The Loss of fidelity due to quantum leakage
for Josephson charge qubits

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Abstract
In this paper we calculate the loss of fidelity due
to quantum leakage for the Josephson charge qubit
(JCQ) in virtue of the Mathieu functions. It is
shown that for an present typical parameters of
JCQ $E_J/E_{ch} \sim 0.02$, the loss of the fidelity per
elementary operation is about $10^{-4}$ which satisfy
the DiVincenzo’s low decoherence criterion. By
appropriately improving the design of the Joseph-
son junction, namely, decreasing $E_J/E_{ch}$ to $\sim 0.01$,
the loss of fidelity per elementary operation can de-
crease to $10^{-5}$ even smaller.

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1 Introduction

A quantum computer can perform certain tasks
which no classical one is able to do in acceptable
times [1]. A quantum bit (qubit) is a quantum
system with two levels which will be a cell for stor-
ing and processing information in a future quantum
computer. So it is a vital task to find out the phys-
ical realizations of the qubits. In last years, physi-
cists have proposed several qubit models which are
based on ion traps [2], QED systems [3], nuclear
spins of large numbers of identical molecules [4],
quantum dots [5], Josephson junction [6] and so
on. Because solid state qubits can be embedded in
electronic circuits as well as scaled up to a large
numbers, they are taken as a particularly promising
The Josephson junction qubit is one of this kinds
of models.

Decoherence, one of the most difficult problems
to be dealt with in quantum computation exists in
all of the qubit models. In general, decoherence
comes from the interaction of the qubits and their
environment. But for some qubit models, for ex-
ample, quantum dots and Josephson junctions, the
decoherence also results from intrinsic source of er-
ror, such as quantum leakage [8] [9]. The quan-
tum leakage is this kind of process that the system
working in the computational Hilbert space leaks
to higher states. The leakage exists in many
quantum processes [10] [11]. It is attracted a par-
ticular attention in the implementations of qubits
and quantum gates for quantum computation [5]
because they must satisfy the DiVincenzo’s checklist five criteria [12] one of which is low decoherence
(so that error correction techniques may be used in
a fault-tolerant manner)—an approximate bench-
mark is the loss of fidelity no more than $10^{-4}$ per
elementary quantum gate operation.

In [8], Fazio et al. investigated the leakage and
fidelity of the Josephson charge qubit (JCQ) oper-
ations. Where the eigenvalues and eigenstates of
the JCQ Hamiltonian are obtained through dia-
ogonalizing the Hamiltonian. In fact, they can be ob-
tained in virtue of the perturbation theory of quan-
tum mechanics [17]. In particular, as pointed in
[8] the eigenvalues and eigenstates can be obtained
by solving the eigen-equation of the JCQ Hamil-
tonian which in fact is the Mathieu equation [14].
The eigenvalues and eigenstates of the JCQ Hamil-
tonian correspond to the characteristic values and
where $C = (C_J + C_g)$. Thus, we can obtain the Lagrangian of the JCQ as
\[
\mathcal{L} = \frac{1}{2} C \left( \frac{\Phi_0}{2\pi} \right)^2 \dot{\varphi}^2 + \frac{\Phi_0 I_c}{2\pi} \cos \varphi. 
\] (4)

The Euler-Lagrange equation can be used to check that the Lagrangian produces the correct classical equations of motion,
\[
\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0. 
\] (5)

By definition, the conjugate variable to $\varphi$ is,
\[
p = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = C \left( \frac{\Phi_0}{2\pi} \right)^2 \dot{\varphi}. 
\] (6)

On the other hand, the charge on the Josephson junction capacitance and the gate capacitance are
\[
q = C_J U_J = C_J \frac{\Phi_0}{2\pi} \varphi \equiv n 2e, 
\]
\[
q_g = C_g U_g = -C_g \frac{\Phi_0}{2\pi} \varphi \equiv n_g 2e, 
\] (7)

where $n$ is the Cooper pairs pass through the Josephson junction, and $n_g$ is the number of two-unit charge $2e$ on the gate capacitance. From Eqs. (6) and (7) we have
\[
p = \Phi_0 \frac{2}{2\pi} \left( q - q_g \right) = C \left( \frac{\Phi_0}{2\pi} \right)^2 \varphi = 2e \frac{2n}{C} \left( n - n_g \right). 
\] (8)

So
\[
\dot{\varphi} = \frac{2 \pi 2e}{\Phi_0 C} \left( n - n_g \right). 
\] (9)

