchanged. Thus by continuously deforming our Alice string’s flux from $h_o$ to $h'_o(1)$, we have obtained a string flux $U(2\pi) = h'_o(1)$ which commutes with all generators; that is, we have removed all Alice behavior of the string. This renders Alice behavior for strings failing our criterion topologically unstable; it can be deformed away, and stabilized only in dynamical, model-dependent ways.

III. TWISTED ALICE LOOPS AS MONOPOLES

Monopoles lie on the vacuum manifold at spatial infinity, with topology given by $\pi_2(G/H)$. We here show that a fully twisted topologically Alice loop is necessarily a topological monopole; that is, an infinite sphere enclosing it has nontrivial $\pi_2(G/H)$.

First, we construct a sensible twisted Alice loop.

Recall that our Alice string has a condensate $\langle \phi \rangle$ which winds asymptotically over the vacuum manifold $G/H$ according to

$$\langle \phi (\varphi) \rangle = U(\varphi) \langle \phi \rangle_o ,$$

where the Wilson line $U(\varphi)$ acts on the vev $\langle \phi \rangle_o$ according to its group representation. $U(\varphi)$ varies continuously over $G$ for $0 < \varphi < 2\pi$, and connects the identity at $\varphi = 0$ to a distinct Wilson loop $U(2\pi)$ in $H$. The string is topological when $U(2\pi) = h_o$ lies in a disconnected component of $H$, with nontrivial $\pi_o(H)$, and is topologically Alice when it meets our criterion (2.1).

Now twist the Alice string: continuously rotate its Wilson line within $G$ by the angle-dependent $H$-group rotation $h^{-1}(\alpha)$:

$$U(\varphi, \alpha) = h^{-1}(\alpha) \ U(\varphi) \ h(\alpha) ,$$

as shown in Figure 1a. This, of course, rotates our condensate among the degenerate vacua on $G/H$:

$$\langle \phi (\varphi, \alpha) \rangle = U(\varphi, \alpha) \langle \phi \rangle_o .$$

Under what conditions may we identify string ends at $\alpha = 0$ and $\alpha = 2\pi$ to form a string loop, as pictured in Figure 1b? First, we require the string configurations to match at the
junction. This is assured if \( h(2\pi) = h(0) \), that is, if \( h(\alpha) \) is a loop. Second, the twisted condensate \( \langle \phi (\varphi, \alpha) \rangle \) must be single-valued. Note that the Wilson line \( U(\varphi, \alpha) \) itself need not be single-valued: indeed, for a monopole configuration, \( U(\varphi, \alpha) \) interpolates from the identity at \( \varphi = 0 \) to a nontrivial loop in \( H \) at \( \varphi = 2\pi \).

First we check singlevaluedness of \( \langle \phi (\varphi, \alpha) \rangle \) at the loop's origin. Here \( \varphi = 0 \) (or \( 2\pi \)) while \( \alpha \) is indeterminate. Note that \( U(0, \alpha) \) is the identity, manifestly single-valued. At \( \varphi = 2\pi \),

\[
U(2\pi, \alpha) = h^{-1}(\alpha) \ U(2\pi) \ h(\alpha) .
\]

This generally does vary with \( \alpha \); however, it is a loop in \( H \), with basepoint \( U(2\pi) = h_\circ \in H \). It thus leaves the condensate invariant, assuming the single value \( \langle \phi \rangle_\circ \) at loop origin.

Elsewhere, we need only show first, that \( \langle \phi (\varphi, \alpha+2\pi) \rangle = \langle \phi (\varphi, \alpha) \rangle \); and second, that \( \langle \phi (\varphi + 2\pi, \alpha) \rangle = \langle \phi (\varphi, \alpha) \rangle \). The first is trivial: since \( h(\alpha) \) is a loop, \( h(\alpha) = h(\alpha + 2\pi) \) and both \( U(\varphi, \alpha) \) and \( \langle \phi (\varphi, \alpha) \rangle \) are single-valued in \( \alpha \).

