One-loop effective action of QCD at high temperature using the heat kernel method

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Abstract. Perturbation theory is an important tool to describe the properties of QCD at very high temperatures. Recently a new technique has been proposed to compute the one-loop effective action of QCD at finite temperature by making a gauge covariant derivative expansion, which is fully consistent with topologically small and large gauge transformations (also time dependent transformations) [1]. This technique is based on the heat kernel expansion, and the thermal Wilson line plays an essential role [2]. We consider a general SU($N_c$) gauge group.

Introduction. The effective action (EA) plays a prominent role in quantum field theory, since it embodies the renormalized properties of the system. To one loop it takes the form $c \text{Tr log}(K)$, where $K$ is the differential operator controlling the quadratic quantum fluctuations above a classical background. This quantity is afflicted by ultraviolet divergences, and it is useful to express it by means of a proper time representation [3],

$$-\text{Tr log}(K) = \int_0^\infty \frac{d\tau}{\tau} \text{Tr} e^{-\tau K} = \int_0^\infty \frac{d\tau}{\tau} \int d^Dx \text{tr} \langle x| e^{-\tau K} |x \rangle. \quad (1)$$

"tr" refers to trace in internal spaces. The (diagonal) heat kernel $\langle x| e^{-\tau K} |x \rangle$ is UV finite.

The heat kernel at finite temperature has been computed for a Klein-Gordon operator, through the so called heat kernel expansion, in the completely general case of non Abelian and non stationary gauge fields and external fields [1, 2]. It is of the form

$$\langle x| e^{-\tau(M-D_\mu^2)} |x \rangle = (4\pi \tau)^{-D/2} \sum_n a_n^T(x) \tau^n. \quad (2)$$

It is an expansion in local and gauge covariant operators. The so called Seeley-DeWitt coefficients, $a_n^T$, are contracted with operators of mass dimension $2n$. The untraced Polyakov loop or thermal Wilson line plays a fundamental role in maintaining manifest gauge invariance at each order. It appears inside $a_n^T$, and it is defined as

$$\Omega(x) = T \exp \left( ig \int_{x_0}^{x_0+\beta} A_0(x_0',\mathbf{x}) dx_0' \right). \quad (3)$$

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2 We consider Klein-Gordon operators of the form, $K = M(x) - D_\mu^2$, where $D_\mu = \partial_\mu - igA_\mu(x)$ is the covariant derivative, $M(x)$ is a scalar (external) field, and $A_\mu(x)$ are the gauge fields.
Effective action of QCD. In Ref. [1] the heat kernel has been applied to obtain the 1-loop EA of QCD at high temperature. The Euclidean partition function of QCD is

\[ Z[A, \bar{q}, q] = \int \mathcal{D}A \prod_{\alpha=1}^{N_f} \mathcal{D}q_\alpha \mathcal{D}\bar{q}_\alpha \exp \left\{ -\frac{1}{2} \int d^4x \text{tr}(F_{\mu\nu}^2) - \int d^4x \sum_{\alpha=1}^{N_f} \bar{q}_\alpha i \not{D} q_\alpha \right\}, \quad (4) \]

where \( F_{\mu\nu} = ig^{-1}[D_{\mu}, D_{\nu}] \) is the field strength. The gauge group of color is \( SU(N_c) \).

The full EA can be obtained upon functional integration of the quark and gluon fields. The quark contribution to the EA at 1-loop is (convention \( Z = e^{-\Gamma[A]} \))

\[ \Gamma_q[A] = -\frac{N_f}{2} \text{Tr} \log(-\not{D}^2) = :\int d^4x \mathcal{L}_q(x). \quad (5) \]

For the gluon fields we use the background field method, in which the gauge field is split into a classical background field plus a quantum fluctuation, i.e. \( A_\mu \rightarrow A_\mu + a_\mu \), in [1]. The standard procedure consists of adding a gauge fixing term and the corresponding Fadded-Popov term in the action. The gluon contribution is then

\[ \Gamma_g[A] = \frac{1}{2} \text{Tr} \log \left( -\delta_{\mu\nu} \hat{D}_\lambda^2 + 2ig\hat{F}_{\mu\nu} \right) - \text{Tr} \log \left( -\hat{D}_\mu^2 \right) = :\int d^4x \mathcal{L}_g(x). \quad (6) \]

The operators in (5) and (6) act in the fundamental and adjoint representations, respectively. They are of the Klein-Gordon form, so we can use the proper time representation. For the different orders of the heat kernel expansion, we obtain

\[ \mathcal{L}_0(x) = \frac{\pi^2 T^4}{3} \left( 1 + \frac{2N_c N_f - 2N_f^2}{15} \right) - \frac{N_f}{4} \text{tr} \left[ (1 - 4\nabla^2)^2 \right] + 2\text{tr} \left[ \hat{\nabla}^2 (1 - \hat{\nabla})^2 \right], \quad (7) \]

\[ \mathcal{L}_2(x) = \left( \frac{1}{2} - g^2 \beta_0 \log(\mu/4\pi T) - \frac{g^2 N_c}{6(4\pi)^2} \right) \text{tr}(F_{\mu\nu}^2) + \frac{11g^4}{12(4\pi)^2} \text{tr} \left[ \left( \psi(\hat{\nabla}) + \psi(1 - \hat{\nabla}) \right) \hat{F}_{\mu\nu}^2 \right] + \frac{g^2 N_f}{3(4\pi)^2} \text{tr} \left[ \left( \psi \left( \frac{1}{2} + \hat{\nabla} \right) + \psi \left( \frac{1}{2} - \hat{\nabla} \right) \right) F_{\mu\nu}^2 \right], \quad (8) \]

