Conformal Null Infinity Does Not Exist for Radiating Solutions in Odd Spacetime Dimensions

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Abstract

We show that for general relativity in odd spacetime dimensions greater than 4, all components of the unphysical Weyl tensor for arbitrary smooth, compact spatial support perturbations of Minkowski spacetime fail to be smooth at null infinity at leading nonvanishing order. This implies that for nearly flat radiating spacetimes, the non-smoothness of the unphysical metric at null infinity manifests itself at the same order as it describes deviations from flatness of the physical metric. Therefore, in odd spacetime dimensions, it does not appear that conformal null infinity can be in any way useful for describing radiation.

The notion of conformal null infinity was introduced more than forty years ago by Penrose [1] in the context of 4-dimensional general relativity, and has provided a remarkably fruitful framework for giving a mathematically precise description of the asymptotic properties of gravitational radiation. In this framework, one defines an “unphysical spacetime” with a metric that is conformally related to the physical metric, in such a way that “asymptotically large distances in null directions” correspond to ordinary points on a boundary (“null infinity”) attached to the unphysical spacetime. One can then define notions such as Bondi mass and energy flux via local definitions on this boundary rather than by taking asymptotic limits in the original, physical spacetime. The appropriate degree of differentiability to assume for the unphysical metric has been been a subject of much discussion and analysis (see [2] and references cited therein). However, in 4-dimensional general relativity it is known that there is a wide class of radiating spacetimes for which the unphysical metric is smooth [3].

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The notion of conformal null infinity and the definitions of Bondi mass and energy flux have recently been investigated by Hollands and Ishibashi [4] in the context of general relativity in spacetimes of dimension greater than 4. It was found in [4] that for even dimensional spacetimes, the notion of asymptotic flatness can be generalized in such a way that notions of Bondi mass and energy flux can be defined. It also was found that this definition of asymptotic flatness is stable to small perturbations in the sense that linearized perturbations with smooth initial data of compact support off of an asymptotically flat solution satisfy the linearization of the asymptotic flatness criteria. (This property had previously been shown to hold in 4-dimensions by Geroch and Xanthopoulos [5].) However, Hollands and Ishibashi also noted that their proposed definition of asymptotic flatness would not work in odd dimensions, since, in their proposal, in odd dimensions the leading order deviation of the unphysical metric from a fixed smooth background metric would be proportional to a half-integral power of the conformal factor $\Omega$. Consequently, the unphysical metric would not have sufficient smoothness to perform any meaningful constructions at null infinity.

The main purpose of this paper is to show that these difficulties are not merely artifacts of the particular proposed definition in [4]. In odd spacetime dimensions, there is no difficulty in conformally mapping Minkowski spacetime into a portion of the Einstein static universe (see below). (Similarly, there is no difficulty in defining a suitable conformal null infinity structure in other stationary spacetimes\(^1\), such as odd dimensional Schwarzschild spacetime). However, severe difficulties with conformal null infinity in odd spacetime dimensions arise when one considers spacetimes containing radiation\(^2\), and these difficulties manifest themselves already at the level of linearized perturbations off of Minkowski spacetime. In this paper, we will show that for perturbations of odd dimensional Minkowski spacetime with initial data that is smooth and of compact support, the fall-off of the physical Weyl tensor is such that the behavior of the unphysical Weyl tensor at null infinity must begin with a half-integral power of $\Omega$. This shows not only that the perturbed unphysical metric cannot be smooth at null infinity, but furthermore that its failure to be smooth manifests itself at the same order as it describes deviations of the physical metric from flatness. It therefore does not seem possible that useful constructions to describe gravitational radiation can be given within a conformal null infinity framework in odd dimensional spacetimes.

To begin, consider $d$-dimensional Minkowski spacetime $(M \equiv \mathbb{R}^d, \eta_{ab})$ for any (even or

\(^1\)It should be noted that in 4 spacetime dimensions, the deviation of the physical metric from the Minkowski metric will fall off at null infinity as $1/r$ for both stationary and radiating solutions. However, in both even and odd dimensional spacetimes with $d > 4$ the deviation of a stationary metric from the Minkowski metric will have a faster fall-off rate than that of a generic asymptotically flat radiating solution [4].

