How can exact and approximate solutions of Einstein’s field equations be compared?

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Abstract

The problem of comparison of the stationary axisymmetric vacuum solutions obtained within the framework of exact and approximate approaches for the description of the same general relativistic systems is considered. We suggest two ways of carrying out such comparison: (i) through the calculation of the Ernst complex potential associated with the approximate solution whose form on the symmetry axis is subsequently used for the identification of the exact solution possessing the same multipole structure, and (ii) the generation of approximate solutions from exact ones by expanding the latter in series of powers of a small parameter. The central result of our paper is the derivation of the correct approximate analogues of the double–Kerr solution possessing the physically meaningful \textit{equilibrium configurations}. We also show that the interpretation of an approximate solution originally attributed to it on the basis of some general physical suppositions may not coincide with its true nature established with the aid of a more accurate technique.

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1. Introduction

Approximation methods had long proved to be an efficient tool for treating specific general relativistic problems. For many years their use was predominant since only a restricted number of physically realistic exact solutions of Einstein’s equations was available before the development of modern solution generating techniques [1–7]. But even now, when many exact solutions have already been constructed, approximation methods are frequently used for attacking some interesting problems not yet solved exactly. Among the relatively recent successful applications of the approximation schemes one can mention the analysis of the frame–dragging effect in the field of a static charged massive magnetic dipole [8] with the aid of the Martin–Pritchett approximate solution [9], or a study of the equilibrium problem of two static charged masses using a test–particle approach [10]. The exact solutions which are in accord with the above approximate findings were constructed later on [11, 12].

In spite of a rather peaceful coexistence of exact and approximation approaches it appears that there does exist a problem upon which the two approaches had produced some not fully concurring results – the vacuum equilibrium problem of two spinning particles. There is only agreement of the exact results with the well–known paper of Wald [13] in which he proved within the framework of a test–particle approach that the spin–spin interaction cannot balance the gravitational attraction of two point–like particles. In his effort to develop a far–reaching successive approximation procedure which would go beyond the paper [13], Bonnor derived an approximate solution [14] which he considered applicable to any two spinning uncharged particles, black holes or superextreme objects. In that solution, however, the balance was not achievable too, this time producing an apparent contradiction with the known exact results. Indeed, it is well known that equilibrium states of two superextreme particles do exist in the double–Kerr solution first constructed by Kramer and Neugebauer [15]. The general analytic solution of the extended double–Kerr equilibrium problem was given in [16, 17], and in [19] a simple relation was found between the coordinate distance at which the balance occurs and the masses and angular momenta of the balancing Kerr constituents.

1Equilibrium states of two identical superextreme Kerr constituents were first obtained and analyzed by Dietz and Hoenselaers [18].
This gives rise to an important question: how can one compare exact and approximate solutions aimed at describing the same physical situations? It is very important to have an answer to this question for being able to interpret approximate solutions correctly, establish limits of their validity, and give a recipe for the rectification of an approximation scheme which for some reasons has turned out to be incompatible with an exact model. In this paper we shall develop an effective approach to the comparison of the exact and approximate solutions based on the following mutually complementary procedures: (i) identification of the exact analogue of an approximate solution through the axis expression of the Ernst potential [20] of the approximate solution, and (ii) generation of the approximate solution from an exact one by expanding the Ernst potential of the exact solution in a series of powers of a small parameter with a subsequent comparison of the result with the known approximate solution. Whereas the procedure (i) provides one with a precise interpretation of the approximate solution (which may not coincide with the interpretation initially attributed to it), the procedure (ii) allows the construction of the approximate solution best matching to the exact one. To illustrate our approach, we shall apply it to the classical problem of two spinning particles, taking as reference points the recent Bonnor’s approximate solution [14] and the exact double–Kerr solution in its extended form [17]. Note that the solution [14] was claimed to be appropriate for the approximate description of two Kerr particles, but we will show that this affirmation is erroneous.

The paper is organized as follows. In Section 2 we derive the Ernst potential of the Bonnor solution and give the interpretation of the latter by means of the corresponding exact solution. In Section 3 we construct the approximate double–Kerr solution in the subextreme case not admitting equilibrium configurations. Section 4 deals with the approximate double–Kerr solutions involving one or two superextreme particles for which the corresponding equilibrium problems can be solved in full analogy with the exact solution. In Section 5 we establish the relation of the parameters entering approximate solutions to the individual Komar masses and angular momenta of the particles. Section 6 presents a discussion of the results obtained. Concluding remarks are contained in Section 7.
2. Interpretation of Bonnor's approximate solution

In this section we will show how a precise physical interpretation of an approximate axisymmetric solution can be established via an appropriately identified exact solution whose properties can be ascribed to the approximate solution. Though the existence of a solid basis for finding such an exact analogue may look questionable at first sight, this is fortunately not the case. The well-known Sibgatullin’s integral method of the construction of exact solutions [7] permits one to generate axisymmetric spacetimes from the axis expressions of their Ernst complex potentials [20] which are used as the initial data in the generation process. This means that all one needs for having at hand an exact analogue of a known approximate solution is to establish the form of the Ernst potential on the symmetry axis of the latter; Sibgatullin’s method will then provide one with the corresponding exact solution possessing the same axis expression of the Ernst potential, and its properties can be studied with the aid of the well established procedures.

This scheme can be used for establishing the physical interpretation of the approximate vacuum solution for two spinning particles [14]. Since this solution was obtained in a somewhat special form, we first of all have to rewrite it using the canonical axisymmetric line element

\[ ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2, \]  

(1)

where the metric coefficients \( f, \gamma \) and \( \omega \) depend on the coordinates \( \rho, z \) only; \( \varphi \) and \( t \) are the azimuthal angle and time, respectively.

