RELATIVISTIC PSEUDOSPIN SYMMETRY AS A SUPERSYMMETRIC PATTERN IN NUCLEI

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Shell-model states involving several pseudospin doublets and “intruder” levels in nuclei, are combined into larger multiplets. The corresponding single-particle spectrum exhibits a supersymmetric pattern whose origin can be traced to the relativistic pseudospin symmetry of a nuclear mean-field Dirac Hamiltonian with scalar and vector potentials.

1. Introduction

Pseudospin doublets\[1\] in nuclei refer to the empirical observation of quasi-degenerate pairs of certain shell-model orbitals with non-relativistic single-nucleon radial, orbital, and total angular momentum quantum numbers:

\[(n, \ell, j = \ell + 1/2) \quad \text{and} \quad (n - 1, \ell + 2, j' = \ell + 3/2). \quad (1)\]

The doublet structure (for \(n \geq 1\)) is expressed in terms of a “pseudo” orbital angular momentum \(\hat{\ell} = \ell + 1\) and “pseudo” spin, \(\hat{s} = 1/2\), coupled to \(j = \hat{\ell} - 1/2\) and \(j' = \hat{\ell} + 1/2\). For example, \([ns_{1/2}, (n - 1) d_{3/2}]\) will have \(\hat{\ell} = 1\), etc. The states in Eq. (1) involve only normal-parity shell-model orbitals. The states \((n = 0, \ell, j = \ell + 1/2)\), with aligned spin and no nodes, are not part of a doublet. This is empirically evident in heavy nuclei, where such states with large \(j\), i.e., \(0g_{9/2}, 0h_{11/2}, 0i_{13/2}\), are the “intruder” abnormal-parity states, which are unique in the major shell.

Pseudospin symmetry is experimentally well corroborated and plays a central role in explaining features of nuclei\[2\] including superdeformation\[3\] and identical bands\[4\]. It has been recently shown to result from a relativistic symmetry of the Dirac Hamiltonian in which the sum of the scalar and vector nuclear mean field potentials cancel\[5\]. The symmetry generators\[6\] combined with known properties of Dirac bound states, provide a natural explanation\[7\] for the structure of pseudospin doublets and for the special status of “intruder” levels in nuclei.
2. Supersymmetric Quantum Mechanics

Supersymmetric quantum mechanics (SUSYQM), initially proposed as a model for supersymmetry (SUSY) breaking in field theory, has by now developed into a field in its own right, with applications in diverse areas of physics. The essential ingredients of SUSYQM are the supersymmetric charges and Hamiltonian

\[
Q_- = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}, \quad Q_+ = \begin{pmatrix} 0 & L^1 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \begin{pmatrix} L^1 L & 0 \\ 0 & LL^1 \end{pmatrix}
\]

which generate the supersymmetric algebra

\[
[\mathcal{H}, Q_\pm] = \{ Q_\pm, Q_\pm \} = 0, \quad \{ Q_-, Q_+ \} = \mathcal{H}.
\] (3)

The partner Hamiltonians \( H_1 \) and \( H_2 \) satisfy an intertwining relation, \( L H_1 = H_2 L \), where in one-dimension the transformation operator \( L = \frac{d}{dx} + W(x) \) is a first-order Darboux transformation expressed in terms of

\[
\kappa_1 = -\ell < 0, \quad \kappa_2 = \ell + 1 > 0.
\]

\[j = \ell + 1/2, \quad j' = \ell + 1/2, \quad \ell = \ell + 1/2, \quad j' = \ell + 3/2, \quad \ell = \ell + 1/2, \quad j' = \ell + 1/2.
\]

Figure 1. Nuclear single-particle spectrum composed of pseudospin doublets and an “intruder” level. All states share a common \( \tilde{\ell} \) and \( \tilde{s} = 1/2 \). The corresponding Dirac \( \kappa \)-quantum numbers are also indicated.

Figure 2. Typical supersymmetric pattern. The Hamiltonians \( H_1 \) and \( H_2 \) have identical spectra with an additional level for \( H_1 \) when SUSY is exact. The operators \( L \) and \( L^\dagger \) connect the partner states.