To show singlevaluedness in \( \varphi \), let us, without loss of generality, diagonalize our string Wilson line \( U(\varphi) \), taking it to be generated by a fixed generator so that \( U(\varphi + 2\pi) = U(\varphi) \ U(2\pi) \). Then our twisted Wilson line obeys

\[
U(\varphi + 2\pi, \alpha) = h^{-1}(\alpha) \ U(\varphi) \ U(2\pi) \ h(\alpha) = U(\varphi, \alpha) \ U(2\pi, \alpha) .
\]

As noted above, \( U(2\pi, \alpha) \) is a loop in \( H \), leaving \( \langle \phi \rangle_\circ \) invariant. Thus

\[
\langle \phi (\varphi, \alpha + 2\pi) \rangle = \langle \phi (\varphi, \alpha) \rangle = U(\varphi, \alpha) \langle \phi \rangle_\circ
\]

and our twisted Alice loop is fully single-valued.

By the exact sequence for \( \pi_2(G/H) \), our twisted Alice loop is a monopole when \( U(\varphi, \alpha) \) interpolates between an element of \( H \) at \( \varphi = 0 \) and a nontrivial loop in \( H \) at \( \varphi = 2\pi \). For convenience, right multiply \( U(\varphi, \alpha) \) by \( h_\circ^{-1} \):

\[
U(\varphi, \alpha) = h^{-1}(\alpha) \ U(\varphi) \ h(\alpha) \ h_\circ^{-1} . \tag{3.1}
\]

Since \( h_\circ^{-1} \in H \), this right multiplication does not change the physical condensate \( \langle \phi (\varphi, \alpha) \rangle \). However, it makes the topology of \( U(\varphi, \alpha) \) clear, for

\[
U(\varphi, \alpha) = \begin{cases} 
  h_\circ^{-1} & \text{for } \varphi = 0 \\
  h^{-1}(\alpha) \ h(\alpha) & \text{for } \varphi = 2\pi
\end{cases}
\]
By definition, if the string is topologically Alice, $\tilde{h}(\alpha) \not\sim h(\alpha)$ so that $h^{-1}(\alpha) \tilde{h}(\alpha)$ is a nontrivial loop in $H$, and the twisted Alice loop carries nontrivial monopole charge. If the string is not topologically Alice, the loop $h^{-1}(\alpha) \tilde{h}(\alpha)$ is trivial in $H$ and the twisted Alice loop carries no monopole charge.

Thus twisted Alice loops carry monopole charge if and only if they obey our topological Alice criterion. We note that the monopole charge displayed, with winding $h^{-1}(\alpha) \tilde{h}(\alpha)$, is exactly that deposited on an initially untwisted Alice loop, when a monopole of winding $h^{-1}(\alpha)$ circumnavigates the string and emerges with winding $\tilde{h}^{-1}(\alpha)$. \[\text{[4]}\] (The inverse twisted Alice loop, generated by $U^{-1}(\varphi, \alpha)$, instead carries monopole charge $h(\alpha) \tilde{h}^{-1}(\alpha)$, deposited in the monopole circumnavigation $h(\alpha) \rightarrow \tilde{h}(\alpha)$.)

We note that our final map $U(2\pi, \alpha)$ for the twisted Alice loop coincides with the flux loop (4.1) and paths $C'_\alpha$ defined in \[\text{[4]}\]. They show that this map coincides with the Lubkin classification of monopole charge for the twisted Alice loop. This reinforces our classification, as identifying twisted topologically Alice loops with physical gauged magnetic monopoles.

We note two key points. First, the topological loop twisting \[\text{[3.1]}\] supports only deposited monopole charge: namely, that deposited when a monopole of winding $h^{-1}(\alpha)$ scatters into one of distinct winding $\tilde{h}^{-1}(\alpha)$. Typical Alice string scattering changes monopole to antimonopole, with monopole number deposited onto the string loop in units of two. Our topological arguments thus leave open the question of whether twisted Alice loops can carry fundamental monopole charge. Second, we implicitly took as twisting function $h(\alpha)$ in equation \[\text{[3.1]}\], the loop in $H$ representing a fundamental monopole. (We call this choice the fully-twisted Alice loop). We remain free to choose a different twisting function $h(\alpha)$ in $H$, so long as it renders $U(\varphi, \alpha)$ singlevalued in the angle $\alpha$. Propitious choice of $h(\alpha)$ in the condensed matter models below allows construction of twisted Alice loops with fundamental monopole charge.