Observing that the line element (1) of [14] can be cast, after changing the signature, into the form

\[ ds^2 = f^{-1}[e^{\nu}(dr^2 + dz^2) + r^2 d\theta^2] - f(dt - n f^{-1} d\theta)^2 - f^{-1}(r^2 - n^2 - l f) d\theta^2, \]  

(2)

we obtain formulae relating notations of the paper [14] and ours:

\[ r = \rho, \quad z = z, \quad \nu = 2\gamma, \quad n = f \omega, \quad r^2 - n^2 - l f = 0. \]  

(3)

Now, bearing in mind [3] and using formulae (14)-(20) of Ref. [14], we are able to rewrite the Bonnor solution in its standard form:

\[ f = 1 - 2l \left( \frac{m_1}{R_1} + \frac{m_2}{R_2} \right) + 2l^2 \left( \frac{m_1}{R_1} + \frac{m_2}{R_2} \right)^2 \]

\[ \text{Throughout the paper units are used in which Newton’s gravitational constant } G \text{ and the speed of light } c \text{ are equal to unity.} \]
\[
\gamma = \frac{1}{2} l^2 \left( \frac{3 j_1 j_2 - 2 b^2 m_1 m_2}{2 b^4} - \frac{m_1^2 \rho^2}{R_1^2} - \frac{m_2 \rho^2}{R_2^2} + \frac{m_1 m_2 (\rho^2 + z^2 - b^2)}{b^2 R_1 R_2} \right)
\]

\[
\omega = -2 l \rho^2 \left( \frac{j_1}{R_1^2} + \frac{j_2}{R_2^2} \right) + l^2 \left[ \frac{m_1 j_2 + m_2 j_1}{b^2} - 2 \rho^2 \left( \frac{m_1 j_1}{R_1^4} + \frac{m_2 j_2}{R_2^4} \right) \right]
\]

\[
R_1 = \sqrt{\rho^2 + (z - b)^2}, \quad R_2 = \sqrt{\rho^2 + (z + b)^2}.
\]

In the above formulae we have left Bonnor’s notations for the masses \(m_1, m_2\) but instead of his \(h_1, h_2\) we have used ours \(j_1 = -h_1, j_2 = -h_2\) as angular momenta of the particles; the parameter \(b\) defines location of the particles on the symmetry axis (the points \(z = \pm b\)). We have introduced an auxiliary nondimensional small parameter \(l\) by the formulae

\[
f = f^{(0)} + l f^{(1)} + l^2 f^{(2)}, \quad \nu = \nu^{(0)} + l \nu^{(1)} + l^2 \nu^{(2)},
\]

\[
n = n^{(0)} + l n^{(1)} + l^2 n^{(2)}
\]

\((f^{(i)}, \nu^{(i)}, n^{(i)})\) are defined in [14], index \(i\) denoting the order of approximation, and it helps to control the order of approximation in Bonnor’s scheme (one can always put \(l = 1\) at the end). Note that the metric functions \((4)\) cannot contain terms with \(l^n, n > 2\), because the solution is given in the second approximation. Mention also that the masses \(m_i\) and angular momenta \(j_i\) are quantities of the same (first) order in \(l\).

The metric coefficients \(f, \gamma, \omega\) defined by \((4)\) are approximate solutions to Einstein’s equations in the stationary axisymmetric case (see, e.g., Section 7.1 of [21])

\[
f \Delta f = (\nabla f)^2 - \rho^{-2} f^4 (\nabla \omega)^2,
\]
\[ \vec{\nabla}(\rho^{-2} f^{2} \vec{\nabla} \omega) = 0, \]
\[ \gamma_{,\rho} = \frac{1}{4} \rho f^{-2} [f_{,\rho}^{2} - f^{2}_{,z} - \rho^{-2} f^{4}(\omega_{,\rho}^{2} - \omega_{,z}^{2})], \]
\[ \gamma_{,z} = \frac{1}{2} \rho f^{-2} (f_{,\rho} f_{,z} - \rho^{-2} f^{4} \omega_{\rho} \omega_{z}^{2}), \]  \hfill (6)

where a comma in subindices denotes partial differentiation, and the operators \( \Delta, \vec{\nabla} \) have the form
\[ \Delta = \frac{\partial^{2}}{\partial \rho^{2}} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^{2}}{\partial z^{2}}; \quad \vec{\nabla} = \vec{\rho}_{0} \frac{\partial}{\partial \rho} + \vec{z}_{0} \frac{\partial}{\partial z} \]  \hfill (7)

(\( \vec{\rho}_{0} \) and \( \vec{z}_{0} \) are unit vectors).

The first two equations in (6) can be rewritten as a single Ernst equation for a complex potential \( \mathcal{E} \) \[ (\mathcal{E} + \bar{\mathcal{E}}) \Delta \mathcal{E} = 2(\vec{\nabla} \mathcal{E})^{2}, \]  \hfill (8)
a bar over a symbol denoting complex conjugation.

The relation of \( \mathcal{E} \) to the functions \( f \) and \( \omega \) is the following:
\[ \mathcal{E} = f + i \Omega, \quad \Omega_{,\rho} = \rho^{-1} f^{2} \omega_{,z}, \quad \Omega_{,z} = -\rho^{-1} f^{2} \omega_{,\rho}. \]  \hfill (9)

Sibgatullin’s method \[ \text{[7]} \] provides a straightforward procedure for the construction of exact solutions of equation (8) corresponding to a given function \( e(z) = \mathcal{E}(\rho = 0, z) \) representing the axis value of the Ernst potential \( \mathcal{E} \). Therefore, in order to identify the exact analogue of Bonnor’s approximate solution \[ \text{[4]} \], we have to find from \[ \text{[4]} \] and \[ \text{[9]} \] the axis value \( e_{B}(z) \) of the corresponding Ernst potential in the second approximation. Having obtained \( e_{B}(z) \), we shall be able to point out what exact solution corresponds to this function and, besides, to use \( e_{B}(z) \) for the calculation of the multipole moments of both the approximate and exact solutions with the aid of the Fodor \( et \) al. procedure \[ \text{[22]} \].

