A No-go Theorem for the Majorana Fermion on the Lattice

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Abstract

A variant of the Nielsen–Ninomiya no-go theorem is formulated. The theorem states that, under several assumptions, it is impossible to write down a doubler-free Euclidean lattice action of a single Majorana fermion in $8k$ and $8k + 1$ dimensions.

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§1. Introduction

In Ref. 1), the authors including the present author examined a compatibility of the Majorana decomposition and the charge conjugation property of lattice Dirac operators by taking the Wilson–Dirac operator\(^2\) and the overlap-Dirac operator\(^3\) as example. There, it was observed that the Majorana decomposition does not work in \(8k\) and \(8k + 1\) dimensions and, as a consequence, it was impossible to obtain a physically acceptable lattice formulation of the Majorana fermion in these dimensions. The authors then argued that this difficulty associated with the Majorana fermion is a manifestation of the global gauge anomaly\(^4\) in \(8k\) dimensions.\(^5\) If this argument is correct, a physically acceptable lattice formulation of the Majorana fermion in these dimensions will be extremely difficult to find, because the difficulty has an intrinsic physical meaning.

In the present paper, we formulate a variant of the Nielsen–Ninomiya no-go theorem\(^6\),\(^7\) which states, under several assumptions, an impossibility of a lattice Majorana fermion in \(8k\) and \(8k + 1\) dimensions. Our result, being independent of a particular choice of a lattice Dirac operator and reflecting specialties of the Clifford algebra in these dimensions, strengthens the above picture that the difficulty associated to the Majorana fermion in these dimensions has a deeper origin. At the same time, the theorem will be useful in attempting to circumvent the difficulty, because our theorem specifies precise assumptions which lead to the difficulty. One has to relax one of these assumptions to find a lattice formulation of the Majorana fermion, just like one usually circumvents the Nielsen–Ninomiya theorem by relaxing the assumption on the chiral symmetry.

§2. Basic assumptions and the theorem

We consider a single free Majorana fermion defined on the infinite-extent Euclidean lattice. For this, we assume the following bi-linear form of a lattice action of the Majorana fermion (\(d\) is the dimensionality of the lattice and \(d = 8k\) or \(d = 8k + 1\))

\[
S_M = a^d \sum_x a^d \sum_y \frac{1}{2} \chi(x)^T B D(x-y) \chi(y),
\]

(2.1)

where the field \(\chi(x)\) that represents Majorana degrees of freedom is Grassmann odd. The matrix \(B\) denotes the “charge conjugation matrix” such that

\[
B \gamma_\mu B^{-1} = (-1)^n \gamma_\mu^* = (-1)^n \gamma_\mu^T,
\]

(2.2)

\[
B^{-1} = B^\dagger, \quad B^T = (-1)^n(n+1)/2 B, \quad B^* B = (-1)^n(n+1)/2,
\]

(2.3)
with \( n = [d/2] \). (This is the matrix \( B_1 \) of Ref. 1.) For even \( d = 2n \), we also define the chiral matrix \( \gamma \) by
\[
\gamma = i^{-n} \gamma_0, \gamma_1 \cdots \gamma_{2n-1}, \quad \{ \gamma, \gamma_\mu \} = 0, \quad \gamma^\dagger = \gamma, \quad \gamma^2 = 1, \quad (2.4)
\]
\[
B \gamma B^{-1} = (-1)^n \gamma^* = (-1)^n \gamma^T. \quad (2.5)
\]
We have assumed that the lattice Dirac operator \( D(x, y) \) is translational invariant, i.e., it depends only on the separation \( x - y \) of positions. We also assume the locality of the Dirac operator; the precise definition of the locality will be given later. To reproduce the Euclidean action of the Majorana fermion in the continuum theory in the classical continuum limit, one has to assume
\[
D(z) = \sum_\mu \gamma_\mu \partial_\mu \delta^d(z) + O(a). \quad (2.6)
\]
See Ref. 1) for a background of the above construction, the so-called “Majorana decomposition”.

