Non-Singular Stationary Global Strings

Y. Verbin\textsuperscript{a} and A.L. Larsen\textsuperscript{b}

\textsuperscript{a} Department of Natural Sciences, The Open University of Israel
P.O.B. 39328, Tel Aviv 61392, Israel

\textsuperscript{b} Department of Physics, University of Southern Denmark,
Campusvej 55, 5230 Odense M, Denmark

Abstract

A field-theoretical model for non-singular global cosmic strings is presented. The model is a non-linear sigma model with a potential term for a self-gravitating complex scalar field. Non-singular stationary solutions with angular momentum and possibly linear momentum are obtained by assuming an oscillatory dependence of the scalar field on $t$, $\varphi$, and $z$. This dependence has an effect similar to gauging the global $U(1)$ symmetry of the model, which is actually a Kaluza-Klein reduction from four to three spacetime dimensions. The method of analysis can be regarded as an extension of the gravito-electromagnetism formalism beyond the weak field limit. Some $D = 3$ self-dual solutions are also discussed.

PACS: 11.27.+d, 11.10Lm, 04.20.Jb, 11.10Kk, 98.80.Cq

1 Introduction

Cosmic strings are a special kind of topological defects \cite{1}, which could have been formed in a symmetry breaking phase transition in the early universe. Often, the broken symmetry is assumed to be local, but the possibility of global strings has also been extensively discussed \cite{1}. The prototype is Cohen and Kaplan’s solution \cite{2, 3, 4} of a complex scalar with a $U(1)$ symmetry breaking potential. The string solution turns out to have a repulsive gravitational field with a curvature singularity at a finite distance from the axis. The singularity is unavoidable as was shown by Gibbons et al. \cite{4} for the standard Lagrangian of a complex field with a non-negative potential term, under the assumptions that the solutions are static and have symmetry under boosts along the string axis. This result may be regarded as the analog in the gravitational case of the $D = 2 + 1$ dimensional version of Derrick’s theorem \cite{5}, excluding the existence of static global strings with finite energy per unit length in flat space.

It was shown explicitly that the singularity of the Cohen-Kaplan solution can be avoided if time dependence is allowed, which results a string solution with de-Sitter like expansion along the string axis \cite{6}. An attempt to get a static non-singular global string was tried \cite{7} by relaxing the symmetry under boosts parallel to the string axis ($z$), but without relaxing the corresponding condition ($T^i_t = T^i_z$) in the string core, so the solution found there has lost its interpretation as a global string gravitational field.

There are additional ways to overcome the “no-go” theorems and find string solutions. All of them involve modifications of the Cohen-Kaplan model. Probably the most well-known is gauging the $U(1)$ symmetry group to become local, thus obtaining the Abelian Higgs model. The main consequence of this modification is that the long range massless Goldstone field transmutes into the longitudinal component of a short range massive vector field. This model admits string solutions (flux tubes) in flat...
space and in the self-gravitating case. However, we will not discuss this possibility here since almost all that can be said about it, has been already said [1]. Other ways are either restricted to the flat space case, or to the gravitating case or are valid in both. One is the Q-ball solution [5], which overcomes Derrick’s theorem by allowing oscillatory time dependence of the complex scalar, so as to avoid the staticity assumption without introducing time dependence into the energy-momentum tensor and other observables. In addition, the potential term is modified in such a way that it has another symmetric local minimum. From a more physical point of view, these modifications result a non-vanishing value of the observables. In addition, the potential term is modified in such a way that it has another symmetric local staticity assumption without introducing time dependence into the energy-momentum tensor and other analogues (“Q-string”) is similarly stabilized by a finite global $U(1)$ charge per unit length (see however [9] for a different situation). Spinning versions have been also recently considered [10]. Adding gravity into the Q-ball system does not change the overall picture of the solutions, but rather yields the so-called Q-stars [11].

A second way which is also based on the existence of a global $U(1)$ charge but in which gravity is indispensable, gives the boson stars [12] [13] [14]. It is also possible to get similar “non-topological solitons” in systems with more than one complex scalar field [14], but since we want to stick to a minimal field content in our model, we will not discuss those as well.

The simplest way to add a topological charge is to assume for the scalar field a “hedgehog configuration”, where it tends asymptotically to symmetry breaking minima of the potential, mapping this way spatial infinity into the vacuum manifold. In the original $U(1)$ system, static string-like solutions fall indeed into homotopy classes characterized by an integer winding number, but such solutions are excluded by the “no go theorems” mentioned above. However, we may think of a second version of a “hedgehog configuration”, where we turn the target space from a trivial vector space into a compact manifold of a non-linear sigma model. Since we are after string-like solutions, the scalar field defines a mapping to target space from the compactified $R^2$, which is transverse with respect to the string axis. If we take the simplest target space $S^2$, the kinetic part of the Lagrangian will be the usual one of the $O(3)$ non-linear sigma model. The potential term will necessarily break explicitly the $O(3)$ symmetry, but we will assume it is still $U(1)$-symmetric. It will be also taken until further notice to be non-negative with minima at $U = 0$. In this form of hedgehog configuration, the scalar field should tend asymptotically to a $U(1)$-symmetric minimum in order to correspond to a linear defect, which will be both localized enough and have a topological charge.