The integration of equations \[ \text{[9]} \] for \( \Omega \), with \( f \) and \( \omega \) defined in \[ \text{[4]} \], can be readily performed using the Mathematica computer programme \[ \text{[23]} \]. The resulting expression, multiplied by \( i \), should then be added to \( f \), yielding the desired Ernst potential of the Bonnor solution
\[ \mathcal{E}_{B} = 1 - 2l M + 2l^{2}(M^{2} - R^{2}) - 2ilR \]
\[ + 4il^{2} \left( \frac{z(m_{1}j_{2} + m_{2}j_{1})(\rho^{2} + z^{2} - b^{2}) + b(m_{1}j_{2} - m_{2}j_{1})(\rho^{2} - z^{2} + b^{2})}{R_{1}^{3}R_{2}^{3}} \right) \]
\[ M = \frac{m_1}{R_1} + \frac{m_2}{R_2}, \quad R = \frac{j_1(z - b)}{R_1^3} + \frac{j_2(z + b)}{R_2^3}. \]  

(10)

On the upper part of the symmetry axis, \( \rho = 0, z > b \), the potential obtained takes the form

\[ e_B(z) = E_B(\rho = 0, z) = 1 - \frac{2lN}{(z^2 - b^2)^2} + \frac{2l^2N^2}{(z^2 - b^2)^4}, \]

\[ N = (z^2 - b^2)[z(m_1 + m_2) + b(m_1 - m_2)] + i(j_1 + j_2)(z^2 + b^2) + 2ibz(j_1 - j_2). \]  

(11)

Expression (11) is the ratio of two polynomials of the eighth order in \( z \), so the exact solution corresponding to this axis data necessarily falls into the family of \( 2N \)–soliton solutions discussed in detail in [24]. Moreover, as the denominator of \( e_B(z) \) is the real function of \( z \), it should be a special member of the \( N = 4 \) subfamily of the soliton solutions which in the general case involves 16 real parameters, contrary to only 5 parameters of the axis data (11).

To see what physical situation describes the exact solution whose Ernst potential on the symmetry axis has the form (11), one has to recall that after the application of Sibgatullin’s method to the data (11) the resulting solution will contain the functions \( r_i = \sqrt{\rho^2 + (z - \alpha_i)^2} \), where \( \alpha_i \) are roots of the algebraic equation

\[ e_B(z) + \bar{e}_B(z) = 0. \]  

(12)

Apparently, this equation has 8 roots which can assume real values or occur in complex conjugate pairs, hence the solution will describe special configurations of 4 spinning objects because each pair of \( \alpha_i \) determines one sub- or one superextreme constituent.

For a more precise analysis we need the explicit form of \( \alpha_i \). Obviously, equation (12) cannot be solved exactly, but its approximate roots up to the second order in \( l \) are available. These should be searched for in the form \( \pm b + z_1l^{1/2} + z_2l + z_3l^{3/2} + z_4l^2 \), the constant coefficients \( z_1, z_2, z_3, z_4 \) to be found from (12). Then one gets the following 8 roots of equation (12):

\[ \alpha_1 = b + \sqrt{j_1}l^{1/2} + A_1l^{3/2} + A_{12}l^2, \]
\[ \alpha_2 = b - i\sqrt{j_1 l^{1/2}} + iA_1 l^{3/2} + A_{12} l^2, \]
\[ \alpha_3 = b + i\sqrt{j_1 l^{1/2}} - iA_1 l^{3/2} + A_{12} l^2, \]
\[ \alpha_4 = b - \sqrt{j_1 l^{1/2}} - A_1 l^{3/2} + A_{12} l^2, \]
\[ \alpha_5 = -b + \sqrt{j_2 l^{1/2}} + A_2 l^{3/2} - A_{12} l^2, \]
\[ \alpha_6 = -b - i\sqrt{j_2 l^{1/2}} + iA_2 l^{3/2} - A_{12} l^2, \]
\[ \alpha_7 = -b + i\sqrt{j_2 l^{1/2}} - iA_2 l^{3/2} - A_{12} l^2, \]
\[ \alpha_8 = -b - \sqrt{j_2 l^{1/2}} - A_2 l^{3/2} - A_{12} l^2, \]
\[ A_1 = \frac{2b^2 m_1^2 + j_1 j_2}{8b^2 \sqrt{j_1}}, \quad A_2 = \frac{2b^2 m_2^2 + j_1 j_2}{8b^2 \sqrt{j_2}}, \quad A_{12} = \frac{2b^2 m_1 m_2 - j_1 j_2}{8b^3}, \]

which means that independently of the sign of \( j_1 \) and \( j_2 \) these \( \alpha \)s determine a system formed by two sub- and two superextreme constituents. More precisely, we have a system of two separated compound objects, each of which is composed of the overlapping sub- and superextreme constituents. One easily arrives at this conclusion by analyzing for instance the expressions for \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \), of which \( \alpha_1 \) and \( \alpha_4 \) define a subextreme constituent (assuming \( j_1 > 0 \)), whereas \( \alpha_2 \) and \( \alpha_3 \) define a superextreme constituent; these two overlapping constituents form a compound spinning object. Another compound object is formed by a sub- and a superextreme constituents defined by the constants \( \alpha_5, \alpha_6, \alpha_7, \alpha_8 \) (see Fig. 1).

Therefore, the physical interpretation of the Bonnor solution (11) supplied by the corresponding exact solution is the following: it represents the second approximation to a special case of the \( N = 4 \) subfamily of multi–soliton exact solutions describing two compound objects located on the symmetry axis and kept apart by a strut in between. Such systems of two compound objects have been considered in the paper [25].