\section{Condensed Matter Alice Strings}

\subsection{The Schwarz, or Nematic, Alice String}

We start with the simplest example, the canonical Schwarz Alice string, \[\text{[1]}\] whose symmetry-breaking pattern coincides with Alice strings in nematic liquid crystals and in
non-chiral Bose condensates.

Here $G$ is $SO(3)$, with Higgs $\phi$ transforming in the adjoint. $\phi$ develops the vev $\langle \phi \rangle = \text{diag} (1, 1, -2)$, breaking $SO(3)$ to the residual symmetry $H = O(2)$, containing $z$-rotations $R_z(\alpha)$ and the discrete symmetry element $h_o = R_x(\pi) = \text{diag} (1, -1, -1)$. Here $\pi_o(H) = Z_2$ and $\pi_1(H) = Z$ so we have topological strings and monopoles. The Alice string has Wilson line $U(\phi) = R_x(\phi/2)$ with $U(2\pi) = h_o$. $U(2\pi)$ fails to commute with unbroken symmetry generator $T_z$; in fact, on parallel transport around the string,

$$T_z \rightarrow U(2\pi) T_z U^{-1}(2\pi) = -T_z .$$

This Schwarz Alice string meets our topological criterion, of changing topological monopole charge on circumnavigation. By the exact sequence for $\pi_2(G/H)$, topological monopoles are associated with nontrivial loops in $O(2)$ which can be unwound in $SO(3)$. Since only even winding loops in $O(2)$ can be unwound in $SO(3)$, the fundamental monopole in this canonical Alice model has a loop in $O(2)$ of winding 2.

To apply our topological criterion, we represent the string by $h_o$, a nontrivial element of $\pi_o(H)$, and the fundamental monopole by $h(\alpha) = R_z(2\alpha)$, a winding 2 element of $\pi_1(H)$. This gives

$$\tilde{h}(\alpha) = h_o \ h(\alpha) \ h_o^{-1} = h^{-1}(\alpha) ,$$

from equation (4.1). Note that $h^{-1}(\alpha)$ has $O(2)$ winding -2, topologically distinct from $h(\alpha)$ of $O(2)$ winding 2. Thus $\tilde{h}(\alpha) \not\sim h(\alpha)$ and our topological criterion is met.

We now construct a monopole as a twisted Alice loop. From Eq. (3.1), the twisted Wilson line

$$U(\phi, \alpha) = h^{-1}(\alpha/2) \ U(\phi) \ h(\alpha/2) \ h_o^{-1}$$

gives an Alice loop with single-valued condensate. (We take $h(\alpha/2)$ because $h$ need only be single-valued in $\alpha$, and $h(\alpha/2)$, the winding 1 loop in $O(2)$, first achieves this.) $U(\phi, \alpha)$ interpolates between $h_o^{-1}$ at $\phi = 0$ and $h^{-1}(\alpha)$ at $\phi = 2\pi$. It is thus the fundamental antimonopole in the model, winding $-2$ in $O(2)$. The inverse twisted Alice loop, with Wilson line $U^{-1}(\phi, \alpha)$, creates the fundamental monopole.

As we show in the next section, twisted Alice loops in $^3\text{He-A}$ and amorphous chiral superconductors also support fundamental monopole charge. Again, this stems from the fact that monopoles must carry even $U(1)$ winding, as only even $U(1)$ loops can unwind inside
the $SO(3)$ factors endemic to $G$ for condensed matter systems. Single-valued half-twisted loops can thus arise, supporting fundamental monopole charge. Embedding $H = O(2)$ in a different $G$, however, can result in Alice loops unable to support fundamental monopole charge, as we see in section V B.

B. The Alice String of $^3$He-A

A more complicated global symmetry-breaking pattern describes the Alice string expected in $^3$He-A, and more recently predicted in amorphous chiral superconductors with p-wave pairing, such as Sr$_2$RuO$_4$. Here $G$ is $SO(3)_L \times SO(3)_S \times U(1)_N$, describing spatial rotations, spin rotations, and a $U(1)$ phase symmetry associated with number conservation of helium atoms. (The $U(1)$ symmetry is approximate, as is independence of spin and orbital rotations due to minimal spin-orbit coupling, but both describe $^3$He-A well.) The matrix order parameter $A$ transforms under symmetry transformations as $A \rightarrow e^{2i\theta} R_S A R_L^{-1}$, where $R_S$ and $R_L$ are spin and orbital rotations, respectively.