To make the discussion more concrete, we write down the action in the general form

$$S = \int d^D x \sqrt{|g|} \left( \frac{1}{2} \mathcal{E}(|\Phi|) \nabla_\mu \Phi^* \nabla^\mu \Phi - U(|\Phi|) + \frac{1}{2\kappa} R \right)$$

(1.1)

where $\kappa$ is the $D$ dimensional gravitational constant (we will discuss only $D = 3, 4$) and $\mathcal{E}(|\Phi|)$ is a non-negative dimensionless function, which serves as a Weyl factor of a conformally-flat target space metric of the non-linear sigma model. The $S^2$ target space is obtained by

$$\mathcal{E}(|\Phi|) = \frac{1}{(1 + |\Phi|^2/\mu^2)^2}$$

(1.2)

where $\mu$ is the diameter of the 2-sphere and sets the energy scale. In $D$ dimensions it has dimensions of energy to the power $(D - 2)/2$. More conventions: $\nabla_\mu$ is a covariant derivative, signature $(+, -, \ldots, -)$ and the Riemann tensor is $R^\lambda_{\mu\nu\rho} = \partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\rho\nu} + \ldots$.

Note that the “linear” Higgs (or rather Goldstone — but we will use the former term) system is recovered either by taking $\mathcal{E}(|\Phi|) = 1$, or using the sigma model Lagrangian in the limit of small fields or formally for $\mu \to \infty$.

In flat spacetime this model has well-known static string-like solutions, but only with a vanishing potential — again a simple consequence of Derrick’s theorem. This is the original $O(3)$ sigma model [5]. We will still use the term “sigma model” for the action (1.1), but refrain from characterizing it by the symmetry group, since it is (explicitly) broken by the potential term to $U(1)$.

The flat space string-like solutions were found to be unstable and may either shrink or spread out [15] [16]. They may be however stabilized by allowing (as above) an oscillatory time dependence and adding a potential term. This produces the so-called “Q-lumps”, which are stable rotating topological
string-like solutions \[17, 18\]. As described, these solutions have oscillatory time as well as angular dependence (around the string axis): \( \Phi = \mu f(r)e^{i(m\omega - \omega t)} \), where \( m \) is an integer and \( \omega \) real. The non-vanishing global \( U(1) \) charge results from the oscillatory time dependence, while it is the mixed angular and time dependence which produces the non-vanishing angular momentum density, \( T_{\varphi}^t \). The topological winding number is \( m \).

The \( O(3) \) sigma model may be also stabilized by a gravitational field in the form of a cosmological expansion \[19\]. Closer to our present purpose is the Comtet-Gibbons solution \[20\], which is a static cylindrically symmetric (self-dual) solution of a self-gravitating \( O(3) \) sigma model (with a vanishing potential).

In this paper, we fill some of the gaps which are evident from the above discussion, and find new non-singular global string solutions, which are topologically non trivial: self-gravitating Q-jumps and generalizations which introduce also a momentum current along the string axis - a possibility which has not been considered before. This is reflected by \( T_{\varphi}^t \neq 0 \) in case of translational motion along the string axis. This will be done by taking the more general oscillatory behavior \( \Phi = \mu f(r)e^{i(qz-\omega t+m\varphi)} \). Some non-topological solutions like self-gravitating spinning Q-strings will be discussed in a future publication.

The expectation that one may get non-singular stationary gravitational fields in this way, is based on the observation that introducing an oscillatory \( z \) and/or \( t \)-dependence is consistent with the existence of two commuting Killing vectors, which generate translations in time and along the string axis, and has the effect of gauging the global symmetry and turning the global sigma model into a gauged one. This way, the original \( D = 4 \) gravitating global sigma model is reduced in a Kaluza-Klein way into a \( D = 3 \) gauged sigma model coupled with \( D = 3 \) gravity and electromagnetism (and a dilaton). Since it is known that the gravitating Nielsen-Olesen flux tube produces a non-singular gravitational field, even when a dilaton coupling is allowed \[21, 22, 23\], there is a good chance it will share this characteristic with the type of solutions described above. As the dilaton introduced by the Kaluza-Klein reduction is also a long range field, the gravitational field of the global string solutions obtained in our model will be finite but not asymptotically flat. Only the pure \( D = 2 + 1 \) system discussed in sec. \[6\] will admit asymptotically flat global vortices.

We proceed and end this section by writing the field equations of our model. They are the following generalization of Klein-Gordon equation

\[
\mathcal{E}(|\Phi|) \nabla_\mu \nabla^\mu \Phi + \frac{\Phi^*}{2|\Phi|} \frac{d\mathcal{E}}{d|\Phi|} \nabla_\mu \Phi \nabla^\mu \Phi + \frac{\Phi}{|\Phi|} \frac{dU}{d|\Phi|} = 0
\] (1.3)

together with the Einstein equations (in term of the Einstein tensor \( G_{\mu\nu} \))

\[-\frac{1}{\kappa} G_{\mu\nu} = T_{\mu\nu} = \frac{1}{2} \mathcal{E}(\Phi) [\partial_\mu \Phi^* \partial_\nu \Phi + \partial_\nu \Phi^* \partial_\mu \Phi - (\nabla_\mu \Phi^*) (\nabla^{\nu} \Phi)] + U(\Phi) g_{\mu\nu} \] (1.4)

which are equivalent to

\[
\frac{1}{\kappa} R_{\mu\nu} = \frac{2}{D-2} U(|\Phi|) g_{\mu\nu} - \frac{1}{2} \mathcal{E}(\Phi) [\partial_\mu \Phi^* \partial_\nu \Phi + \partial_\nu \Phi^* \partial_\mu \Phi]
\] (1.5)

2 Field Equations for Sigma Model Global Strings

We start with the \( D \)-dimensional action \[14\] and look for string-like solutions, where the metric tensor is written in the form

\[
ds^2 = g_{\mu\nu}(x^\lambda) dx'^\alpha dx'^\beta = e^{2\sigma(x^\gamma)} h_{\alpha\beta}(x^\gamma) dx^\alpha dx^\beta - e^{-2(\varphi-3)\sigma(x^\gamma)} [dz + A_\alpha(x^\gamma) dx^\alpha]^2.
\] (2.1)