\(^2\)The fact that Huygens’ principle fails for the wave equation in odd dimensional Minkowski spacetime suggests the possibility that the failure of Huygens’ principle might be responsible for the difficulties with the conformal null infinity framework. However, our analysis does not reveal any obvious direct connection between the failure of Huygens’ principle and the failure of the conformal null infinity framework.
odd) $d > 3$, where $\eta_{ab}$ denotes the flat metric

$$ ds^2 = -dt^2 + dx_1^2 + \cdots + dx_{d-1}^2, $$

$$ ds^2 = -dt^2 + dr^2 + r^2 d\omega_{d-2}^2 , \quad (1) $$

where $d\omega_{d-2}^2$ denotes the metric on the unit, round $(d-2)$-dimensional sphere. The $d$-dimensional Einstein static universe is the manifold $E \equiv \mathbb{R} \times S^{d-1}$ with metric $\tilde{g}_{ab}$ given by

$$ d\tilde{s}^2 = -dT^2 + dR^2 + \sin^2 R d\omega_{d-2}^2 . \quad (2) $$

As in 4-dimensions (see, e.g., [6]), we can conformally map $d$-dimensional Minkowski spacetime into a portion of the $d$-dimensional Einstein static universe as follows: We define $u = t - r$, $v = t + r$, and we define the map $\psi : M \to E$ so as to preserve the $(d-2)$-sphere angular coordinates and take the point labeled by $(u, v)$ into the point labeled by

$$ T = \tan^{-1} v + \tan^{-1} u, \quad (3) \quad R = \tan^{-1} v - \tan^{-1} u. \quad (4) $$

Then it can be verified that

$$ \tilde{g}_{ab} = \Omega^2 \psi^* \eta_{ab} , \quad (5) $$

where

$$ \Omega^2 = 4(1 + v^2)^{-1}(1 + u^2)^{-1} . \quad (6) $$

Note that this construction works equally well for $d$ odd or even.

The null portion of the boundary of Minkowski spacetime in the Einstein static universe—namely, the null surfaces $\mathcal{I}^+$ and $\mathcal{I}^-$, defined, respectively, by $T = \pi - R$ and $T = R - \pi$, with $0 < R < \pi$—provides the prototype for the construction of conformal null infinity for asymptotically flat spacetimes. However, in order for this construction to be useful for describing spacetimes where gravitational radiation is present, it is necessary for this construction to continue to work if we perturb Minkowski spacetime to another solution of the vacuum Einstein equation $R_{ab} = 0$. This leads to the following requirement at the level of linear perturbation theory: Let $\gamma_{ab}$ be a solution of the linearized Einstein equation with suitably regular initial data (say, smooth and of compact support). Then, modulo a gauge transformation of $\gamma_{ab}$ and a perturbation, $\delta \Omega$, of the conformal factor, we require that the perturbed unphysical metric extend to conformal null infinity so as to be sufficiently differentiable there. Here, by “sufficiently differentiable”, we mean (at the very least) that some non-Minkowskian asymptotic properties of the physical metric $\eta_{ab} + \gamma_{ab}$ can be expressed in terms of the curvature of the unphysical metric (or derivatives of its curvature) evaluated at conformal null infinity. Thus, given $\gamma_{ab}$, we seek a vector field $\xi^a$ on Minkowski spacetime and a smooth function $\delta \Omega$ on the Einstein static universe (with $\delta \Omega = 0$ at conformal null infinity) such that the perturbed unphysical metric

$$ \tilde{\gamma}_{ab} = \Omega^2 \psi^*(\gamma_{ab} + \partial_a \xi_b + \partial_b \xi_a) + 2\Omega \delta \Omega \psi^* \eta_{ab} \quad (7) $$

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extends to conformal null infinity so as to be sufficiently differentiable there. (Here $\partial_a$ denotes the derivative operator associated with $\eta_{ab}$ and indices are raised and lowered with $\eta_{ab}$.) If the spacetime dimension $d$ is even, it has been shown \[5\], \[4\] that—by a suitable choice of $\xi^a$ and $\delta\Omega$—the perturbed unphysical metric $\tilde{\gamma}_{ab}$ can always be made to be smooth. Indeed, as already mentioned above, smoothness holds not only for perturbations of Minkowski spacetime but for perturbations of any asymptotically flat vacuum solution. However, we now shall show that this is not the case when $d$ is odd.