The order parameter develops the form

$$ A_{ij} = \Delta_A \hat{d}_i (\hat{m}_j + i \hat{n}_j) , $$

where $\hat{m}$ and $\hat{n}$ are perpendicular, determining $\hat{l} = \hat{m} \times \hat{n}$, the direction of the condensate’s angular momentum vector. This breaks $G$ to the residual symmetry $H = U(1)_{S_d} \times U(1)_{L_{i-N/2}} \times Z_2$, consisting of spin rotations about the $\hat{d}$ axis; spatial rotations about the $\hat{l}$ axis when compensated by a matching $U(1)_N$ phase rotation, and the discrete $Z_2$ transformation $h_o$, with $h_o : \hat{d} \rightarrow -\hat{d}, \quad \hat{m} + i \hat{n} \rightarrow - (\hat{m} + i \hat{n})$.

Identifying the defect topology requires care in this setting, as the exact sequences relating $\pi_2(G/H)$ and $\pi_1(G/H)$, the monopole and string homotopy groups, to $\pi_1(H)$ and $\pi_0(H)$ are highly nontrivial. Note that $\pi_2(G/H) = Z$, corresponding to the loops $\pi_1(U(1)_{S_d})$, which can be unwound in $G$. (Loops of the other $U(1)$ factor cannot be unwound in $G$, as they contain unshrinkable $U(1)_N$ loops.) $\pi_1(G/H) = Z_4$, which describes strings of two different origins. First, the Alice strings, called half-quantum vortices, have Wilson lines ending in a disconnected component of $H$, getting topological stability from $\pi_0(H)$. Second, a $Z_2$ winding one vortex, nontrivial in $SO(3)_L$ in $G$, induces as its image a $Z_2$ winding one vortex...
in $G/H$, with topological stability inherited from $\pi_1(G)$. These two classes of vortices are not independent: instead winding twice about a half-quantum vortex is equivalent to once around a winding one vortex, and the full string homotopy is $\pi_1(G/H) = Z_4$, or windings 0, $\pm 1/2$, and 1 modulo 2, with Alice strings corresponding to windings $\pm 1/2$.

The Volovik-Mineev Alice string, of winding $\pm 1/2$, has order parameter $A_{ij}$ with $\hat{d} = \hat{x}$ in spin space, and $\{\hat{l}, \hat{m}, \hat{n}\} = \{\hat{x}, \hat{y}, \hat{z}\}$ in ordinary space. This is acted on by Wilson line

$$U(\varphi) = e^{\pm i\varphi/2} R_{S_\hat{x}}(\varphi/2)$$

to give, asymptotically in $r$,

$$A_{ij}(\varphi) = \Delta A e^{\pm i\varphi/2} (\cos(\varphi/2) \hat{x}_j + \sin(\varphi/2) \hat{y}_j) S (\hat{x}_j + i\hat{y}_j)_L ,$$

single-valued in $\varphi$. Note that $U(2\pi) = -R_{S_\hat{x}}(\pi)$ lies in the same homotopy class as $h_o$. This string is Alice, making unbroken symmetry generator $T_{S_\hat{x}}$ double-valued. Physically, this means that a particle flips its spin, and hence its magnetization, on circumnavigating the Alice string.

This long-studied Alice string meets our topological criterion, of changing topological monopole charge upon circumnavigation. By the exact sequence for $\pi_2(G/H)$, topological monopoles are associated with nontrivial loops in $U(1)_{S_\hat{x}}$ which can be unwound in $SO(3)_S$. As in the nematic case, only even winding loops in $U(1)_{S_\hat{x}}$ can be unwound in $SO(3)_S$. Thus the fundamental monopole in $^3$He-A corresponds to a loop in $U(1)_{S_\hat{x}}$ of winding 2.