The indices \( \alpha, \beta \)... go from 0 to \( D-2 \), and we denote \( x^{D-1} = z \). This is just the Kaluza-Klein reduction from \( D \) to \( D-1 \) dimensions. If we also take \( \Phi = \psi(x^\gamma) e^{iqz} \), the \( D \)-dimensional gravitating sigma model with a global \( U(1) \) symmetry turns into a gravitating gauged sigma-Maxwell-dilaton system in \( D-1 \) dimensions, as it is evident from the Lagrangian which becomes:

\[
\sqrt{|g|} \left( \frac{1}{2} \mathcal{E}(\Phi|) \nabla_{\mu} \Phi \nabla^{\mu} \Phi - U(|\Phi|) + \frac{1}{2\kappa} R(g) \right) = \]

3
\[ \sqrt{|h|} \left( \frac{1}{2} \mathcal{E}(\Phi)|D_\alpha \psi^* D^\alpha \psi - q^2 e^{2(\ell-2)\sigma} |\psi|^2 - e^{2\sigma} U(|\psi|) \right) + \frac{\sqrt{|h|}}{2\kappa} \left( R(h) + (D-2)(D-3) \nabla_\alpha \sigma \nabla^\alpha \sigma - \frac{1}{4} e^{-2(D-2)\sigma} \sigma F_{\alpha \beta} F^{\alpha \beta} \right) \] (2.2)

after we define a usual covariant derivative \( D_\alpha = \partial_\alpha - iqA_\alpha \) and field strength \( F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \). The last term can obviously be omitted. All geometric quantities are related to the \((D-1)\) metric \( h_{\alpha \beta} \).

In the next step we assume stationarity, i.e. parametrize the scalar field and the \(D-1\) metric as:

\[ \psi = \phi(x^k)e^{-i\omega t}, \quad h_{\alpha \beta}(x^k) dx^\alpha dx^\beta = N^2(x^k)|dt + L_i(x^k) dx^i|^2 - \gamma_{ij}(x^k) dx^i dx^j \] (2.3)

where the indices \(i,j,...\) go from \(1\) to \(D-2\). Now we do not write the Lagrangian or the action in the \(D-2\) dimensional notation, but assume \(D = 4\) and write directly the field equations in terms of the following:

- \( \nabla_\iota \) the covariant derivative with respect to the 2-dimensional metric \( \gamma_{ij} \),
- \( R_{ij}(\gamma) \) the corresponding Ricci tensor,
- \( L_i = \partial_\iota L_\iota - \partial_\iota L_\iota = \ell \sqrt{\gamma} \epsilon_{ij} \).
- \( F_{i\ell} = NE_i, \quad F^{0\ell} = -D_\ell/N \)
- \( F_{ij} = -B \sqrt{\gamma} \epsilon_{ij}, \quad F^{ij} = -He^{ij}/\sqrt{\gamma} \)
- \( \bar{D}_j = \partial_j - iqA_j - (\omega + qA_0)L_j = \partial_j - iqA_j \)
- \( \bar{J}_k = -\frac{i}{2}\mathcal{E}(\phi^*)(\phi^* D_k \phi - \phi D_k \phi^*) \)

We use the usual convention of raising and lowering indices by \(\gamma_{ij}\) and its inverse. Note the distinction between \(D^\iota\) which is an “electric” (displacement) field (obeying \(D_\iota = E_\iota + NH \sqrt{\gamma} \epsilon_{ij} L_j\)), and \(\bar{D}^\iota\) which is a covariant derivative. We will also use \(E^2\) to denote \(E_i E^i\). This way of dealing with the field equations is essentially an extension of the gravito-electromagnetism formalism beyond the weak field limit, where it is usually used\([24,25]\). The components of \(F_{\alpha \beta}\) will be referred to as gravito-electric and gravito-magnetic.

Einstein equations split therefore according to the components \((00), (zz), (i0), (iz), (oz)\) and \((ij)\). The first 5 are:

\[ \frac{1}{N} \nabla_i \nabla^i N + \frac{N^2 e^2}{2} - \frac{H^2}{2} e^{-4\sigma} = -\kappa \left( 2e^{2\sigma} U(|\phi|) + q^2 e^{4\sigma} - \frac{(\omega + qA_0)^2}{N^2} \right) |\phi|^2 \mathcal{E}(|\phi|) \] (2.4)

\[ \frac{1}{N} \nabla_i (N \nabla^i \sigma) + \frac{H^2 - E^2}{2} e^{-4\sigma} = \kappa \left( e^{2\sigma} U(|\phi|) + q^2 e^{4\sigma} |\phi|^2 \mathcal{E}(|\phi|) \right) \] (2.5)

\[ \frac{\epsilon^{ij}}{N \sqrt{\gamma}} \partial_j (N^3 \ell) - \frac{e^{-4\sigma} HN \epsilon^{ij}}{\sqrt{\gamma}} E_j = 2\kappa (\omega + qA_0) \bar{J}_j \] (2.6)

\[ \frac{\epsilon^{ij}}{N \sqrt{\gamma}} \partial_j (NH e^{-4\sigma}) = -2\kappa q \bar{J}_j \] (2.7)

\[ \frac{1}{N} \nabla_i (e^{-4\sigma} D^i) = 2\kappa q \left( L_i \bar{J}^i - \frac{\omega + qA_0}{N^2} |\phi|^2 \mathcal{E}(|\phi|) \right) \] (2.8)

and the \((ij)\) components are:

\[ R_{ij}(\gamma) + \frac{1}{N} \nabla_i \nabla_j N - \frac{N^2 e^2}{2} \gamma_{ij} + 2\partial_i \sigma \partial_j \sigma + \frac{1}{2} (E^2 \gamma_{ij} - E_i E_j) e^{-4\sigma} = -\kappa \left( 2e^{2\sigma} U(|\phi|) + q^2 e^{4\sigma} |\phi|^2 \mathcal{E}(|\phi|) \right) \gamma_{ij} + \frac{1}{2} \mathcal{E}(|\phi|)(\bar{D}_i \phi^* \bar{D}_j \phi + \bar{D}_j \phi^* \bar{D}_i \phi) \] (2.9)
The equation for the scalar field is

\[ \mathcal{D}_i \mathcal{D}^i \phi + \frac{\phi^*}{2|\phi|E} \frac{dE}{d|\phi|} \mathcal{D}_i \phi \mathcal{D}^i \phi + \left( \frac{(\omega + qA_0)^2}{N^2} - q^2 e^{4\sigma} \right) \left( 1 + \frac{|\phi|}{2E} \frac{dE}{d|\phi|} \right) \phi - \frac{e^{2\sigma}}{|\phi|E} \frac{dU}{d|\phi|} \phi = 0 \quad (2.10) \]

Another equation which is not independent, but will be useful is the \((ij)\) component of (1.4):

\[ \frac{1}{N}(\nabla_i \nabla_j N - \gamma_{ij} \nabla_k \nabla^k N) - \frac{N^2 \ell^2}{4} \gamma_{ij} + 2\partial_i \sigma \partial_j \sigma - \gamma_{ij} \gamma^{kl} \partial_k \sigma \partial_l \sigma + \frac{e^{-4\sigma}}{2} \left( \frac{H^2 + E^2}{2} \gamma_{ij} - E_i E_j \right) = -\frac{\kappa}{2} \mathcal{E}(|\phi|) (\bar{D}_i \phi^* \bar{D}_j \phi + \bar{D}_j \phi^* \bar{D}_i \phi) + \frac{\kappa}{2} (\mathcal{E}(\phi) \bar{D}_k \phi^* \bar{D}_k \phi + 2e^2 U(|\phi|) + \left( 4 e^{4\sigma} - \frac{(\omega + qA_0)^2}{N^2} \right) |\phi|^2 \mathcal{E}(\phi) ) \gamma_{ij} \quad (2.11) \]

### 3 Three-Dimensional System

In order to get a clearer picture of the system, it is useful to deal with a lower dimensional case if possible. Although this set of field equations is genuinely four-dimensional, it contains also in a special limit the field equations for the \(D = 3\) system. These are equations (2.4), (2.6), (2.9), (2.10) with the substitutions: \( q = 0, \sigma = 0 \) and \( A_\alpha = 0 \), which imply \( E_i = D_i = 0 \) and \( B = H = 0 \). We get therefore:

\[ \frac{1}{N} \nabla_i \nabla^i N + \frac{N^2 \ell^2}{2} = -\kappa \left( 2U(|\phi|) - \frac{\omega^2}{N^2} |\phi|^2 \mathcal{E}(|\phi|) \right) \quad (3.1) \]

\[ \frac{\epsilon^{ij}}{N\sqrt{|\gamma|}} \partial_j (N^3 \ell) = 2\kappa \omega \bar{J}^i \quad (3.2) \]

\[ R_{ij}(\gamma) + \frac{1}{N} \nabla_i \nabla_j N - \frac{N^2 \ell^2}{2} \gamma_{ij} = -\kappa \left( 2U(|\phi|) \gamma_{ij} + \frac{1}{2} \mathcal{E}(\phi) (\bar{D}_i \phi^* \bar{D}_j \phi + \bar{D}_j \phi^* \bar{D}_i \phi) \right) \quad (3.3) \]

and for the scalar field:

\[ \mathcal{D}_i \mathcal{D}^i \phi + \frac{\phi^*}{2|\phi|E} \frac{dE}{d|\phi|} \mathcal{D}_i \phi \mathcal{D}^i \phi + \frac{\omega^2}{N^2} \left( 1 + \frac{|\phi|}{2E} \frac{dE}{d|\phi|} \right) \phi - \frac{1}{|\phi|E} \frac{dU}{d|\phi|} \phi = 0 \quad (3.4) \]

Note that a solution of the \(D = 3\) system is not automatically a solution to the \(D = 4\) system with the above-mentioned substitutions. The main reason is eq (2.5) which gives an additional condition, which is consistent with the \(D = 3\) theory only in the absence of self-interaction (i.e. \( U = 0 \)). On the other hand, the fact that there is an explicit dependence in the Einstein equations upon the dimensionality \(D\), does not hinder the possibility to use them for the \(D = 3\) field equations. Accordingly, the \(D = 3\) analogue of eq (2.11) is:

\[ \frac{1}{N}(\nabla_i \nabla_j N - \gamma_{ij} \nabla_k \nabla^k N) - \frac{N^2 \ell^2}{4} \gamma_{ij} = -\kappa \mathcal{E}(\phi) (\bar{D}_i \phi^* \bar{D}_j \phi + \bar{D}_j \phi^* \bar{D}_i \phi) + \frac{\kappa}{2} (\mathcal{E}(\phi) \bar{D}_k \phi^* \bar{D}_k \phi + 2U(|\phi|) - \frac{\omega^2}{N^2} |\phi|^2 \mathcal{E}(\phi) ) \gamma_{ij} \quad (3.5) \]