Rather than analyze the behavior of linearized perturbations in terms of the metric perturbation $\gamma_{ab}$, it is much more convenient to work directly with the perturbed Weyl tensor, $\delta C_{abcd}$. This has the dual advantage that $\delta C_{abcd}$ is both gauge invariant and conformally invariant. Consequently, by working with $\delta C_{abcd}$ we will be able to immediately determine the behavior of the unphysical Weyl tensor $\tilde{\delta} C_{abcd}$ near conformal null infinity. We will show that in odd dimensions, the leading order behavior of $\tilde{\delta} C_{abcd}$ must always begin with a half-integral power of $\Omega$, and thus it is non-smooth to the same extent as it is nonvanishing.

First, consider any spacetime that satisfies the vacuum Einstein equation $R_{ab} = 0$. The uncontracted Bianchi identity immediately yields

$$\nabla_a C_{bcde} = 0.$$  \hspace{1cm} (8)

Contracting over $a$ and $d$ and using the tracelessness of the Weyl tensor, we obtain

$$\nabla^a C_{bcde} = 0.$$  \hspace{1cm} (9)

Applying $\nabla^a$ to eq.(8), we obtain

$$0 = \nabla^a \nabla_{[a} C_{bc]de} = \frac{1}{3} \nabla^a \nabla_a C_{bcde} + \frac{1}{3} \nabla^a \nabla_b C_{cade} + \frac{1}{3} \nabla^a \nabla_c C_{abde}.$$  \hspace{1cm} (10)

But, we also have

$$\nabla^a \nabla_b C_{cade} = \nabla_a C_{cade} + C^a_{\ bc} C_{fade} + C^a_{\ ba} C_{cfde} + C^a_{\ bd} C_{cafe} + C^a_{\ be} C_{cadf}$$

$$= C^a_{\ bc} f C_{fade} + C^a_{\ bd} f C_{cafe} + C^a_{\ be} f C_{cadf},$$ \hspace{1cm} (11)

and similarly for the last term in eq.(10). Consequently, we obtain

$$\nabla^a \nabla_a C_{bcde} = \text{terms quadratic in } C.$$  \hspace{1cm} (12)

It follows immediately from the above formulas that for any linearized perturbation $\gamma_{ab}$ off of Minkowski spacetime that satisfies the linearized Einstein equation $\delta R_{ab} = 0$, the linearized Weyl tensor $\delta C_{abcd}$ satisfies

$$\nabla^a \delta C_{abcd} = 0.$$  \hspace{1cm} (13)

and

$$\nabla^a \nabla_a \delta C_{abcd} = 0.$$  \hspace{1cm} (14)
For each \( \mu = 0, 1, \ldots, d - 1 \) let \( K_\mu^a \) denote the translational Killing field in Minkowski spacetime associated with the \( \mu \)th coordinate basis field of the global inertial coordinates \((t = x^0, x^1, \ldots, x^{d-1})\) of eq. (11), i.e., let \( K_\mu^a = (\partial / \partial x^\mu)^a \). Then we have \( \partial_a K_\mu^b = 0 \). Let \( W^a \) denote the dilational conformal Killing field of Minkowski spacetime whose components in the global inertial coordinates are given by \( W^\mu = x^\mu \). Then we have

\[
\partial_a W^b = \delta_a^b .
\] (15)

For \( i = 1, 2, 3, 4 \), let

\[
V_i^a = \sum_\mu a_i^\mu K_\mu^a + b_i W^a
\] (16)

where the \( a_i^\mu \) and \( b_i \) are arbitrary constants. Let

\[
\phi = \delta C_{abcd} V_1^a V_2^b V_3^c V_4^d .
\] (17)

Then, we have

\[
\partial_c \phi = (\partial_c \delta C_{abcd}) V_1^a V_2^b V_3^c V_4^d + b_1 \delta C_{ebcd} V_2^b V_3^c V_4^d + b_2 \delta C_{acde} V_1^a V_3^c V_4^d + b_3 \delta C_{abcd} V_1^a V_2^b V_4^d + b_4 \delta C_{abc} V_1^a V_2^b V_3^c .
\] (18)

Applying \( \partial^e \) to this equation and using eqs. (13), (14), (15), and the tracelessness of the Weyl tensor, we obtain

\[
\partial^e \partial_c \phi = 0 .
\] (19)