Figure 1 portrays the level scheme of an ensemble of pseudospin doublets, Eq. (1), with fixed \( \ell, j, j' \) and \( n = 1, 2, 3, \ldots \) together with the “intruder” level \( (n = 0, \ell, j = \ell + 1/2) \). The single-particle spectrum exhibits towers of pair-wise degenerate states, sharing a common \( \ell \), and an additional non-degenerate nodeless “intruder” state at the bottom of the spin-aligned tower. A comparison with Fig. 2 reveals a striking similarity with a supersymmetric pattern. In the present contribution we identify the underlying supersymmetric structure associated with a Dirac Hamiltonian possessing a relativistic pseudospin symmetry.
a superpotential $W(x)$. The intertwining relation ensures that if $\Psi_1$ is an eigenstate of $H_1$, then also $\Psi_2 = L\Psi_1$ is an eigenstate of $H_2$ with the same energy, unless $L\Psi_1$ vanishes or produces an unphysical state (e.g., non-normalizable). Consequently, as shown in Fig. 2, the SUSY partner Hamiltonians $H_1$ and $H_2$ are isospectral in the sense that their spectra consist of pair-wise degenerate levels with a possible non-degenerate single state in one sector (when the supersymmetry is exact). The wave functions of the degenerate levels are simply related in terms of $L$ and $L^\dagger$. Such characteristic features define a supersymmetric pattern. In what follows we show that a Dirac Hamiltonian with pseudospin symmetry obeys an intertwining relation and consequently gives rise to a supersymmetric pattern.

3. Dirac Hamiltonian with Central Fields

A relativistic mean field description of nuclei employs a Dirac Hamiltonian, $H = \hat{\alpha} \cdot p + \beta (M + V_S) + V_V$, for a nucleon of mass $M$ moving in external scalar, $V_S$, and vector, $V_V$, potentials. When the potentials are spherically symmetric: $V_S = V_S(r)$, $V_V = V_V(r)$, the operator $\hat{K} = -\beta (\sigma \cdot \ell + 1)$, (with $\sigma$ the Pauli matrices and $\ell = -i r \times \nabla$), commutes with $H$ and its non-zero integer eigenvalues $\kappa = \pm (j + 1/2)$ are used to label the Dirac wave functions

$$\Psi_{\kappa, m} = \frac{1}{r} \left( \begin{array}{c} G_{\kappa} [Y_{\ell} \chi^{(j)}_{m}] \\ iF_{\kappa} [Y_{\ell'} \chi^{(j)}_{m}] \end{array} \right).$$

Here $G_{\kappa}(r)$ and $F_{\kappa}(r)$ are the radial wave functions of the upper and lower components respectively, $Y_{\ell}$ and $\chi$ are the spherical harmonic and spin function which are coupled to angular momentum $j$ with projection $m$. The labels $\kappa = -(j + 1/2) < 0$ and $\ell' = \ell + 1$ hold for aligned spin $j = \ell + 1/2$ ($s_{1/2}, p_{3/2}$, etc.), while $\kappa = (j + 1/2) > 0$ and $\ell' = \ell - 1$ hold for unaligned spin $j = \ell - 1/2$ ($p_{1/2}, d_{3/2}$, etc.). Denoting the pair of radial wave functions by

$$\Phi_{\kappa} = \left( \begin{array}{c} G_{\kappa} \\ F_{\kappa} \end{array} \right),$$

the radial Dirac equations can be cast in Hamiltonian form,

$$H_{\kappa} \Phi_{\kappa} = E \Phi_{\kappa},$$

with

$$H_{\kappa} = \left( \begin{array}{cc} M + \Delta & -\frac{d}{dr} + \frac{\kappa}{r} \\ \frac{d}{dr} + \frac{\kappa}{r} & -(M + \Sigma) \end{array} \right)$$

$$\Delta(r) = V_S + V_V , \quad \Sigma(r) = V_S - V_V.$$
The nuclear single-particle spectrum is obtained from the valence bound-state solutions of Eq. (6) with positive binding energy \((M - E) > 0\) and total energy \(E > 0\). The non-relativistic shell-model wave functions are identified with the upper components of the Dirac wave functions (4). For relativistic mean fields relevant to nuclei, \(V_S\) is attractive and \(V_V\) is repulsive with typical values \(V_S(0) \sim -400\), \(V_V(0) \sim 350\), MeV. The potentials satisfy \(rV_S, rV_V \to 0\) for \(r \to 0\), and \(V_S, V_V \to 0\) for \(r \to \infty\). Under such circumstances one can prove the following properties which are relevant for understanding the nodal structure of pseudospin doublets and intruder levels in nuclei.

(a) The radial nodes of \(F_\kappa (n_F)\) and \(G_\kappa (n_G)\) are related. Specifically,
\[
\begin{align*}
n_F &= n_G & \text{for } \kappa < 0, \\
n_F &= n_G + 1 & \text{for } \kappa > 0.
\end{align*}
\]
(b) Bound states with \(n_F = n_G = 0\) can occur only for \(\kappa < 0\).