It should be also pointed out that Bonnor’s solution cannot be considered as an approximation to the well–known double–Kerr spacetime [15]. Although the general \( N = 4 \) soliton metric (the quadruple–Kerr solution) possesses 16 arbitrary real parameters and permits in principle the reduction to the simpler \( N = 2 \) double–Kerr case, such a reduction is not possible for \( \alpha \)s defined in (13) because of the insufficient number of constants required for performing the desired reduction. Correct approximations to the double–Kerr solution will be obtained in the next two sections.

To conclude this part, below we write down the first five complex mul-
tipole moments of the Bonnor solution calculated from \( \text{11} \) with the aid of the Fodor et al. procedure \cite{22}:

\[
\begin{align*}
P_0 &= l(m_1 + m_2), \\
P_1 &= l[b(m_1 - m_2) + i(j_1 + j_2)], \\
P_2 &= bl[b(m_1 + m_2) + 2i(j_1 - j_2)], \\
P_3 &= b^2l[b(m_1 - m_2) + 3i(j_1 + j_2)], \\
P_4 &= b^3l[b(m_1 + m_2) + 4i(j_1 - j_2)].
\end{align*}
\] (14)

The form of these multipoles confirms the interpretation of \( m_1, m_2 \) as masses, and \( j_1, j_2 \) as angular momenta. However, a more subtle analysis is needed to see whether \( m_i \) and \( j_i \) coincide with the individual Komar masses and angular momenta \cite{26} of the particles (see our Section 5).

3. Approximate double–Kerr solution in the black–hole case

When a problem has an exact solution of a complicated form, one may think about the conversion of that solution into an approximate one under some simplifying assumptions. The generation of approximate solutions from the already known exact solutions has obvious advantages compared to the development of approximation schemes exclusively on the basis of some general suppositions and physical intuition, first of all because the approximation in the former case will not contradict the existing exact results but will only permit their presentation in a simpler form. On the other hand, there is always a danger that an approximate method not orientated at the exact solution will lead to a result contradicting the exact findings and, besides, will be unable to identify the origin of the contradiction.

Usually the exact solutions constructed with the aid of the modern solution generating techniques, especially those describing the many–body configurations, involve the parameter sets most adjusted to the mathematical structure of the solutions but which may look “unphysical” to non–experts of the generation methods. At the same time, a rather sophisticated relation of the parameters of the generated solutions to the physical characteristics of spacetimes they describe certainly may course difficulties for introducing the simplifying physical assumptions for a possible approximate analysis of the generated solutions.

The aim of this section and of the next one is to demonstrate that within the framework of Sibgatullin’s method it is possible not only to construct exact solutions but also develop a consistent approach to the analysis of their
physical properties on the basis of reliable approximation schemes. Below we shall obtain approximate analogues of the well–known double–Kerr solution [15] making use of the unique opportunities offered by Sibgatullin’s method – the possibility of the unified treatment of the sub- and superextreme cases of spinning particles and a very clear relations of all the parameters of the solution to the axis data and, consequently, to the relativistic multipole moments. The Ernst potential of the extended double–Kerr solution has the form [17]

\[ E = \frac{\Lambda + \Gamma}{\Lambda - \Gamma}, \quad \Lambda = \sum_{1 \leq i < j \leq 4} \lambda_{ij} r_i r_j, \quad \Gamma = \sum_{i=1}^{4} \nu_i r_i, \]

\[ \lambda_{ij} = (-1)^{i+j} (\alpha_i - \alpha_j)(\alpha_{i'} - \alpha_{j'}) X_i X_j, \quad (i', j' \neq i, j; \ i' < j'), \]

\[ \nu_i = (-1)^i (\alpha_i - \alpha_{i'})(\alpha_{i'} - \alpha_{k'}) (\alpha_{j'} - \alpha_{k'}) X_i, \quad (i', j', k' \neq i; \ i' < j' < k') \]

\[ X_i = \frac{(\alpha_i - \beta_1)(\alpha_i - \beta_2)}{(\alpha_i - \beta_1)(\alpha_i - \beta_2)}, \quad r_i = \sqrt{\rho^2 + (z - \alpha_i)^2}, \quad (15) \]

The parameters entering (15) are \( \beta_1, \beta_2 \) which can assume arbitrary complex values, and \( \alpha_i, \ i = 1, 2, 3, 4, \) which can take arbitrary real values or occur in complex conjugate pairs (without loss of generality we can assign to \( \alpha_i \) the following order: \( \text{Re} \alpha_1 \geq \text{Re} \alpha_2 \geq \text{Re} \alpha_3 \geq \text{Re} \alpha_4 \)). On the symmetry axis \( E \) is defined by the expression

\[ e(z) = 1 + \frac{e_1}{z - \beta_1} + \frac{e_2}{z - \beta_2}, \quad (16) \]

where \( e_1 \) and \( e_2 \) are two arbitrary complex constants.

Note that \( \alpha_i \) are roots of Sibgatullin’s algebraic condition

\[ e(z) + \bar{e}(z) = 0, \quad (17) \]

which means that the following relation holds:

\[ 2 + \sum_{l=1}^{2} \left( \frac{e_l}{z - \beta_l} + \frac{\bar{e}_l}{z - \beta_l} \right) = \frac{2 \prod_{n=1}^{4} (z - \alpha_n)}{\prod_{k=1}^{2} (z - \beta_k)(z - \beta_k)}, \quad (18) \]

whence one obtains the connection between the parameters \( e_l \) and \( \alpha_n \)

\[ e_1 = \frac{2 \prod_{n=1}^{4} (\beta_1 - \alpha_n)}{(\beta_1 - \beta_2)(\beta_1 - \beta_3)(\beta_1 - \beta_4)}, \quad e_2 = \frac{2 \prod_{n=1}^{4} (\beta_2 - \alpha_n)}{(\beta_2 - \beta_1)(\beta_2 - \beta_3)(\beta_2 - \beta_4)}. \quad (19) \]
In the papers [17, 19] we presented the general solution to the equilibrium problem of two Kerr particles and found a simple relation between the coordinate distance at which equilibrium takes place and individual Komar masses and angular momenta of the balancing constituents. The question which now would be of interest to answer in view of the paper [14] is whether the double–Kerr equilibrium problem can be solved with the aid of some approximation procedure? Below we are going to show that the answer to this question is ‘yes’ and it is supported by an approximation scheme which is able to reproduce the exact results.