Consequently, on the Einstein static universe, the quantity

\[
\tilde{\phi} \equiv \Omega^{1-d/2} \phi
\] (20)

satisfies the conformally invariant wave equation

\[
\tilde{\nabla}^e \tilde{\nabla}_e \tilde{\phi} - \frac{d - 2}{4(d - 1)} \tilde{R} \tilde{\phi} = 0 ,
\] (21)

(see, e.g., appendix D of [6]) where \( \tilde{R} \) denotes the scalar curvature of the Einstein static universe and, in this equation, indices are raised and lowered with the unphysical metric, \( \tilde{g}_{ab} \) (see eq. (2)). It then follows immediately from the well posedness of the initial value formulation of this equation on the Einstein static universe that \( \tilde{\phi} \) extends smoothly to conformal null infinity and cannot vanish everywhere on \( I^+ \) or on \( I^- \) unless \( \phi \) itself vanishes identically. Since we have

\[
\delta \tilde{C}_{abcd} = \Omega^2 \delta C_{abcd}
\] (22)

(where \( \delta \tilde{C}_{abcd} \) denotes the perturbed unphysical Weyl tensor with its index lowered by \( \tilde{g}_{ab} \)) we have learned that for any perturbation of Minkowski spacetime with smooth initial data of compact support, and for any choice of the constants \( a_i^\mu \) and \( b_i \) (see eq.(16) above), the quantity

\[
\chi \equiv \Omega^{-1-d/2} \delta \tilde{C}_{abcd} V_1^a V_2^b V_3^c V_4^d
\] (23)

is a conformally invariant wave equation.
is smooth everywhere in the Einstein static universe, and it cannot vanish everywhere on \( I^+ \) (or on \( I^- \)) unless \( \chi \) vanishes identically.

By direct calculation, it can be verified that in the Einstein static universe, we have

\[
K^a_\mu = \alpha_\mu n^a - \Omega \tilde{\nabla}^a \alpha_\mu. \tag{24}
\]

Here \( n^a \equiv \tilde{\nabla}^a \Omega \), each \( \alpha_\mu \) is smooth, and again indices are raised and lowered with \( \tilde{g}_{ab} \). Since \( \alpha_0 \) is nonvanishing in a neighborhood of \( I^+ \), we may substitute for \( n^a \) in terms of \( t^a = K^a_0 \) and rewrite this equation as

\[
K^a_\mu = \beta_\mu t^a - \alpha_0 \Omega \tilde{\nabla}^a \beta_\mu, \tag{25}
\]

where \( \beta_\mu = \alpha_\mu / \alpha_0 \), so, in particular, we have \( \beta_0 = 1 \). For \( \mu = 1, \ldots, d - 1 \) we define \( s^a_\mu = \tilde{\nabla}^a \beta_\mu \). Then it is not difficult to verify that on \( I^+ \) the vector fields \( t^a, s^a_1, \ldots, s^a_{d-1} \) span the subspace of the tangent space that is tangent to \( I^+ \).

Similarly, we obtain

\[
W^a = ut^a + \Omega l^a, \tag{26}
\]

where \( u = t - r \) and \( l^a \) is a smooth null vector field that is nonvanishing at \( I^+ \) but is not normal to \( I^+ \). Since \( l^a \) is transverse to \( I^+ \), it follows immediately that the vector fields \( t^a, l^a, s^a_1, \ldots, s^a_{d-1} \) span the full tangent space at each point of a neighborhood of \( I^+ \).

Substituting eqs. (25) and (26) into eq. (16) to get the form of \( V_i^a \) and substituting this result into eq. (23), we see that \( \chi \) takes the form

\[
\chi = \Omega^{-1-d/2} [\Omega^2 F_1 + \Omega^3 F_2 + \Omega^4 F_3]. \tag{27}
\]