In applying our topological criterion, we choose $U(2\pi)$ as our representative of the string, a nontrivial element of $\pi_o(H)$, and $h(\alpha) = R_{S_\hat{x}}(2\alpha)$ as our representative of the fundamental monopole, a winding 2 element of $\pi_1(U(1)_{S_\hat{x}})$. This gives

$$\tilde{h}(\alpha) = h_o \ h(\alpha) \ h_o^{-1} = h^{-1}(\alpha) ,$$

since $T_{S_\hat{x}} \rightarrow -T_{S_\hat{x}}$. Note that $h^{-1}(\alpha)$ has $U(1)_{S_\hat{x}}$ winding -2, topologically distinct from $h(\alpha)$ of $U(1)_{S_\hat{x}}$ with winding 2. Thus $\tilde{h}(\alpha) \not\sim h(\alpha)$, meeting our topological criterion.

As in the nematic case, we construct a monopole as a twisted Alice loop. From Eq. (3.1),

the twisted Wilson line

$$U(\varphi, \alpha) = h^{-1}(\alpha/2) \ U(\varphi) \ h(\alpha/2) \ h_o^{-1}$$

generates an Alice loop with single-valued condensate. (Again $h(\alpha/2)$ appears, the winding 1 loop in $U(1)_{S_\hat{x}}$, as our minimal single-valued choice in constructing $U(\varphi, \alpha)$.) $U(\varphi, \alpha)$ interpolates between $h_o^{-1}$ at $\varphi = 0$ and $h^{-1}(\alpha)$ at $\varphi = 2\pi$. It is thus the fundamental
antimonopole in the model, which contains monopoles and antimonopoles of even winding in $U(1)_{S_4}$ only. Note that the inverse twisted Alice loop, with twisted Wilson line $U^{-1}(\varphi, \alpha)$, again generates the fundamental monopole.

V. PARTICLE PHYSICS ALICE STRINGS

A. A Nontopologically Alice string

Consider the nontopologically Alice string introduced in [2]: a Higgs $\phi$, transforming in the adjoint under $G = SO(6)$, acquires the vev $\langle \phi \rangle = \mathrm{diag}(1^3, -1^3)$. This condensate leaves unbroken an $SO(3) \times SO(3)$ subgroup of $SO(6)$ and a discrete $Z_2$ transformation $h_1 = -\mathbb{1}_6$, so $H = SO(3) \times SO(3) \times Z_2$. Here $\pi_o(H) = Z_2$ and $\pi_1(H) = Z_2 \times Z_2$, so topological strings and monopoles form, with monopoles and antimonopoles identified. Alice characteristics of the string depend on $U(2\pi)$. For $U(2\pi) = h_1$, all unbroken generators $T_{ij}$ are single-valued under parallel transport around the string, and the string is not Alice. However, for the topologically equivalent choice $U(2\pi) = \mathrm{diag}(1^2, (-1)^4) = -R_{12}(\pi)$, the string is algebraically Alice, making generators $T_{13}$ and $T_{23}$ of rotations in the 13- and 23-planes double-valued. Since this Alice behavior is removable by deforming to the topologically equivalent string with $U(2\pi) = h_1$, it must be nontopological. However, it is instructive to see how the two strings fail our topological criterion for Alice behavior. We take as our nontrivial monopole loop $h(\alpha) = R_{13}(\alpha)$, with monopole charge (1,0). We find, for the string with $h_o = h_1$, $\tilde{h}(\alpha) = h_o h(\alpha) h_o^{-1} = h(\alpha)$. That is, our topological criterion fails, as the monopole remains unchanged in circumnavigating the Alice string. Instead, for the Alice string with $h_o = -R_{12}(\pi)$, $\tilde{h}(\alpha) = h_o h(\alpha) h_o^{-1} = h^{-1}(\alpha)$, since $T_{13} \rightarrow -T_{13}$. Here the (1,0) monopole transforms into its antimonopole on traversing the string. However, that transformation is nontopological, as monopoles and antimonopoles are topologically equivalent. So, despite algebraic Alice behavior, this string also is not topologically Alice. Loops $h(\alpha)$ alter on string traversals, but in a topologically trivial way.