Now we can construct the Hamiltonian via Legendre transformation,
\[
\mathcal{H} = \dot{\varphi} p - \mathcal{L} = C \left( \frac{\Phi_0}{2\pi} \right)^2 \dot{\varphi}^2 - \frac{\Phi_0 I_c}{2\pi} \cos \varphi = \frac{2e^2}{C} \left( n - n_g \right)^2 - \frac{\Phi_0 I_c}{2\pi} \cos \varphi. 
\] (10)

Setting $E_{ch} = \frac{2e^2}{C}$, $E_J = \frac{\Phi_0 I_c}{2\pi}$, we have
\[
\mathcal{H} = E_{ch} \left( n - n_g \right)^2 - E_J \cos \varphi. 
\] (11)

So we have $[p, \varphi] = -i\hbar$, and $[n, \varphi] = -i\hbar / \Phi_0 e$. As Ref. [10] to shorten notations we use units where
\[ e = 1, \ h = 1. \] So \([n, \varphi] = -i.\] Due to the conjugate relationship we have \(n = -i \frac{d}{d\varphi}.\) Therefore the Hamiltonian becomes

\[
\mathcal{H} = E_{ch} \left( -\frac{d^2}{d\varphi^2} + 2n_g \frac{d}{d\varphi} + n_o^2 \right) - E_J \cos \varphi.
\]

Setting the eigenstates of \(\mathcal{H}\) be \(\Psi_n\) and according to the eigen-equation, we have

\[
\frac{d^2 \Psi_n}{d\varphi^2} + 2n_g \frac{d\Psi_n}{d\varphi} \left( n_o^2 - \frac{E_n}{E_{ch}} - \frac{E_J \cos \varphi}{E_{ch}} \right) \Psi_n = 0.
\]

Setting \(\Psi_n = e^{i\varphi} \psi_n(\varphi),\) and \(x = 1/2,\) we have

\[
\frac{d^2 \psi_n}{d\varphi^2} + 2k \frac{d\psi_n}{d\varphi} \left( k^2 - \frac{E_n}{E_{ch}} - \frac{E_J \cos \varphi}{E_{ch}} \right) \psi_n = 0,
\]

where \(k = \frac{1}{2} - n_x.\) If we modulate the controllable gate voltage \(V_g\) and make \(n_g = x = \frac{1}{2},\) then Eq. (14) become

\[
\frac{d^2 \psi_n}{d\varphi^2} + (\lambda - 2v \cos \varphi) \psi_n = 0,
\]

where

\[
\lambda = \frac{E_n}{E_{ch}}, \quad v = -\frac{E_J}{2E_{ch}}.
\]

Eq. (15) is the canonical form of the Mathieu equation [14], its characteristic functions called Mathieu functions. The Mathieu functions were introduced by Mathieu [14] when analyzing the movements of membranes of elliptical shape. Since then the characters of the Mathieu functions have been investigated by Mathieu and others [14]. In recent years, the functions have been attracted much attention because they have some applications in many fields of physics [15]. The Mathieu equation has the well known periodic solutions

\[
\begin{cases}
  c_{2n} (\varphi, v) & \text{even solutions with period } \pi \\
  s_{2n+1} (\varphi, v) & \text{odd solutions with period } \pi \\
  c_{2n+1} (\varphi, v) & \text{even solutions with period } 2\pi \\
  s_{2n+1} (\varphi, v) & \text{odd solutions with period } 2\pi
\end{cases}
\]

with eigenvalues \(\alpha_{2n} (v),\)

It has been pointed that the periodic boundary conditions \(\psi_n (\varphi = 0) = \psi_n (\varphi = \pi)\) singles out only the \(2\pi\) periodic Mathieu eigenfunctions \(c_{2n}, s_{2n}\) for an integer \(x\) and the \(\pi\)-anti-periodic Mathieu eigenfunctions \(c_{2n+1}, s_{2n+1}\) for a half-integer \(x.\) So when one suddenly switch the offset charge from idle point to the degeneracy point \(n_g = \frac{1}{2} (\text{or another half-integer}),\) we can obtain the eigenvalues (leave over the first five terms) and the eigenfunctions (leave over the first three terms) of the Hamiltonian \(\mathcal{H}\) as

\[
\begin{align*}
E_1^c &= E_{ch} \left( 1 + \frac{v^2}{8} + \frac{v^3}{64} - \frac{v^4}{1536} - \cdots \right), \\
E_1^s &= E_{ch} \left( 1 - \frac{v^2}{8} + \frac{v^3}{64} - \frac{v^4}{1536} + \cdots \right)
\end{align*}
\]

Enlightened by [18], we have

\[
\begin{align*}
|\psi_1^c\rangle &= \left[ \left| 0 \right\rangle + \left| 1 \right\rangle \right] + \frac{v}{8} \left[ \left| -1 \right\rangle + \left| 2 \right\rangle \right] + \cdots, \\
|\psi_1^s\rangle &= \left[ \left| 0 \right\rangle - \left| 1 \right\rangle \right] + \frac{v}{8} \left[ \left| -1 \right\rangle - \left| 2 \right\rangle \right] + \cdots
\end{align*}
\]