4. Relativistic Pseudospin Symmetry in Nuclei

A relativistic pseudospin symmetry occurs when the sum of the scalar and vector potentials is a constant
\[
\Delta(r) = V_S(r) + V_V(r) = \Delta_0.
\]
A Dirac Hamiltonian satisfying (9) has an invariant \(SU(2)\) algebra generated by \[\hat{s}_\mu = \left( \begin{array}{cc} \hat{s}_\mu & 0 \\ 0 & \hat{s}_\mu \end{array} \right) \]
\[
\hat{s}_\mu = \frac{\sigma_\mu}{2} \text{ are the usual spin operators, } \hat{s}_\mu = U_p \hat{s}_\mu U_p \text{ and } U_p = \frac{\sigma \cdot p}{p} \text{ is a momentum-helicity unitary operator.}
\]
In the symmetry limit the Dirac eigenfunctions belong to the spinor representation of \(SU(2)\). The relativistic pseudospin symmetry determines the form of the eigenfunctions in the doublet to be
\[
\Psi_{\kappa_1 < 0, m} = \frac{1}{r} \left( \begin{array}{c} G_{\kappa_1} [Y_{\ell-1}^m \chi_j^{(j)}] \\ iF_{\kappa_1} [Y_{\ell}^m \chi_j^{(j)}] \end{array} \right) \quad \kappa_1 = -\ell < 0 , \ j = -1/2 , \quad (11a)
\]
\[
\Psi_{\kappa_2 > 0, m} = \frac{1}{r} \left( \begin{array}{c} G_{\kappa_2} [Y_{\ell+1}^m \chi_j^{(j')} = \ell + 1 > 0 , \ j' = \ell + 1/2 , \quad (11b)
\]
and imposes the following conditions on their radial amplitudes:
Figure 3. Top left panel: the upper components \( g(r) = rG_\kappa(r) \) of the \( 2s_{1/2} \) (solid line) and \( 1d_{3/2} \) (dashed line) Dirac eigenfunctions in \(^{208}\text{Pb}\). Top right panel: testing the differential relation of Eq. (12b) for the upper components of \( 2s_{1/2} \) \((\kappa_1 = -1)\) and \( 1d_{3/2} \) \((\kappa_2 = 2)\). Bottom panel: the lower components \( f(r) = rF_\kappa(r) \) of \( 2s_{1/2} \) and \( 1d_{3/2} \), testing relation (12a). Based on calculations in [12,13].

\[
F_{\kappa_1} = F_{\kappa_2}, \tag{12a}
\]

\[
\frac{dG_{\kappa_1}}{dr} + \frac{\kappa_1}{r} G_{\kappa_1} = \frac{dG_{\kappa_2}}{dr} + \frac{\kappa_2}{r} G_{\kappa_2}. \tag{12b}
\]

The two eigenstates in the doublet are connected by the pseudospin generators \( \hat{S}_\mu \) (10). The lower components are connected by the usual spin operators and, therefore, have the same spatial wave functions. Consequently, the two states of the doublet share a common \( \ell \) which is the orbital angular momentum of the lower component. The Dirac structure then ensures that the orbital angular momentum of the upper components in Eq. (11) is \( \ell = \ell' - 1 \) for \( j = \ell' - 1/2 = \ell + 1/2 \), and \( \ell + 2 = \ell + 1 \) for \( j' = \ell + 1/2 = \ell + 3/2 \).
This explains the particular angular momenta defining the pseudospin doublets in Eq. (1). The radial amplitudes of the lower components are equal \( n_F = n \). Property (a) of the previous section then ensures that \( G_{\kappa_1} \) has \( n \) nodes and \( G_{\kappa_2} \) has \( n - 1 \) nodes, in agreement with Eq. (1). Property (b) ensures that the Dirac state with \( n_F = n_G = 0 \), corresponding to the “intruder” shell-model state, has a wave function as in Eq. (11a) with \( \kappa < 0 \), and does not have a partner eigenstate (with \( \kappa > 0 \)).

Realistic mean fields in nuclei approximately satisfy condition (9) with \( \Delta_0 \approx 0 \). The required breaking of pseudospin symmetry in nuclei is small. Quasi-degenerate doublets of normal-parity states and abnormal-parity levels without a partner eigenstate persist in the spectra. The relations (12) between wave functions have been tested in numerous realistic mean field calculations in a variety of nuclei, and were found to be obeyed to a good approximation, especially for doublets near the Fermi surface \( ^{12}_1,^{13}_1 \). A representative example for neutrons in \(^{208}\)Pb is shown in Fig. 3.