Now, since the field \( \chi(x) \) is Grassmann odd, the Dirac operator in Eq. (2.1) must be skew-symmetric, \( D(-z)^T B^T = -BD(z) \) or*
\[
D(-z)^T = -BD(z)B^{-1}. \quad (2.7)
\]
According to the Nielsen–Ninomiya theorem, the Dirac operator \( D \) cannot be chiral invariant because otherwise the species doubling occurs under physically reasonable assumptions. Thus we have to assume \( \{ \gamma, D \} \neq 0 \). As a possible nature of this breaking of the chiral symmetry, we postulate
\[
\{ \gamma, D(-z) \}^T = +B\{ \gamma, D(z) \}B^{-1}, \quad (2.8)
\]
in addition to the fundamental requirement (2.7). We refer to the property (2.8) as the “pseudo-chiral invariance”. This property is motivated from the fact that the Wilson–Dirac operator and the overlap-Dirac operator in 8k dimensions satisfy the relation (2.8), because these Dirac operators behave as \( D(-z)^T = +B\gamma D(z)\gamma B^{-1} \).

We immediately find, however, if one requires both the skewness (2.7) and the pseudo-chiral invariance (2.8), the Dirac operator \( D \) is chiral invariant, \( \{ \gamma, D \} = 0 \). In other words, once having a Dirac operator \( D \) that possesses the pseudo-chiral invariance, one can always enforce the skewness by “anti-symmetrizing” the operator as
\[
D_A(z) = \frac{1}{2}[D(z) - B^{-1}D(-z)^TB]. \quad (2.9)
\]

*) In this paper, we understand that the transpose operation acts only on the spinor indices. The transpose operation in Ref. 1) such as \( D^T \) that acts also on position-space indices corresponds to \( D(-z)^T \) in our present notation.
Then one has $D_A(-z)^T = -BD_A(z)B^{-1}$ and $\{\gamma, D_A\} = 0$ (the latter is a special case of the pseudo-chiral invariance). For example, the anti-symmetrization of the Wilson–Dirac operator removes the Wilson term and makes it chiral invariant as observed in Ref. 1). One can also verify that, under an assumption of the pseudo-chiral invariance (2.8), the lattice action (2.1) has the chiral invariance, $\delta S_M = 0$ for $\delta \chi = \epsilon \gamma \chi$. Thus, recalling the Nielsen–Ninomiya theorem, we expect the species doubling. The precise statement is given by

**Theorem 2.1** *For the momentum representation of the free lattice Dirac operator*

$$\tilde{D}(p) = a^d \sum_z e^{-ipz} D(z), \quad (2.10)$$

*in $8k$ dimensions, the following five properties are incompatible:*

1. Locality: $\tilde{D}(p)$ is a smooth function of $p_\mu$ with the period $2\pi/a$.  
2. Correct dispersion: $\tilde{D}(p) = i \sum_\mu \gamma_\mu p_\mu + O(a p^2)$.  
3. No species doubling: $\tilde{D}(p)$ is invertible for all $p \neq 0$.  
4. Skewness: $\tilde{D}(-p)^T = -B \tilde{D}(p)B^{-1}$.  
5. Pseudo-chiral invariance: $\{\gamma, \tilde{D}(-p)\}^T = +B\{\gamma, \tilde{D}(p)\}B^{-1}$.

The proof of the theorem is trivial if one invokes the Nielsen–Ninomiya theorem. From the properties 4 and 5, one has $\{\gamma, \tilde{D}(p)\} = 0$. This chiral invariance is incompatible with the rest of properties, 1, 2 and 3, according to the Nielsen–Ninomiya theorem. □