\[ \mathcal{L}_3(x) = -\frac{2g^2}{(4\pi)^4} \frac{N_f}{T^2} \text{tr} \left[ \left( \psi''(\frac{1}{2} + \hat{\nabla}) + \psi''(\frac{1}{2} - \hat{\nabla}) \right) \left( \frac{1}{60} [D_\mu, F_{\mu\nu}]^2 - \frac{1}{24} [D_\lambda, F_{\mu\nu}]^2 \right) + \frac{1}{45} igF_{\mu\nu} F_{\nu\lambda} F_{\lambda\mu} - \frac{1}{20} [D_0, F_{\mu\nu}]^2 + \frac{1}{30} [D_i, E_i]^2 + \frac{1}{15} igE_i F_{ij} E_j \right] \]

\[ + \frac{g}{2(4\pi)^4} \frac{1}{T^2} \text{tr} \left[ \left( \psi''(\hat{\nabla}) + \psi''(1 - \hat{\nabla}) \right) \left( \frac{1}{30} [\hat{D}_\mu, \hat{F}_{\mu\nu}]^2 - \frac{1}{3} [\hat{D}_\lambda, \hat{F}_{\mu\nu}]^2 \right) + \frac{61}{45} ig\hat{F}_{\mu\nu} \hat{F}_{\nu\lambda} \hat{F}_{\lambda\mu} - \frac{3}{5} [\hat{D}_0, \hat{F}_{\mu\nu}]^2 + \frac{1}{15} [\hat{D}_i, \hat{E}_i]^2 + \frac{2}{15} ig\hat{E}_i \hat{F}_{ij} \hat{E}_j \right], \quad (9) \]

3 The integrals over \( \tau \) are one-valued functions of the Polyakov loop, so gauge invariance is manifest.
where $-\frac{1}{2} < \bar{\nu} < \frac{1}{2}$ and $0 < \tilde{\nu} < 1$. $\psi(q)$ is the digamma function. $E_i = F_{0i}$ is the electric field. $g$ is the running coupling defined in the $\overline{\text{MS}}$ scheme. $\beta_0 = (11N_c - 2N_f)/(3(4\pi)^2)$.

A new technique has been proposed recently for 1-loop QCD at high temperature \cite{Diakonov:2002}. It goes beyond ours in that all orders in $A$ in \cite{Schwinger:1968} for the gluon sector, and in \cite{Megias:2003} for the quark sector and for SU(3) in the absence of chromomagnetic field ($A_i = 0$). Our results agree with these.

**Dimensional reduced effective theory.** In the high temperature limit non stationary fluctuations become heavy and are therefore suppressed, and QCD behaves as an effective three-dimensional theory for the stationary configurations only. This effective theory is obtained by i) using stationary backgrounds and ii) taking purely non-stationary fluctuations only, that is, removing the static Matsubara mode. Doing this, one obtains\footnote{Dimensional reduced effective theory.}

$$
\mathcal{L}_0^i(x) = g^2 \left(\frac{N_c}{3} + \frac{N_f}{6}\right) T\langle A_0^2 \rangle + \frac{g^4}{4\pi^2 T} \langle A_0^2 \rangle^2 + \frac{g^4}{12\pi^2 T} \langle N_c - N_f \rangle \langle A_0^4 \rangle,
$$

$$
\mathcal{L}_4^i(x) = \frac{1}{2T} \langle F_{\mu\nu}^2 \rangle,
$$

$$
\mathcal{L}_6^i(x) = - \frac{2}{15} \frac{g^2 \zeta(3)}{(4\pi)^2} \left[ i g \left( \frac{2}{3} N_c - \frac{14}{3} N_f \right) \langle F_{\mu \nu} F_{\nu \lambda} F_{\lambda \mu} \rangle - (19N_c - 28N_f) \langle |D_{\mu} F_{\mu \nu}|^2 \rangle - (18N_c - 21N_f) \langle |D_0 F_{\mu \nu}|^2 \rangle + (2N_c - 14N_f) \langle |D_i E_i|^2 \rangle + 110g^2 \langle A_0^2 \rangle \langle F_{\mu \nu}^2 \rangle + ig \langle 4N_c - 28N_f \rangle \langle E_i F_{i j} E_j \rangle + g^2 \langle 110N_c - 140N_f \rangle \langle A_0^2 F_{\mu \nu}^2 \rangle + 220g^2 \langle A_0 F_{\mu \nu}^2 \rangle \right]
$$

where $\langle X \rangle := \text{tr}(X)$, and $B_i = \frac{i}{2} \epsilon_{ijk} F_{jik}$ is the magnetic field. $\mathcal{L}_6^i$ has been computed in \cite{Megias:2003} for the gluon sector, and in \cite{Megias:2003} for the quark sector and for SU(3) in the absence of chromomagnetic field ($A_i = 0$). Our results agree with these.

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\footnote{Dimensional reduced effective theory.}

\footnote{In that formulas we have rescaled $A_i$ and $A_0$ with different renormalization factors, so that $\mathcal{L}_4^i$ looks like the zero temperature renormalized tree level: $g \to Z_g g$, $A_i \to Z_{M}^{1/2} A_i$, $A_0 \to Z^{1/2}_E A_0$; with
\[Z_M = Z_q = 1 + 2g^2 \beta_0 \log(\mu/4\pi T) + \gamma_E] - \frac{g^2}{3(4\pi)^2} \left[ -N_c + 8N_f \log 2 \right] ; \quad Z_E = Z_M - \frac{2g^2}{3(4\pi)^2} \left(N_c - N_f \right).\]}

\[\Omega(x) = e^{2\pi x} \text{ is in the fundamental representation, and } \hat{\Omega}(x) = e^{2\pi \tilde{\nu}} \text{ is in the adjoint one.}\]