### 4 Self-Dual Solutions

It is known that the field equations simplify considerably if self-duality is possible (see e.g. ref. 26 and references therein). So we will start in the simplest case of self-dual solutions in the \(D = 3\) system. The self-dual solutions obey

\[ N = 1, \quad \bar{D}_i \phi = i \eta^{ij} \frac{\epsilon^{ij}}{\sqrt{|\gamma|}} \bar{D}_j \phi \quad (4.1) \]
where \( \eta = \pm 1 \) corresponds to self-dual or anti-self-dual solutions. In the following we will refer to both as “self-dual”. This is consistent with eq. (3.1) or (3.2), provided the following relation holds:

\[
\frac{\ell^2}{2\kappa} + 2U(|\phi|) - \omega^2|\phi|^2E(|\phi|) = 0. \tag{4.2}
\]

This relation is also consistent with eq. (3.3). Since the current also simplifies considerably for self-dual solutions, we get from eq. (3.2):

\[
\partial_t \ell = 2\eta \kappa |\phi| |E(|\phi|)| \partial_\phi |\phi| \tag{4.3}
\]

This equation can be integrated to give \( \ell(|\phi|) \) for any given \( E(|\phi|) \). In the most familiar case of the \( S^2 \) sigma model defined by eq. (1.2), the integration is trivial and we find the following form of potential term, which will give rise to localized self-dual solutions:

\[
U_{SD}(|\phi|) = \frac{\omega^2\mu^2(|\phi|^2/\mu^2 - \kappa \mu^2/2)}{2(1 + |\phi|^2/\mu^2)^2} \tag{4.4}
\]

We chose the integration constant such that \( \ell \) will vanish asymptotically, i.e. for fields that satisfy \( |\Phi| \to \infty \) as \( r \to \infty \). It is interesting to note that the linear sigma model (\( E(|\phi|) = 1 \)) needs for self-duality a potential which is unbounded from below, which is a property shared by other systems [26].

Since target space is a sphere, it is natural and useful to use an “angular” field defined by \( |\Phi| = \mu \tan(\Theta/2) \). In terms of this field, we get for example \( E = \cos^4(\Theta/2) \) and see that \( |\Phi| \to \infty \) is actually just the “south pole” of target space, \( \Theta = \pi \). Thus, the self-duality potential can be rewritten as:

\[
U_{SD}(\Theta) = \frac{\omega^2\mu^2}{8} \left( \sin^2\Theta - 2\kappa \mu^2 \cos^4(\Theta/2) \right) \tag{4.5}
\]

This generalizes the flat space self-dual Q-lumps [17, 18] to be self-gravitating. The flat space limit is reproduced by taking \( \kappa = 0 \). From a different direction, this can be regarded as a spinning generalization of the self-dual Comtet-Gibbons solutions [20]. Those are reproduced for \( \omega = 0 \). Explicit solutions are presented in section 5.

## 5 Rotational Symmetry

We now return to \( D = 4 \) and specialize to rotational symmetry. We are left with the 2-metric

\[
\gamma_{ij} dx^i dx^j = \alpha^2(r) dr^2 + \beta^2(r) d\varphi^2 \tag{5.1}
\]

where we postpone the gauge fixing to a later stage. The vector \( A_\alpha \) will have only two non-vanishing components \( A_0(r) \) and \( A_\varphi(r) \), from which we derive the gravito-electric and gravito-magnetic fields \( E_i(r) \) and \( B_i(r) \). Similarly, \( L_i \) will have only one non-vanishing component namely \( L_\varphi(r) \); thus \( \ell \) will also depend on \( r \) only. We will also use \( \tilde{A}_\varphi \) defined by \( q \tilde{A}_\varphi = q A_\varphi - (\omega + q A_0) L_\varphi \).

We define a dimensionless potential function \( u \) by \( U = \lambda \mu^4 u(\Theta) \) where \( \lambda \) is a dimensionless coupling constant (in \( D = 4 \)), and a dimensionless Newton’s constant \( \gamma = \kappa \mu^2 \) and get therefore the following set of field equations for the \( S^2 \) model. From equations (2.1 - 2.8) we get

\[
\frac{1}{N\alpha \beta} \left( \frac{\beta N!}{\alpha} \right)’’ + \frac{N^2 \ell^2}{2} - \frac{H^2}{2} e^{-4\sigma} = \gamma \left[ \left( \frac{\omega + q A_0}{N^2} - q^2 e^{4\sigma} \right) \sin^2 \Theta \right] - 2\lambda \mu^2 e^{2\sigma} u(\Theta) \tag{5.2}
\]

\[
\frac{1}{N\alpha \beta} \left( \frac{\beta N}{\alpha} \right)’’ + \frac{H^2}{2} - \frac{(E_i/\alpha)^2}{2} e^{-4\sigma} = \gamma \left[ q^2 e^{4\sigma} \sin^2 \Theta \right] + \lambda \mu^2 e^{2\sigma} u(\Theta) \tag{5.3}
\]

\[
\frac{\beta}{N\alpha} (N^3 \ell)^’ - \frac{\beta N}{\alpha} H E \sigma e^{-4\sigma} = -\gamma^2 (\omega + q A_0)(m - q A_\varphi) \sin^2 \Theta \tag{5.4}
\]

\[
\frac{\beta}{N\alpha} (N \tilde{A}_\sigma)^’ = \frac{\gamma}{2} q (m - q \tilde{A}_\varphi) \sin^2 \Theta \tag{5.5}
\]

\[
\frac{1}{N\alpha \beta} \left( \frac{\beta}{\alpha} E \sigma e^{-4\sigma} \right)^’ + H \sigma e^{-4\sigma} = -\frac{\gamma}{2N^2} q (\omega + q A_0) \sin^2 \Theta \tag{5.6}
\]
where the last one is a combination of \(2.7\) and \(2.8\). From eq \(2.9\) we take the equation for \(\gamma\) and we have also to add the following relations:

\[
\begin{align*}
F_{\gamma} & = 0, \\
\Theta & = \text{any finite constant while}
\end{align*}
\]