In order to develop the desired physically meaningful approximation procedure for the double–Kerr solution it is likely first of all to rewrite the formulae (15) in a more “physical” parameter set which would be mathematically equivalent to the parameter set used in (15). The reparametrization of the axis data (16) can be done in analogy with a single Kerr solution [27] whose axis expression of the potential \( \xi = (1 - E)/(1 + E) \) has the form\(^3\)

\[
x(z) = \frac{1 - e(z)}{1 + e(z)} = \frac{m}{z - ia}, \tag{20}
\]

where \( m \) is the total mass and \( a \) is the angular momentum per unit mass of the Kerr source.

In the case of the double–Kerr solution we simply generalize (20) to the expression

\[
x(z) = \frac{m_1 + iv}{z - b - ia_1} + \frac{m_2 - iv}{z + b - ia_2}, \tag{21}
\]

taking \( m_1, m_2 \) as the “masses” of the Kerr particles, and \( a_1, a_2 \) as their “angular momenta per unit mass”; the parameter \( b \) denotes displacements of the particles from the origin of coordinates along the \( z \)–axis, and \( \nu \) represents the angular momentum of the part of the symmetry axis separating the particles.

From (21) we obtain the corresponding expression of the Ernst potential \( \mathcal{E} \)

\[
e(z) = \frac{1 - x(z)}{1 + x(z)} = \frac{e_-}{e_+},
\]

\(^3\)The potential \( \xi = (\mathcal{E} - 1)/(\mathcal{E} + 1) \) was first introduced by Ernst in his famous paper [20]. Changing the sign in the original expression is explained by the successful use of the modified potential in the Fodor et al. procedure [22] for the calculation of multipole moments.
\[ e_{\pm} = (z - b - ia_1)(z + b - ia_2) \]
\[ \pm (m_1 + iv)(z + b - ia_2) \pm (m_2 - iv)(z - b - ia_1). \] (22)

From (21) the expressions of the total mass \( M \) and total angular momentum \( J \) of the system can be readily found with the aid of the Fodor et al. procedure:
\[ M = m_1 + m_2, \quad J = m_1 a_1 + m_2 a_2 + 2bv. \] (23)

This supports our physical interpretation of the parameters, even though their precise relation to the individual Komar masses and angular momenta will be established later on in Section 5. It is worth pointing out that the parametrization (22) of the Ernst potential involves 6 arbitrary real parameters and it has one advantage compared to the axis expression (16) which is the absence of the unphysical NUT parameter that guarantees the asymptotic flatness of the solution from the very beginning. Up to this NUT parameter, the two parametrizations are totally equivalent in a sense that both of them represent all possible stationary asymptotically flat configurations of two Kerr particles.

Now it is possible to develop an approximation scheme as follows. We can assume the values of the parameters \( m_1, m_2, a_1, a_2, \nu \) to be small compared to the values of \( b \), which is equivalent to considering them proportional to some auxiliary non–dimensional small parameter \( l \). Then an approximate expression of the potential \( E \) can be found as an expansion in powers of \( l \) up to the desired order. Technically this consists in (a) finding approximate expressions for \( \alpha_n \) from equation (17), (b) identifying the expansions for \( X_n \) from the formulae (15), (22) in which the approximate expressions for \( \alpha_n \) should be used, (c) obtaining the expansions for \( r_n \) corresponding to approximate \( \alpha_n \), and finally (d) the expansion of the potential \( E \) in powers of \( l \) after the substitution of \( \alpha_n, X_n, r_n \) previously obtained into the formulae (15). Once the potential \( E \) is found, the approximate expressions for the metric coefficients \( \omega \) and \( \gamma \) can be derived from equations (6) and (9) (note that in spite of the availability of the exact expressions for \( \omega \) and \( \gamma \) it looks more attractive to carry out additional integration instead of working out the expansions of the known formulae).

We have to distinguish between the following three major approximation schemes which must be treated separately.

**Scheme 1.** All five parameters \( m_1, m_2, a_1, a_2, \nu \) are assumed to be small compared to \( b \). In this case, making the substitutions \( m_i \to lm_i, a_i \to la_i, \)
\( \nu \rightarrow l\nu \) in \([22]\), we see that Sibgatullin’s condition \([17]\) at the zeroth order in \( l \) yields the equation
\[
(z^2 - b^2)^2 = 0,
\]
whose double roots \(+b\) and \(-b\) should be taken as the zeroth approximation for the parameters \( \alpha_i \): \( \alpha_1 = \alpha_2 = b \), \( \alpha_3 = \alpha_4 = -b \) (see Fig. 2a).

**Scheme 2.** In this scheme, \( m_1, m_2, \nu \) are small parameters compared to \( b \), while \( a_1 \) and \( a_2 \) are of the zeroth order in \( l \), i.e., ‘comparable’ with \( b \). After the substitutions \( m_i \rightarrow lm_i, \nu \rightarrow l\nu \), the zeroth order in \( l \) of equation \([17]\) takes the form
\[
(z^2 - 2bz + b^2 + a_1^2)(z^2 + 2bz + b^2 + a_2^2) = 0,
\]
and the roots of this equation provide us with the zeroth approximation of \( \alpha_i \): \( \alpha_1 = b - ia_1 \), \( \alpha_2 = b + ia_1 \), \( \alpha_3 = -b - ia_2 \), \( \alpha_4 = -b + ia_2 \) (Fig. 2b).