Here, \(\delta\) denotes the higher-order effects of \(v^2.\) When \(E_J \ll E_{ch},\) we can set \(\delta \to 0.\) So after a time \(t,\) the initial state \(|\beta_0\rangle = \cos \theta \left| 0 \right\rangle + \sin \theta \left| 1 \right\rangle\) in the system becomes

\[
|\Psi\rangle_R = U_R |\beta_0\rangle
\]

\[
= \sum_{j=\pm 1} e^{-iE_j^c t} |\psi_j^c\rangle \langle \psi_j^c | \beta_0\rangle
\]

\[
= e^{-iE_1^c t} |\psi_1^c\rangle \langle \psi_1^c | \beta_0\rangle + e^{-iE_1^s t} |\psi_0^s\rangle \langle \psi_0^s | \beta_0\rangle
\]

It is shown that by using the Mathieu functions we can obtain more exact results of the eigenvalues and eigenstates of \(\mathcal{H}\) than previous researches. In particular, the eigenvalues and eigenstates can approximate to a arbitrary higher order of the Mathieu functions can be obtained.
On the other hand, because the Josephson energy $E_J$ is much smaller than the charging energy $E_{ch}$, and both of them are smaller than the superconducting energy gap $\Delta$, the Hamiltonian Eq. (11) can be parameterized by the number of the Cooper pairs $n$ through the junction as

$$\mathcal{H} = \sum_{n} \left\{ E_{ch} (n-n_g)^2 |n\rangle \langle n| - \frac{1}{2} E_J [ |n\rangle \langle n+1| + |n+1\rangle \langle n| ] \right\}. \quad (21)$$

When $n_g$ is modulated to a half-integer, say $n_g = 1/2$ and the charging energies of two adjacent states are closed each other, the Josephson tunneling mixes them strongly. Then, the system can be reduced to a two-state system (qubit) because all other charge states have much higher energy and they can be neglected, the Hamiltonian is approximately reads

$$\mathcal{H}_I = E_{ch} \left( n - \frac{1}{2} \right)^2 \sigma_z - \frac{1}{2} E_J \sigma_x. \quad (22)$$

This is an ideal Hamiltonian of the qubit. By choosing the reference point of the energy at $E_0 = E_{ch}/4$, the Hamiltonian can deduce to

$$\mathcal{H}_I = -\frac{E_J}{2} \sigma_x, \quad (23)$$

which has the eigenvalues and eigenstates as

$$E_0 = -\frac{E_J}{2}, \quad |\psi_0\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle),$$

$$E_1 = \frac{E_J}{2}, \quad |\psi_1\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle). \quad (24)$$

So after a time $t$ the initial state $|\beta_0\rangle$ becomes

$$|\Psi\rangle_I = U_I |\beta_0\rangle = \sum_{i=1,2} e^{-i\mathcal{E}_i t/2} |\varphi_i\rangle \langle \varphi_i| \beta_0\rangle$$

$$= e^{iE_1 t/2} |\varphi_0\rangle \langle \varphi_0| \beta_0\rangle + e^{-iE_1 t/2} |\varphi_1\rangle \langle \varphi_1| \beta_0\rangle. \quad (25)$$

It is shown that $|\Psi\rangle_R$ and $|\Psi\rangle_I$ are different. The difference derives from the quantum leakage. In the following, we shall investigate the loss of fidelity due to the quantum leakage for the JCQ.

### 3 Leakage and Fidelity of JCQ

The leakage of the JCQ is in fact the probability of initial state $|\beta_0\rangle$ leaks out to higher states after some time in the practical system. It can be defined as

$$L = \max \sum_{i\neq 0,1} R \langle \Psi | \Pi_i | \Psi \rangle_R$$

$$= 1 - \min \sum_{i=0,1} R \langle \Psi | \Pi_i | \Psi \rangle_R \quad (26)$$

where $\Pi_0 = |0\rangle \langle 0|, \Pi_1 = |1\rangle \langle 1|, \ldots, \Pi_i = |i\rangle \langle i| (i)$ are project operators; $|\Psi\rangle_R$ is the final state. The loss of fidelity is the probability by measuring the state $|\Psi\rangle_R$ with the project operators $\Pi_i = |i\rangle \langle i|$. By use of Eq. (20) we have

$$\sum_{i=0,1} R \langle \Psi | \Pi_i | \Psi \rangle_R = \sum_{i=0,1} \langle \beta_0 | U_R^i \Pi_i U_R | \beta_0 \rangle$$

$$= \langle \psi_1^I \psi_1^R \rangle \langle \beta_0 | \psi_1^R \rangle^2$$

$$+ \langle \psi_0^I \psi_0^R \rangle \langle \beta_0 | \psi_0^R \rangle^2 \quad (27)$$