Here \( F_1 \) denotes a linear combination of contractions of \( \delta \tilde{C}_{abcd} \) with the vectors \( t^a, l^a, s^a_\mu \) that include precisely two \( t^a \)'s, \( F_2 \) denotes a linear combination of these contractions that include precisely one \( t^a \), and \( F_3 \) denotes a linear combination of these contractions that does not include any \( t^a \). The precise linear combinations occurring in \( F_1, F_2, \) and \( F_3 \) depend, of course, upon the choice of \( V_i^a \). By choosing \( V_1^a = V_3^a = t^a \) and taking each of \( V_2^a \) and \( V_4^a \) to be either \( W^a \) or a \( K^a_\mu \), we obtain \( F_2 = F_3 = 0 \) and we can make \( F_1 \) be any contraction of \( \delta \tilde{C}_{abcd} t^a t^c \) with \( l^a \)'s and \( s^a_\mu \)'s. This proves that the quantity

\[
\Omega^{1-d/2} \delta \tilde{C}_{abcd} t^a t^c \tag{28}
\]

must be smooth at \( I^+ \). Given this result, we now may take \( V_1^a = t^a \) and choose \( V_2^a, V_3^a, V_4^a \) to be any combinations of \( W^a \) and \( K^a_\mu \) to conclude that

\[
\Omega^{2-d/2} \delta \tilde{C}_{abcd} \tag{29}
\]

must be smooth at \( I^+ \). Finally, given that expressions (28) and (29) are smooth at \( I^+ \), if we then choose \( V_1^a, V_2^a, V_3^a, V_4^a \) to be any combinations of \( W^a \) and \( K^a_\mu \) we find that

\[
\Omega^{3-d/2} \delta \tilde{C}_{abcd} \tag{30}
\]

This set of \( d \) vectors is “overcomplete”; the vectors \( s^a_1, \ldots, s^a_{d-1} \) are not linearly independent at \( I^+ \).
must be smooth at $I^+$. 

Since $t^a = \alpha_0 n^a - \Omega \tilde{\nabla}^a \alpha_0$, we may restate the above result as follows: Let $\gamma_{ab}$ be an arbitrary solution to the linearized Einstein equation (with smooth initial data of compact support) off of $d$-dimensional Minkowski spacetime for any $d > 3$. Then the corresponding perturbed unphysical Weyl tensor $\delta \tilde{C}_{abcd}$ in the Einstein static universe is such that the three quantities

$$
\Omega^{3-d/2} \delta \tilde{C}_{abcd}, \quad \Omega^{2-d/2} \delta \tilde{C}_{abcd} n^a, \quad \Omega^{1-d/2} \delta \tilde{C}_{abcd} n^a n^c
$$

(31)

are all smooth at $I^+$. Furthermore, since $\chi$ cannot vanish identically for all $\delta_i^\mu$ and $b_i$ (unless $\gamma_{ab}$ is pure gauge), it follows that at least one of the above quantities must be nonvanishing somewhere on $I^+$.

In $d = 4$ dimensions, it turns out that $\delta \tilde{C}_{abcd}$ vanishes at $I^+$, so, in particular, the first two terms in eq. (31) vanish. The fact that, generically, in 4 dimensions all components of the perturbed unphysical Weyl tensor are $O(\Omega)$ at $I^+$ gives rise to the familiar asymptotic “peeling properties” of the Weyl tensor in the physical spacetime. However, the arguments leading to the vanishing of the first two terms in eq. (31) (see [7]) appear to be very special to the 4-dimensional case. We see no reason why these terms should vanish when $d > 4$. If so, the “peeling behavior” of the physical Weyl tensor for $d > 4$ would be qualitatively different from the 4-dimensional case.

In any case, our above results establish that, $\Omega^{3-d/2} \delta \tilde{C}_{abcd}$ is smooth at $I^+$, and $\Omega^{1-d/2} \delta \tilde{C}_{abcd}$ cannot vanish at $I^+$. This proves that in odd-dimensional spacetimes, the leading order behavior of the perturbed unphysical Weyl tensor always begins with a half-integral power of $\Omega$. Consequently, the unphysical metric fails to be smooth at $I^+$ (and $I^-$) to the same degree as it describes deviations of the physical metric from Minkowski spacetime.

It should be emphasized that our analysis does not show that it is impossible to define a suitable notion of asymptotic flatness in odd dimensions that admits a notion of Bondi mass and energy flux. Indeed, it seems entirely possible that such a definition of asymptotic flatness could be formulated in terms of an asymptotic expansion of the physical metric in (integral and half-integral) powers of $1/r$, analogous to the definition of asymptotic flatness in 4-dimensions originally given by Bondi et al [3]. However, our analysis clearly indicates that it will not be fruitful to seek a definition of asymptotic flatness within the conformal null infinity framework for odd dimensional spacetimes.

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References