For odd dimensions $d = 8k + 1$, if the Dirac operator $D$ has the “parity invariance” $D(-z) = -D(z)$ or $D(z) + D(-z) = 0$, one immediately encounters the species doubling as we will see shortly. Thus we instead postulate the following “pseudo-parity invariance”

$$[D(z) + D(-z)]^T = +B[D(z) + D(-z)]B^{-1}. \quad (2.11)$$

In $8k + 1$ dimensions, the Wilson–Dirac operator and the overlap-Dirac operator possess this property, because these Dirac operators satisfy $D(z)^T = +BD(z)B^{-1}$. Requiring both the skewness (2.7) and the pseudo-parity invariance (2.11), however, the parity invariance $D(-z) = -D(z)$ is resulted. In other words, having a Dirac operator $D$ that possesses the pseudo-parity invariance (2.11), one can always enforce the skewness by Eq. (2.3). Then the anti-symmetrized Dirac operator satisfies $D_A(-z) = -D_A(z)$. Our statement is

**Theorem 2.2** *For the momentum representation of the free lattice Dirac operator*

$$\tilde{D}(p) = a^d \sum_z e^{-ipz} D(z), \quad (2.12)$$

*in $8k + 1$ dimensions, the following four properties are incompatible:
1. Locality: $\tilde{D}(p)$ is a smooth function of $p_\mu$ with the period $2\pi/a$.
2. No species doubling: $\tilde{D}(p)$ is invertible for all $p \neq 0$.
3. Skewness: $\tilde{D}(-p)^T = -B\tilde{D}(p)B^{-1}$.
4. Pseudo-parity invariance: $[\tilde{D}(p) + \tilde{D}(-p)]^T = +B[\tilde{D}(p) + \tilde{D}(-p)]B^{-1}$.

One has $\tilde{D}(-p) = -\tilde{D}(p)$ from the properties 3 and 4. By substituting, say, $p_d = (\pi/a, 0, \cdots, 0)$ into this relation, one concludes that $D(p_d) = \tilde{D}(-p_d) = -\tilde{D}(p_d) = 0$ from the periodicity in the property 1. This is in contradiction with the property 2. \qed

§3. Discussion

We have to explain why a similar no-go theorem does not apply to other dimensions than $d = 8k$ and $d = 8k + 1$. The Majorana decomposition in Euclidean field theory is possible only for dimensions, $d = 0, 1, 2, 3, 4 \text{ mod } 8$. See Ref. 1). In $8k + 2$ dimensions, the skewness reads $D(-z)^T = +BD(z)B^{-1}$ and, corresponding to the pseudo-chiral invariance (2.28), we may postulate $\{\gamma, D(-z)\}^T = +B\{\gamma, D(z)\}B^{-1}$; the Wilson and the overlap Dirac operators possess this property due to $D(-z)^T = +BD(z)B^{-1}$. However, a combination of these two properties does not imply the chiral invariance. In fact, one sees that these two properties are the same relation. Similarly, in $8k + 4$ dimensions, the skewness reads $D(-z)^T = +B\gamma D(z)\gamma B^{-1}$ and the pseudo-chiral invariance will be replaced by $\{\gamma, D(-z)\}^T = +B\gamma\{\gamma, D(z)\}\gamma B^{-1} = +B\{\gamma, D(z)\}B^{-1}$; the Wilson and the overlap Dirac operators in fact possess this property due to $D(-z)^T = +B\gamma D(z)\gamma B^{-1}$. A combination of these two again does not imply the chiral invariance because these two are the same relation. For odd $8k + 3$ dimensions, the skewness reads $D(-z)^T = +BD(z)B^{-1}$. The pseudo-parity invariance (2.11) is replaced by $[D(z) + D(-z)]^T = +B[D(z) + D(-z)]B^{-1}$; the Wilson–Dirac operator possesses this property due to $D(-z)^T = +BD(z)B^{-1}$; the overlap-Dirac operator does not have such a simple property and cannot be utilized in these dimensions.\(^1\) These two properties are however the same and does not lead to the parity invariance. All the above facts are reflections of properties of gamma matrices in each dimension and are consistent with the fact that we had a successful lattice Majorana decomposition in dimensions other than $8k$ and $8k + 1$.\(^1\)