**Scheme 3.** In this scheme we assume that only one of the constants \( a_1, a_2 \) is comparable with \( b \), say, \( a_2 \), whereas \( m_1, m_2, a_1, \nu \) are small parameters. Equation \([17]\) in this case leads to
\[
(z - b)^2(z^2 + 2bz + b^2 + a_2^2) = 0,
\]
giving the following zero-order approximations for \( \alpha_i \): \( \alpha_1 = \alpha_2 = b \), \( \alpha_3 = -b - ia_2 \), \( \alpha_4 = -b + ia_2 \) (Fig. 2c).

Remarkably, the schemes 1–3 correspond to the configurations composed of two subextreme constituents, of two superextreme constituents, and of one sub- and one superextreme constituents of the extended exact double–Kerr solution, respectively (we remind that in the exact solution a pair of real–valued \( \alpha \)s defines a subextreme constituent, and a pair of complex conjugate \( \alpha \)s defines a superextreme Kerr constituent). Mention also that only the first scheme can be considered as a sort of a “point–like–particles approximation” since, as will be seen later on, it involves two real distances \( R_1 = \sqrt{\rho^2 + (z - b)^2} \) and \( R_2 = \sqrt{\rho^2 + (z + b)^2} \), whereas the remaining schemes involve, respectively, four and three different square roots \( R_i \) due to the presence of two (scheme 2) and one (scheme 3) superextreme objects whose absolute values of the angular momenta per unit mass exceed the individual masses.

Below we shall work out in detail the approximation scheme 1; schemes 2 and 3 will be considered in the next section.

**Approximate double–Kerr solution via scheme 1.** Without loss of generality we can set \( b = 1 \) in the previous formulae (this simply means the rescaling...
of the parameters and coordinates) for simplifying a bit the expressions we are going to obtain. Like the paper [14], we shall be interested in the expansion of \( \mathcal{E} \) and corresponding metric functions in powers of \( l \) up to \( l^2 \) (the second approximation). Then we have to calculate \( \alpha_i \) up to \( l^3 \) for obtaining the correct expressions for \( X_i \) in the second order of \( l \). After introducing the factor \( l \) into the parameters \( m_i, a_i, \nu \), equation (17) takes the form

\[
(z^2 - 1)^2 - l^2[(m_1^2 - a_1^2)(z + 1)^2 + (m_2^2 - a_2^2)(z - 1)^2 \\
+ 2m_1m_2(z^2 - 1) + 4\nu^2] + 2\nu l^3(m_1 + m_2)[a_1(z + 1) \\
- a_2(z - 1)] - l^4[m_1^2 a_2^2 + m_2^2 a_1^2 + \nu^2(a_1 - a_2)^2 \\
+ a_1 a_2(2m_1 m_2 - a_1 a_2)] = 0. \tag{27}
\]

The roots of this equation can be searched for in the form \( \pm 1 + z_1 l + z_2 l^2 + z_3 l^3 \), the coefficients \( z_i \) to be determined from (27). The resulting expressions for \( \alpha_i \) are the following (in view of the cumbersome coefficients \( z_3 \) we write out \( \alpha_i \) as only up to the terms \( l^2 \), but it should be remembered that the coefficients \( z_3 \) cannot be omitted during the calculations):

\[
\alpha_1 = 1 + l\sqrt{m_1^2 + \nu^2 - a_1^2} + \frac{l^2}{2} \left( m_1 m_2 - \nu^2 - \frac{a_1 \nu(m_1 + m_2)}{\sqrt{m_1^2 + \nu^2 - a_1^2}} \right),
\]

\[
\alpha_2 = 1 - l\sqrt{m_1^2 + \nu^2 - a_1^2} + \frac{l^2}{2} \left( m_1 m_2 - \nu^2 + \frac{a_1 \nu(m_1 + m_2)}{\sqrt{m_1^2 + \nu^2 - a_1^2}} \right),
\]

\[
\alpha_3 = -1 + l\sqrt{m_2^2 + \nu^2 - a_2^2} - \frac{l^2}{2} \left( m_1 m_2 - \nu^2 + \frac{a_2 \nu(m_1 + m_2)}{\sqrt{m_2^2 + \nu^2 - a_2^2}} \right),
\]

\[
\alpha_4 = -1 - l\sqrt{m_2^2 + \nu^2 - a_2^2} - \frac{l^2}{2} \left( m_1 m_2 - \nu^2 - \frac{a_2 \nu(m_1 + m_2)}{\sqrt{m_2^2 + \nu^2 - a_2^2}} \right). \tag{28}
\]

The expansion of the constant objects \( X_i \) from (15) associated with the parameters \( \alpha_i \) can be most conveniently found with the aid of the formula

\[
X_i = \left. \frac{\bar{e}_+}{e_+} \right|_{z=\alpha_i}, \tag{29}
\]

where \( e_+ \) is defined in (22).
The expansion of $X_i$ should be performed up to the second power of $l$; this yields (to avoid the complicated expressions, we give explicitly only the terms linear in $l$)

\[
X_1 = \frac{\sqrt{m_1^2 + \nu^2 - a_1^2} + ia_1}{m_1 + i\nu} - \frac{l}{2(m_1 + i\nu)} \left( 2a_1a_2 - i\nu(m_1 + m_2) \right)
- 2ia_2\sqrt{m_1^2 + \nu^2 - a_1^2} + \frac{a_1\nu(m_1 + m_2)}{\sqrt{m_1^2 + \nu^2 - a_1^2}},
\]