### 5. Relativistic Pseudospin Symmetry and SUSY

In the pseudospin limit, Eq. (9), the two Dirac states \( \Psi_{\kappa_1<0,m} \) and \( \Psi_{\kappa_2>0,m} \) of Eq. (11) with \( \kappa_1 + \kappa_2 = 1 \) are degenerate, unless both the upper and lower components have no nodes, in which case only \( \Psi_{\kappa_1<0,m} \) is a bound state. Altogether, as shown in Fig. 4, the ensemble of Dirac states with \( \kappa_1 + \kappa_2 = 1 \) exhibits a supersymmetric pattern of twin towers with pair-wise degenerate pseudospin doublets sharing a common \( \ell \), and an additional non-degenerate nodeless state at the bottom of the \( \kappa_1 < 0 \) tower. An exception to this rule is the tower with \( \kappa_2 = 1 \) (\( p_{1/2} \) states with \( \ell = 0 \)), which is on its own, because states with \( \kappa_1 = 0 \) do not exist. The supersymmetric structure arises because in the pseudospin limit the respective radial Dirac Hamiltonians, \( H_{\kappa_1} \) and \( H_{\kappa_2} \), satisfy an intertwining relation of the form

\[
LH_{\kappa_1} = H_{\kappa_2}L,
\]

with \( \kappa_1 + \kappa_2 = 1 \). The transformation operator is found to be

\[
L = b \left( \frac{\partial}{\partial r} - \frac{\kappa_2}{r} - \frac{\kappa_2}{r} \right) \left( 2M + \Sigma + \Delta_0 \right).
\]

\( L \) connects the two doublet states

\[
L \Phi_{\kappa_1} = b(M + \Delta_0 - E) \Phi_{\kappa_2} \quad (\kappa_1 + \kappa_2 = 1) ,
\]

\( (15) \)
and identifies the two states as supersymmetric partners. Eq. (15) relies on the input that the two states are eigenstates of the Dirac Hamiltonian with $E_{\kappa_1} = E_{\kappa_2} = E$ and their wave functions satisfy the relations in Eq. (12). The fact that nuclear wave functions, obtained in realistic mean-field calculations, obey these relations to a good approximation, confirms the relevance of supersymmetry to these nuclear states.

Constructing supersymmetric charges $Q_{\pm}$ and Hamiltonian $H_{\kappa}$ from $L_{\kappa}$, $H_{\kappa_1}$, $H_{\kappa_2}$ as in Eq. (2), ensures the fulfillment of Eq. (3), except for the last relation which now reads

$$\{Q_-, Q_+\} = b^2 [H - (M + \Delta_0)] [H - (M + \Delta_0)] .$$

(16)

Eq. (16) involves a polynomial of $H$, indicating a quadratic deformation of the conventional supersymmetric algebra. The latter arises because both the Dirac Hamiltonian, $H_{\kappa}$, and the transformation operator, $L$, are of first order. In real nuclei, the relativistic pseudospin symmetry is slightly broken, implying $\Delta(r) \neq \Delta_0$ in Eq. (9). Taking $H_{\kappa}$ as in Eq. (7) and $L$ as in Eq. (14) but with $\Delta_0 \rightarrow \Delta(r)$, we now find that

$$L H_{\kappa_1} - H_{\kappa_2} L = i b \frac{d\Delta}{dr} \sigma_2$$

(17)

Furthermore, $\{Q_-, Q_+\}$ has the same formal form as in Eq. (16), but the appearance of $\Delta(r)$ instead of $\Delta_0$ implies that the anticommutator is no longer just a polynomial of $H$. Figure 4. Schematic supersymmetric pattern in the pseudospin limit of the Dirac Hamiltonian [8].
6. Summary

We discussed a possible grouping of shell-model single-particle states into larger multiplets, exhibiting a supersymmetric pattern. The multiplets involve several quasi-degenerate pseudospin doublets and intruder levels without a partner eigenstate. In contrast to previous studies of pseudospin in nuclei, the suggested grouping of nuclear states treats the intruder levels and pseudospin doublets on equal footing. The underlying supersymmetric structure is linked with an approximate relativistic pseudospin symmetry of the nuclear mean-field Dirac Hamiltonian. The relativistic pseudospin symmetry imposes relations between the upper and lower components of the two Dirac states forming the doublet. These relations, which are obeyed to a good approximation by realistic mean-field wave functions, imply that the Dirac Hamiltonian with pseudospin symmetry obeys an intertwining relation which gives rise to the indicated supersymmetric pattern.

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References