In Ref. 1), it was argued that the difficulty of the Majorana decomposition in $8k$ and $8k + 1$ dimensions is a manifestation of the global gauge anomaly in $8k$ dimensions. For this argument, an equivalence of the Majorana fermion and the Weyl fermion in a real representation in $8k$ dimensions, that holds in the Minkowski spacetime and in the unregularized Euclidean theory, was crucial. (A similar equivalence in $8k + 4$ dimensions holds even in Euclidean lattice gauge theory.\(^8\)) In Euclidean theories, the following correspondence is
suggested:
\[ \chi(x) = \psi(x) + B^{-1}\overline{\psi}(x)^T, \quad (3.1) \]
where \( \psi(x) \) is the (left-handed) Weyl fermion,
\[ \frac{1 - \gamma}{2} \psi(x) = \psi(x), \quad \overline{\psi(x)} \frac{1 + \gamma}{2} = \overline{\psi(x)}. \quad (3.2) \]
If one substitutes the expression (3.1) into the action (2.1) and assumes the pseudo-chiral invariance (2.8), one then ends up with the action of the Weyl fermion
\[ S_M = a^d \sum_x a^d \sum_y \overline{\psi(x)}D(x-y)\psi(y). \quad (3.3) \]
This is consistent with the expected equivalence between the Majorana fermion and the Weyl fermion in 8k dimensions. As we have observed, however, the above action suffers from the species doubling (because we have assumed the pseudo-chiral invariance) and cannot be utilized as it stands.

On the basis of this expected equivalence between Majorana and Weyl fermions in 8k dimensions, one might think that it could be possible to define a theory of the Majorana fermion in 8k dimensions by using the lattice Weyl fermion through Eq. (3.1). For a review on recent developments on lattice Weyl fermions with an extensive list of references, see, Ref. 9). In this approach, the partition function of the Majorana fermion is given by the partition function of the Weyl fermion and the correlation functions of the Majorana fermion are defined through Eq. (3.1) from correlation functions of the Weyl fermion. The two-point function of the Weyl fermion is given by
\[ \langle \psi(x)\overline{\psi(y)} \rangle = \hat{P}_- \frac{1}{D} P_+(x,y), \quad (3.4) \]
where \( \hat{P}_- = (1 - \hat{\gamma})/2 \) and \( \hat{\gamma} = \gamma(1 - aD) \); the Dirac operator \( D \) is supposed to satisfy the Ginsparg–Wilson relation \( \gamma D + D\gamma = aD\gamma D. \) By using Eq. (3.1), we have
\[ \langle \chi(x)\chi(y)^T \rangle = \frac{1}{BD} \frac{1}{2} a^{-d} \delta_{x,y}, \quad (3.5) \]
after some manipulation, or by taking the anti-symmetric part of the right hands side,
\[ \langle \chi(x)\chi(y)^T \rangle = \frac{1}{BD_A} (x,y). \quad (3.6) \]
If the overlap-Dirac operator is utilized as the Dirac operator \( D \), the propagator (3.6) acquires doubler’s poles as we have observed. So this natural approach based on the lattice Weyl fermion as it stands does not remove the difficulty.
We hope that our no-go theorem will be useful in investigating a possible solution to the difficulty concerning lattice Majorana fermions in $8k$ and $8k+1$ dimensions.\textsuperscript{1}) It seems rather non-trivial to avoid this difficulty of lattice Majorana fermions, due to its possible connection to the global gauge anomaly. We should always keep in mind, however, an epigram, “No-go theorems, however, are frequently circumvented in an unexpected way”.\textsuperscript{11)}

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References

1) T. Inagaki and H. Suzuki, hep-lat/0406026.
11) P. Hasenfratz, hep-lat/0406033.

\textsuperscript{1)} For this, it would be useful to note the fact that, in $8k$ and $8k+1$ dimensions, there exists a representation of the Clifford algebra such that all the gamma matrices are real symmetric and $B = 1$. 

\textsuperscript{11)}