\[
X_2 = -\frac{\sqrt{m_2^2 + \nu^2 - a_2^2} - ia_1}{m_2 - i\nu} - \frac{l}{2(m_2 - i\nu)} \left( 2a_1a_2 + i\nu(m_1 + m_2) \right)
+ 2ia_2\sqrt{m_2^2 + \nu^2 - a_2^2} - \frac{a_1\nu(m_1 + m_2)}{\sqrt{m_2^2 + \nu^2 - a_2^2}},
\]

\[
X_3 = \frac{\sqrt{m_2^2 + \nu^2 - a_2^2} + ia_2}{m_2 - i\nu} + \frac{l}{2(m_2 - i\nu)} \left( 2a_1a_2 + i\nu(m_1 + m_2) \right)
- 2ia_1\sqrt{m_2^2 + \nu^2 - a_2^2} - \frac{a_2\nu(m_1 + m_2)}{\sqrt{m_2^2 + \nu^2 - a_2^2}},
\]

\[
X_4 = -\frac{\sqrt{m_2^2 + \nu^2 - a_2^2} - ia_2}{m_2 - i\nu} + \frac{l}{2(m_2 - i\nu)} \left( 2a_1a_2 + i\nu(m_1 + m_2) \right)
+ 2ia_1\sqrt{m_2^2 + \nu^2 - a_2^2} + \frac{a_2\nu(m_1 + m_2)}{\sqrt{m_2^2 + \nu^2 - a_2^2}}. \tag{30}
\]

Before passing to $\mathcal{E}$ there still remains to find the expansion of $r_i = \sqrt{\rho^2 + (z - \alpha_i)^2}$. The calculation of $r_i$ with the aid of (28) does not exhibit difficulty and leads to the expressions

\[
r_1 = R_1 - l\frac{(z - 1)\sqrt{m_1^2 + \nu^2 - a_1^2}}{R_1} - l^2 \left( \frac{(m_1^2 + \nu^2 - a_1^2)(z - 1)^2}{2R_1^3} \right)
+ \frac{z(m_1m_2 - \nu^2) - m_1^2 - m_1m_2 + a_1^2}{2R_1} - \frac{a_1\nu(m_1 + m_2)(z - 1)}{2R_1\sqrt{m_1^2 + \nu^2 - a_1^2}},
\]

\[
r_2 = R_1 + l\frac{(z - 1)\sqrt{m_1^2 + \nu^2 - a_1^2}}{R_1} - l^2 \left( \frac{(m_1^2 + \nu^2 - a_1^2)(z - 1)^2}{2R_1^3} \right).
\]

15
where we have introduced

\[ R_1 = \sqrt{\rho^2 + (z - 1)^2}, \quad R_2 = \sqrt{\rho^2 + (z + 1)^2}. \]  

We have realized all the necessary steps for eventually being able to obtain the form of \( \mathcal{E} \) in our approximation scheme. The substitution of the approximate values of \( \alpha_i, X_i \) and \( r_i \) into (15) yields, after expanding \( \mathcal{E} \) in powers of \( l \),

\[
\mathcal{E} = 1 - 2l \left( \frac{m_1 + i\nu}{R_1} + \frac{m_2 - i\nu}{R_2} \right) + 2l^2 \left[ \left( \frac{m_1 + i\nu}{R_1} + \frac{m_2 - i\nu}{R_2} \right)^2 - i a_1 (m_1 + i\nu)(z - 1) \frac{1}{R_1^3} - i a_2 (m_2 - i\nu)(z + 1) \frac{1}{R_2^3} \right].
\]

For completeness we also give the expanded version of the potential \( \xi \) associates with \( \mathcal{E} \):

\[
\xi = \frac{1 - \mathcal{E}}{1 + \mathcal{E}} = l \left( \frac{m_1 + i\nu}{R_1} + \frac{m_2 - i\nu}{R_2} \right) + il^2 \left( \frac{a_1 (m_1 + i\nu)(z - 1)}{R_1^3} + \frac{a_2 (m_2 - i\nu)(z + 1)}{R_2^3} \right).
\]

The knowledge of the potential \( \mathcal{E} \) is sufficient for obtaining the corresponding metric functions \( f, \omega, \gamma \) in (11). The function \( f \) is simply the real
part of $\mathcal{E}$, hence it has the form

$$f = 1 - 2l \left( \frac{m_1}{R_1} + \frac{m_2}{R_2} \right) + 2l^2 \left[ \left( \frac{m_1}{R_1} + \frac{m_2}{R_2} \right)^2 - \nu^2 \left( \frac{1}{R_1} - \frac{1}{R_2} \right)^2 \right] + \frac{a_1 \nu(z-1)}{R_1^2} - \frac{a_2 \nu(z+1)}{R_2^2} \right]. \quad (35)$$

The metric coefficients $\omega$ and $\gamma$ can be found by integrating equations (9) and (6), respectively. The function $\Omega$ entering equations (9) is defined by the imaginary part of $E$. With this, the integration of (9) gives

$$\omega = 2l \nu \left( \frac{z-1}{R_1} - \frac{z+1}{R_2} \right) - 2l^2 \left[ \nu(m_1 + m_2) \right.$$

$$+ \rho^2 \left( \frac{m_1 a_1}{R_1^3} + \frac{m_1 a_2}{R_2^3} \right) - \nu(m_1 + m_2)(z^2 + \rho^2 - 1) \left. \right] \right]. \quad (36)$$

The substitution of (35), (36) into equations (6) and the integration of the latter system yields the form of the remaining function $\gamma$:

$$\gamma = \frac{l^2}{2} \left[ \frac{(m_1 m_2 - \nu^2)(z^2 + \rho^2 - 1)}{R_1 R_2} - m_1 m_2 + \nu^2 \right.$$

$$- \rho^2 \left( \frac{m_1^2 + \nu^2}{R_1^2} + \frac{m_2^2 + \nu^2}{R_2^2} \right) \left. \right]. \quad (37)$$

Mention that the integration constants in the above expressions for $\omega$ and $\gamma$ are chosen in such a way that both functions vanish at spatial infinity.