Because of

$$\langle \psi_1^I \beta_0 \rangle = \sqrt{\frac{32}{64 + v^2 + \delta}} (\cos \theta + \sin \theta),$$

$$\langle \psi_0^I \beta_0 \rangle = \sqrt{\frac{32}{64 + v^2 + \delta}} (\cos \theta - \sin \theta),$$

$$\langle \psi_1^I \psi_1^R \rangle = \langle \psi_1^I \psi_1^R \rangle = \frac{64}{64 + v^2 + \delta},$$

$$\langle \psi_0^I \psi_0^R \rangle = 0. \quad (28)$$

we have

$$L = 1 - \left( \frac{64}{64 + v^2 + \delta} \right)^2 \approx 1 - \left( \frac{64}{64 + v^2} \right)^2. \quad (29)$$

On the other hand, in general, the fidelity is defined as $F(\rho, \sigma) = tr \sqrt{\rho \sigma \rho \sigma}$ for two arbitrary states $\rho$ and $\sigma$, and $F(|\psi\rangle, |\varphi\rangle) = \langle \varphi | \psi \rangle$ for two pure states $|\psi\rangle$ and $|\varphi\rangle$. By using $|\Psi\rangle_I$ and $|\Psi\rangle_R$ we can straightforwardly calculate the loss of fidelity, it is also

$$L = 1 - |I \langle \Psi | \Pi | \Psi \rangle_R|^2 \approx 1 - \left( \frac{64}{64 + v^2} \right)^2. \quad (30)$$

It means that if we do not consider the interaction of the qubit and the environment, the loss of fidelity
is just the quantum leakage $L$. So the relation of the fidelity and leakage is $F = 1 - L$.

For present typical parameters of Josephson junction $E_J/E_{ch} = 0.02$ [3], we can obtain the fidelity $F = 0.9999875000$ (calculated by using our formula), which is agreement with $F = 0.9999823223$ (calculated by using the Eq.(7) of Ref.[3]) very well. It can be easily seen that the loss of fidelity due to the leakage will be decreased by an appropriate choice of the device parameters. For example, the fidelity will increase to $F = 0.9999968750$ for $E_J/E_{ch} = 0.01$, and to $F = 0.9999992188$ for $E_J/E_{ch} = 0.005$, which shows that the loss of the fidelity is about $10^{-6}$ per elementary gate operation as $E_J/E_{ch} \sim 0.005$. According to DiVincenzo’s low decoherence criterion the loss of fidelity is tolerable. From the subsection II we know $E_{ch} = \frac{2 \pi^2}{c}$, $E_J = \frac{\Phi_0 L}{2 \pi}$, and we use units where $e = 1$, $\hbar = 1$. So $\Phi_0 = \pi \hbar / e = \pi$ in the units. Therefore $E_{ch} = \frac{2}{\pi}$, $E_J = \frac{L}{2}$. To decrease $E_J/E_{ch}$ one should decrease the critical current $I_c$ OR the total capacitance $C$. However, in Ref.[20] we investigate the short-time decoherence of the JCQ, where the increasing of the critical current $I_c$ AND decreasing the total capacitance $C$ are needed for decreasing the decoherence derived from the interaction of the system and its environment. From the analysis of the two papers we know that to decrease the decoherence not only from quantum leakage but from the environment one may improve the design of the JCQ through increasing the critical current $I_c$ AND decreasing the total capacitance $C$. But the design should keep $E_J/E_{ch} \leq 0.005 \sim 0.02$.

4 Conclusions

In this paper we investigated the loss of fidelity due to the quantum leakage for JCQ. Our researches are based on a lot of results of Mathieu functions. It is shown that our results agree with previous corresponding investigations very well. However, our work can expand to higher order approximation easily because of the well researched Mathieu functions. In particular, our results provide a feedback on how to improve the design of the JCQ for quantum computation. It is shown that decreasing the critical current and decreasing the Josephson capacitance and gate capacitance can decrease the decoherence from the quantum leakage. However, in order to decrease the total decoherence one may improve the design by increasing the critical current $I_c$ AND decreasing the total capacitance $C$. But the design should keep $E_J/E_{ch} \leq 0.005 \sim 0.02$. So we think that it is necessary to develop the technology of increasing the Josephson critical current and decreasing the capacitances in the small Josephson junctions in order to make the JCQ suitable for quantum computation.

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