We might still hope to construct a (1,0) monopole as a twisted string loop, taking for our string the algebraic, but nontopologically Alice string $U(\varphi) = R_{34}(\varphi/2) R_{56}(\varphi/2)$, with algebraic Alice flux $U(2\pi) = h_o = -R_{12}(\pi)$ as above. From Eq. (3.1), the twisted Wilson
\[ U(\varphi, \alpha) = h^{-1}(\alpha) U(\varphi) h(\alpha) h_o^{-1} \]
generates an Alice loop with single-valued condensate. \( U(\varphi, \alpha) \) interpolates between \( h_o^{-1} \) at \( \varphi = 0 \) and \( h^{-2}(\alpha) \) at \( \varphi = 2\pi \). This is a loop in \( H \) of winding \((-2,0)\); however, winding \((-2,0)\) loops are deformable to the identity in \( H \), so this fully twisted nontopologically Alice loop fails to carry topological monopole charge.

Recall that, in building singlevalued twisted Alice loops, we required \( U(\varphi, \alpha) \) to be singlevalued in \( \alpha \); we thus identified \( h(\alpha) \) as a loop in \( H \). Strictly, we do not need \( h(\alpha) \) to be a loop; all we need is
\[ h^{-1}(2\pi) \ U(\varphi) \ h(2\pi) = U(\varphi) . \]
We might still hope to build the fundamental \((1,0)\) monopole as a twisted Alice loop, exploiting this freedom in \( h(\alpha) \). Were the twisted loop
\[ U(\varphi, \alpha) = h^{-1}(\alpha/2) U(\varphi) h(\alpha/2) h_o^{-1} \]
single-valued, it would carry fundamental \((-1,0)\) monopole charge, as it interpolates between \( h_o^{-1} \) at \( \varphi = 0 \) and the nontrivial \((-1,0)\) antimonopole \( h^{-1}(\alpha) \) at \( \varphi = 2\pi \). However, this twisted loop candidate is not single-valued; it obeys instead \( h^{-1}(2\pi) \ U(\varphi) \ h(2\pi) = U^{-1}(\varphi) \).

We thus cannot build a fundamental \((1,0)\) monopole as a twisted Alice loop in this model, where Alice behavior is nontopological and monopole charge is \( Z_2 \times Z_2 \).

This possibility to construct \( U(\varphi, \alpha) \) single-valued in \( \alpha \), without forcing \( h(\alpha) \) to be a loop, always merits investigating. Indeed, in [6, 12], one of us constructed what is essentially the fundamental monopole in this model, by exploiting exactly such an accidental algebraic singlevaluedness. That construction (most clearly in section IIIA of [6], taking \( F(r) \), \( \varphi \) as the spherical coordinates \( \theta, \varphi \) at spatial infinity),is quite similar to the twisting constructions here. However, it describes a fundamentally point-like defect, and cannot be interpreted as a twisted loop.

**B. A Topologically Alice Loop Carrying only Deposited Monopole Charge**

We consider a slightly modified canonical Alice string. Take \( G \) to be \( SU(3) \), with Higgs \( \phi \) transforming according to \( \phi \rightarrow g \phi g^T \) (giving fermions in this model a Majorana mass). When \( \phi \) develops the vev \( \langle \phi \rangle = \text{diag}(1,1,-2) \), \( SU(3) \) breaks to the residual symmetry
$H = O(2)$, identical to that of the canonical Schwarz Alice string. Again we have $\pi_o(H) = Z_2$ and $\pi_1(H) = Z$, with topological strings and monopoles. We have the same Alice string as in the canonical case, making the $O(2)$ generator $T_z$ double-valued. This Alice behavior is again topological, as our topological criterion, that $\pi_o(H)$ acts nontrivially on $\pi_1(H)$, depends only on the unbroken symmetry group $H$.

Where we deviate from the canonical Alice string model is in the identification of twisted Alice loops as monopoles. Here, by the exact sequence for $\pi_2(G/H)$, topological monopoles are associated with nontrivial loops in $O(2)$ which can be unwound in $G$, here $SU(3)$. All nontrivial loops in $O(2)$ can be unwound in $SU(3)$; thus the fundamental monopole in this model has a loop in $O(2)$ of winding 1.