For the scalar field we get:

\[
\omega \bigg[ 20 \bigg] \text{is a self-dual solution with vanishing potential.}
\]

Few soliton-like solutions to this system are already known in the gravitating case – all of them static \((\omega = q = 0, \varphi = A_\varphi = A_0 = 0)\). For the “linear” case \((\mathcal{E}(\Phi) = 1)\) with a potential with a \(U(1)\) symmetry breaking minimum, there exists Cohen and Kaplan’s global string \([20]\) which is singular as already mentioned in the introduction. In the non-linear sector, the Comtet-Gibbons string-like solution \([20]\) is a self-dual solution with vanishing potential.

Remotely related solutions are the dilatonic-Melvin solution \([27, 22]\), and the three-dimensional dilaton-Maxwell-Einstein system that were discussed by some authors \([28, 29, 30, 31]\). The main difference between our model and the above mentioned dilaton-Maxwell-Einstein papers, is that none of them has suggested a possible source for the angular momentum or electromagnetic fields of those solutions. It may be thought that those dilaton-Maxwell-Einstein papers may be considered relevant to our model in the asymptotic regions, where the scalar field settles to its vacuum value, but even from this point of view there are significant differences. The dilaton in the present system is a metric component and not an additional field, so its coupling to other fields is not arbitrary. Similarly, the “electric” or “magnetic” fields are also of gravitational origin, and since they have specific source, the field equations are inconsistent with the separate conditions \(E_r = 0\) or \(H = 0\). Consequently, the solutions presented here cannot be classified as either electric or magnetic but always (for non-vanishing self-interaction) have both kinds of fields or none of them. A third difference is that most of the above-mentioned papers introduce a non-vanishing cosmological constant, which we avoid here.

We solve the equations numerically taking the “Kasner gauge” \(\alpha(r)e^\sigma(r) = 1\) so that \(q_{rr} = -1\). In order to get localized solutions, we solve the equations with the boundary conditions which ensure regularity at the origin and will be as close as possible to asymptotically vacuum. That is, we impose on the 3-metric

\[
N(0) = 1, \quad N'(0) = 0, \quad \beta(0) = 0, \quad \beta'(0) = 1, \quad L_\varphi(0) = 0, \quad L'_\varphi(0) = 0.
\]

As for the gravito-electromagnetic components, \(A_0(0)\) can be any finite constant while

\[
A'_0(0) = 0, \quad \lim_{r \to \infty} A_0(r) = 0, \quad A_\varphi(0) = 0
\]

(for \(A_\varphi(\infty)\) see below) and for the “dilaton” and the scalar field

\[
\sigma(0) = 0, \quad \sigma'(0) = 0, \quad \Theta(0) = 0, \quad \lim_{r \to \infty} \Theta(r) = \Theta_0
\]
where $\Theta_0$ is the location of the minimum of the potential. The value of $\Theta_0$ distinguishes between two types of hedgehog configurations: a “sigma model hedgehog” and a “Higgs hedgehog”. The first can be realized if the scalar tends asymptotically to the south pole, i.e. $\Theta_0 = \pi$. In this case $A_0(\infty)$ can be any finite constant and the gravito-magnetic flux is not quantized. The second corresponds to any other value $0 < \Theta_0 < \pi$ which is associated with a quantized flux: $A_0(\infty) = m/q$. In order to discuss both kinds of solutions, we may use the following simple potential function:

$$u_1(\Theta) = \left(\frac{\sin^2(\Theta/2) - \sin^2(\Theta_0/2)}{\sin^2(\Theta_0/2)}\right)^2$$

(5.14)

where $\Theta_0$ is, as mentioned above, the location of the potential minimum. It is actually the standard Higgs potential multiplied by the conformal factor $E(|\Phi|)$ and written in terms of the angular field. The case $\Theta_0 = \pi$ corresponds to a “single well potential”. We will also use the “double well potential”, which is a special case of (5.14) with two minima at the two poles $\Theta = 0, \pi$:

$$u_2(\Theta) = \sin^2 \Theta$$

(5.15)

These minima do not break the symmetry so they allow topological solutions of the “sigma model hedgehog” type. This is the potential used by Leese [17] in $D = 3$ flat spacetime for obtaining the self-dual Q-lumps. Ward [18] used another form given by $u(\Theta) = \sin^2 \Theta (1 + \cos^2 \Theta)$, which we will not use here. All the other potentials are shown in figure 1.

6 Special Solutions: Strings with Linear and/or Angular Momentum

We solve numerically the field equations, using the MATHEMATICA package, after rewriting them as a system of 14 coupled first order ordinary differential equations. We use the dimensionless radial variable $x = \mu r$, and rescale accordingly the other dimensional quantities $q/\mu \to q$, $\omega/\mu \to \omega$, $\mu \beta(r) \to \beta(r)$ and $\mu L_p(r) \to L_p(r)$.

The following sections describe the main characteristics of the solutions. In all cases we take $m = 1$.

6.1 Higgs-like Potentials

We start by discussing a family of solutions for potentials of a Higgs type, whose minima break spontaneously the $U(1)$ symmetry – eq (5.14). The main disadvantage of the Higgs type models is that the solutions are not localized “enough”, so the gravitational field is still singular, and we present them here only as a basis for improvement. Figure 2 depicts the fields of a static solution for the case $\Theta_0 = \pi/2$. It is very similar to the Cohen-Kaplan solution. For small $\gamma$ the singularity appears for very large $r$ (or $x$ for $\gamma = 0.2$ and $\omega = 0$ the singularity is already at $x \sim 10^9$), so for demonstrative reasons we took a large $\gamma$. There are two kinds of singularities. The first is the one appearing in figure 2 where $N$, $e^{-\sigma}$ and $\beta$ vanish while $\beta e^{\sigma}$ diverges for some finite $r$. For $\omega = 0$ $Ne^{\sigma} = e^{-\sigma}$ so $N$ and $e^{-\sigma}$ vanish at the same point. If $|\omega| > 0$ but smaller than a certain bound (which depends on $\gamma$), $Ne^{\sigma}$ decreases faster than $e^{-\sigma}$, so it is the zero of $Ne^{\sigma}$ or rather $N$ which determines the singularity. If however $|\omega|$ is large enough, $\beta e^{\sigma}$ changes its trend and vanishes while $N$ and $Ne^{\sigma}$ diverge. This last behavior is seen in figure 3. A similar behavior is found for $q \neq 0$ as well.

This behavior is a direct result from the existence of the long range Goldstone fields that exist in this model. This is the origin for the singularity of the static metric of the global string, and the other long range effects in other global defects.

As in the Cohen-Kaplan solution, it is the contribution to the energy density from the angular dependence which does not vanish fast enough to produce a localized source. In addition to the angular dependence, we allow here also time dependence and $z$-dependence which yield similar contributions all present in the right-hand-side of (5.14). Since all these terms are proportional to $\sin^2 \Theta$, it is quite easy to identify the way to improve the asymptotic behavior namely, taking $\Theta_0 = \pi$ or $|\Phi| \to \infty$ as $r \to \infty$. Note that the opposite direction, $\Theta_0 \ll 1$ gives the Higgs limit, as already mentioned.
6.2 Single Well Potential

In order to have solutions with $\Theta_0 = \pi$ (or $|\Phi| \rightarrow \infty$), we take the following simple monotonically decreasing potential $U(|\Phi|) = \lambda u^4/(1 + |\Phi|^2/\mu^2)$, which is one of the “vacuumless” potentials of Cho and Vilenkin [32, 33, 34], who also found only singular string-like solutions. However, since we use it within the sigma model framework, this potential, unlike those of Cho and Vilenkin, is not vacuumless but a single-well potential with a single minimum at the south pole ($\Theta = \pi$) as is evident from its behavior in terms of the angular field:

$$u(\Theta) = \cos^2(\Theta/2)$$

Thus, we may get non-singular global strings and indeed we find non-singular stationary solutions, though not static ones. There are two kinds of string-like solutions: purely rotating (PR in what follows) where no gravito-electromagnetic fields exist, and rotating solutions with additional gravito-electromagnetic fields (RGEM). On a first sight, one may think that the fields of the latter may be either gravito-magnetic or gravito-electric, according to the field components produced by the momentum along the string axis. However, this is not the case (if self-interaction exists) since the field equations are inconsistent with either $E_r = 0$ or $H = 0$. Therefore, only solutions with gravito-electromagnetic fields exist.

Both kinds of solutions (PR and RGEM) have $\omega \neq 0$ and the PR solutions have $q = 0$. On the other hand, RGEM solutions exist for either vanishing or non-vanishing values of $q$. The fact that $q = 0$ allows non-vanishing gravito-electromagnetic fields is a direct result of the “Maxwell” equation for the “displacement” field $D^i$ eq (2.8), which shows that it is $D^i$ (rather than $E^i$) that vanishes if $q = 0$. Since there is no proportionality between these two gravito-electric fields, $D^i = 0$ does not imply $E^i = 0$ nor $B = 0$ (or $H = 0$) but only a specific relation between them, somewhat similarly to a medium with electric and magnetic permeabilities. Therefore, these fields can be non-zero even when $q = 0$. Another way to see that there should exist RGEM solutions when $q = 0$ is from the fact that it is possible to “gauge away” $q$ by a coordinate transformation which however will not transform to zero the gravito-electromagnetic fields. The case $q = 0$ splits therefore into two branches: a PR branch and a RGEM branch.

The fields of a purely rotating solution are depicted in figure 4. There is only a narrow range of $\omega$ values, outside of which only singular solutions exist. Those of a rotating gravito-electromagnetic solution are presented in the two parts of figure 5. RGEM solutions with $q = 0$ are similar and are not shown. The fact that RGEM non-singular solutions exist with smaller $\omega$ is a general one, and may be understood by the fact that less angular momentum is required to balance the scalar and gravitational self-attraction, in the presence of gravito-electromagnetic fields. Another significant difference between the two kinds of solutions, is the asymptotic behavior of $e^{-\sigma}$. While it is monotonically decreasing function in the PR case, it is monotonically increasing for RGEM solutions. This difference is a direct consequence from the field equations: it is obvious from equation (5.3) that for vanishing gravito-electromagnetic fields, the function $\sigma(r)$ is monotonically increasing so $e^{-\sigma(r)}$ is decreasing. On the other hand, for RGEM solutions, the magnetic contribution is the dominant term for large distances and it comes with an opposite sign, so $e^{-\sigma(r)}$ has the opposite large distance behavior - although it may start decreasing near the axis.

Another kind of potential which we can consider is

$$u(\Theta) = \cos^4(\Theta/2)$$

which is just the special case $\Theta_0 = \pi$ of (6.1). We find non-singular PR solutions, which are quite similar to those of the previous single well potential, thus we do not present them. However, we were able to obtain only singular RGEM solutions for this potential.