We now construct a monopole as a twisted Alice loop. We take $U(\varphi) = R_x(\varphi/2)$, as in the canonical Alice case, and $h(\alpha) = R_z(\alpha)$, a loop of winding 1. From Eq. (3.1), the twisted Wilson line

$$U(\varphi, \alpha) = h^{-1}(\alpha) \ U(\varphi) \ h(\alpha) \ h_o^{-1}$$

generates a twisted Alice loop with single-valued condensate. $U(\varphi, \alpha)$ interpolates between $h_o^{-1}$ at $\varphi = 0$ and $h^{-2}(\alpha)$ at $\varphi = 2\pi$. This twisted Alice loop carries monopole charge of $-2$, which while nontrivial is not the fundamental antimonopole in this model. (Similarly, the inverse twisted Alice loop, with Wilson line $U^{-1}(\varphi, \alpha)$, carries monopole charge $+2$).

Again, we might still hope to build a fundamental monopole as a twisted Alice loop, by allowing $h(\alpha)$ above to be not a loop, but a curve obeying Eq. (5.1). This looser constraint still guarantees singlevaluedness in $\alpha$ of $U(\varphi, \alpha)$. Indeed, were the half-twisted loop

$$U(\varphi, \alpha) = h^{-1}(\alpha/2) \ U(\varphi) \ h(\alpha/2) \ h_o^{-1}$$

single-valued in $\alpha$, with $h(\alpha) = R_z(\alpha)$ as above, it would carry fundamental antimonopole charge. This is because it interpolates between $h_o^{-1}$ at $\varphi = 0$ and the winding $-1$ loop $h^{-1}(\alpha)$ at $\varphi = 2\pi$. However, this half-twisted loop candidate is not single-valued in $\alpha$; it obeys instead $U(\varphi, 2\pi) = R_z^{-1}(\varphi) \ U(\varphi, 0)$. We thus cannot build a fundamental monopole as a twisted Alice loop in this model. Instead twisted Alice loops carry only the monopole charge which topological arguments ensure they must carry: because monopoles scatter into antimonopoles on transiting Alice loops, Alice loops must support deposited monopole charge, which arises in units of 2. Our full twisting construction creates twisted Alice loops supporting exactly that deposited charge.
VI. CONCLUSIONS

We have established a topological criterion for strings to display Alice behavior. This criterion, that $\pi_o(H)$ acts nontrivially on $\pi_1(H)$, depends only on the residual symmetry group $H$. Alice strings must form in models obeying this criterion, while Alice behavior can be deformed away for strings failing the criterion. Particularly, the criterion requires that topological monopoles always accompany topologically Alice strings; and furthermore, that topologically Alice strings alter the topological charge of monopoles that circumnavigate them. We construct monopoles as twisted loops of Alice string, and show that such twisted loops can always support deposited monopole charge. Whether twisted Alice loops can support fundamental monopole charge depends on the symmetry-breaking pattern more closely, as we examined in condensed matter and illustrative particle physics models. Specifically, it depends on the initial symmetry group $G$, through the identification of loops in $H$ with monopoles via the exact sequence for $\pi_2(G/H)$. When fundamental monopoles correspond to nonminimal-winding loops in $H$, single-valued half-twisted Alice loops may arise, carrying fundamental monopole charge. This occurs generically for condensed matter topologically Alice strings, since only winding 2 loops in $H$ unwind in the $SO(3)$ initial symmetry groups associated with angular momenta. When fundamental monopoles, instead, have minimal winding in $H$, only algebraic accident allows a half-twisted Alice loop to be single-valued. Generically, in such minimal embeddings, twisted Alice loops can support only deposited, not fundamental, monopole charge.

Acknowledgments

Early stages of this work were supported by NSF grant PHY-9631182 and by the University Research Committee of Emory University. KB thanks the KITP (under NSF grant PHY99-07949) for hospitality during the writing of an early version of this paper. The work of T.I. was supported in part by the U.S. Department of Energy.
\[ U(\alpha, \varphi) = h^{-1}(\alpha) U(\varphi) h(\alpha) \]
\[ \alpha = 0 \quad U(\varphi) \]

FIG. 1: a) A twisted Alice string. b) Identifying twisted Alice string ends to form a twisted Alice loop.