6.3 Double Well Potential

We take the double well potential (5.15). The two minima do not break the symmetry, so they allow topological solutions of the “sigma model hedgehog” type. The fields of a purely rotating solution appear

\[1\] we thank the referee for pointing out towards this direction.
in figure 4. As for the previous potential, there is a range of \( \omega \) values outside of which only singular solutions exist, while \( \gamma \) can be quite large. The fields of a rotating gravito-electromagnetic global string are presented in figure 4. As before, the \( q = 0 \) RGEM solutions are similar as they are related to the \( q \neq 0 \) ones by a coordinate transformation. The gravito-electromagnetic solutions become closed above a certain critical value of \( \gamma \). The behavior of \( e^{-\sigma} \) is also analogous to that of the single well potential: decreasing for the PR case and increasing for the RGEM case.

6.4 The Self-Dual Potential in \( D = 3, 4 \)

Now we take the potential (4.5) which, as we saw already, allows \( D = 3 \) self-dual rotating strings or rather vortices. We write it as in all other cases in terms of the dimensionless potential function \( u(\Theta) \), but with the \( D = 3 \) coefficient \( \lambda \) as \( U = \lambda \mu^6 u(\Theta) \). The necessary condition for self-duality is therefore \( \omega^2 = 8\lambda \mu^4 \). In order to obtain vortex solutions, we may directly solve the second order system of equations (5.2), (5.4), (5.7) and (5.9) with the substitutions \( q = 0, \ N = 1, \ \sigma = 0, \ A_0 = 0 \) and \( A_\phi = 0 \) which imply \( E_\theta = 0 \) and \( H = 0 \). Alternatively, we may solve a first order system which include (among other equations which we do not present), the equations (4.1) and (4.3) used for rotational symmetry. A typical solution is shown in figure 8.

Now we can proceed and search for solutions with the same potential (up to powers of \( \mu \)) in \( D = 4 \). It is not surprising that we don’t find self-dual solutions. Yet there exist non-singular solutions, which we do not show since they are quite similar to those of the double well potential.

6.5 No Self-Interaction

For \( U = 0 \) we reproduce the well-known Comtet-Gibbons string-like solutions [20], which have asymptotically a conical geometry. Figure 9 shows the fields of a solution with a very large value of \( \gamma \) that was taken in order to stress the angular deficit, which is directly seen from the asymptotic slope of \( \beta(r) \).

A new type of (purely) rotating solutions is obtained for non-vanishing \( \omega \). The fields of one of these solutions are depicted in figure 10. All these solutions have asymptotic cylindrical two-geometry, while the metric component \( N(r) \) increases linearly as \( r \to \infty \). Solutions exist for any \( |\omega| > 0 \). The limit \( \omega \to 0 \) is singular, and does not reproduce the conical \( \omega = 0 \) solutions.

The fields of a rotating gravito-electromagnetic global string are presented in the two parts of figure 11. A non-rotating solution with gravito-magnetic field is also found and presented in figure 12. Because \( \omega = 0 \) we have here, as in the Comtet-Gibbons solution, \( N = e^{-\sigma} \) but now different from 1. This kind of purely gravito-magnetic solution is special to the case of vanishing potential and cannot exist otherwise.

7 Conclusion and Outlook

We have shown that non-singular global cosmic string solutions exist in a simple sigma model, if we also allow for rotation or momentum current. Another result is the self-gravitating generalization of the \( D = 3 \) self-dual Q-lumps. Further investigation is required in order to fully understand these solutions and their consequences. The geometry of the solutions should be clarified by geodesic analysis, and their possible role in a cosmological context also deserves further study. Another direction is a higher dimensional generalization of these solutions to be codimension 2 branes. These may be relevant to the extra dimensional resolution of the hierarchy problem, which was advanced recently by several authors [35, 36, 37, 38] using similar kind of solutions.

\(^2\)Note that \( \kappa \mu^2 = \gamma \) is still dimensionless as in \( D = 4 \) (or higher).
References


Figure 1: Dimensionless potential functions: a “Higgs-like” $u_1(\Theta)$ with $\Theta_0 = \pi/2$ – dashed, the “single-well” $u_1(\Theta)$ with $\Theta_0 = \pi$, the double well $u_2(\Theta)$ and the self-dual $u_{SD}(\Theta)$.

Figure 2: The fields of a static global string solution for a Higgs-like potential.

$\gamma=1, \lambda=1, q=0, \omega=0$
Figure 3: The fields of a purely rotating global string solution for a Higgs-like potential.

Figure 4: The fields of a purely rotating global string solution for a single well potential.
Figure 5: The fields of a rotating gravito-electromagnetic global string solution for a single well potential.
Figure 6: The fields of a purely rotating global string solution for a double well potential.

\[ \gamma = 0.5, \lambda = 1, q = 0, \omega = 6 \]
Figure 7: The fields of a rotating gravito-electromagnetic global string solution for a double well potential.

\( \gamma = 0.5, \lambda = 1, q = 0.5, \omega = 3.56354 \)
$\gamma = 0.5, \lambda = 1, \omega = \sqrt{8}$

Figure 8: The fields of a rotating self-dual global string solution in $D = 3$. Note that $N(x) = 1$.

$\gamma = 1.9, \lambda = 0, q = 0, \omega = 0$

Figure 9: The fields of the Comtet-Gibbons (static) global string solution.
Figure 10: The fields of a purely rotating global string solution without self-interaction. Here $e^{-\sigma(x)} = 1$. 

\[
\gamma = 1.9, \lambda = 0, q = 0, \omega = 2
\]
Figure 11: The fields of a rotating gravito-electromagnetic global string solution without self-interaction.
Figure 12: The fields of a non-rotating gravito-magnetic global string solution without self-interaction.