The behavior of $\omega$ and $\gamma$ on the symmetry axis is crucial for establishing whether equilibrium of two Kerr particles can be achieved due to balance of the gravitational attraction and spin–spin repulsion forces. We recall that for two particles to be in equilibrium, the functions $\omega$ and $\gamma$ should vanish on the parts of the $z$–axis outside and between the particles (the particles are situated at the points $z = \pm 1$). A simple inspection of the formulae (36), (37) shows that $\omega = \gamma = 0$ when $|z| > 1$. By demanding $\omega = \gamma = 0$ on the part $|z| < 1$ of the symmetry axis, we arrive at two balance conditions

$$- 4\nu l [1 + l(m_1 + m_2)] = 0 \quad (38)$$

(condition of vanishing $\omega$), and

$$- l^2 (m_1 m_2 - \nu^2) = 0 \quad (39)$$
Bearing in mind that the parameter \( l \) is an auxiliary parameter which can be always put equal to 1 in the final formulae, we see that the system (38)–(39) has no physically acceptable solutions since \( \nu = 0 \) satisfying (38) implies vanishing of one of the masses \( m_1, m_2 \).

This result about the absence of equilibrium states in the approximate double–Kerr solution is consistent with our theorem \([17]\) on the absence of balance of two Kerr black holes with positive masses in the exact double–Kerr solution.

We turn now to two other approximation schemes within which equilibrium configurations of two spinning particles are available.

4. Two approximation schemes leading to the balance of Kerr particles

The approximation scheme 2 involves two superextreme Kerr particles. In the exact double–Kerr solution a superextreme constituent is defined by a cut joining two complex conjugate \( \alpha \). As we have seen in the previous section, by supposing the angular momenta per unit mass to be greater than the masses, already in the zeroth approximation we obtained two pairs of complex conjugate \( \alpha \) as roots of equation (25). Hence, in the approximation scheme too a superextreme particle is represented by a cut, and not by a point on the symmetry axis. This obviously will lead to the appearance in scheme 2 of four different square roots \( R_i \), \( i = 1,4 \), involving complex quantities, unlike in scheme 1 where we had only two real distances \( R_1 \) and \( R_2 \) defining the location of the particles on the symmetry axis.

Here we shall consider a slightly modified scheme 2 in which the parameter \( \nu \) will be assigned the order 2 in \( l \) (the case when \( \nu \) is proportional to \( l \), not \( l^2 \), does not lead to equilibrium configurations). The roots of equation (17) after the substitution \( m_1 \rightarrow lm_1, m_2 \rightarrow lm_2, \nu \rightarrow l^2\nu \) into (22) have been found to have the following form (technically they are obtainable as in scheme 1):

\[
\alpha_1 = \bar{\alpha}_2 = 1 - ia_1 + il^2 \left( \frac{m_1^2}{2a_1} + \frac{m_1m_2}{a_1 + a_2 + 2i} \right) + l^3 \frac{\nu(m_1 + m_2)}{a_1 + a_2 + 2i},
\]

\[
\alpha_3 = \bar{\alpha}_4 = -1 - ia_2 + il^2 \left( \frac{m_2^2}{2a_2} + \frac{m_1m_2}{a_1 + a_2 - 2i} \right) - l^3 \frac{\nu(m_1 + m_2)}{a_1 + a_2 - 2i}.
\]

We have found it more advantageous for schemes 2 and 3 to calculate the quantities \( lX_i \) instead of \( X_i \), that simplifies the subsequent expansion of \( \mathcal{E} \).
The resulting expressions for \( lX_i \) are \( (lX_2 \text{ and } lX_4 \text{ are written without their quadratic terms in } l \text{ because of their complexity}) \)

\[
\begin{align*}
lX_1 &= \frac{il^2 m_1(a_1 - a_2 + 2i)}{2a_1(a_1 + a_2 + 2i)}, \\
lX_2 &= \frac{2ia_1(a_1 + a_2 - 2i)}{m_1(a_1 - a_2 - 2i)} + \frac{2a_1 \nu(a_1 + a_2 - 2i)}{m_1^2(a_1 - a_2 - 2i)}, \\
lX_3 &= -\frac{il^2 m_2(a_1 - a_2 + 2i)}{2a_2(a_1 + a_2 - 2i)}, \\
lX_4 &= -\frac{2ia_2(a_1 + a_2 + 2i)}{m_2(a_1 - a_2 - 2i)} + \frac{2a_2 \nu(a_1 + a_2 + 2i)}{m_2^2(a_1 - a_2 - 2i)}.
\end{align*}
\]

(41)

The expansion of \( r_i \) leads to the following result:

\[
\begin{align*}
r_1 &= R_1 - il^2 \frac{z - 1 + ia_1}{R_1} \left( \frac{m_1^2}{2a_1} + \frac{m_1 m_2}{a_1 + a_2 + 2i} \right), \\
r_2 &= R_2 + il^2 \frac{z - 1 - ia_1}{R_2} \left( \frac{m_1^2}{2a_1} + \frac{m_1 m_2}{a_1 + a_2 - 2i} \right), \\
r_3 &= R_3 - il^2 \frac{z + 1 + ia_2}{R_3} \left( \frac{m_2^2}{2a_2} + \frac{m_1 m_2}{a_1 + a_2 - 2i} \right), \\
r_4 &= R_4 + il^2 \frac{z + 1 - ia_2}{R_4} \left( \frac{m_2^2}{2a_2} + \frac{m_1 m_2}{a_1 + a_2 + 2i} \right),
\end{align*}
\]

(42)

where

\[ R_1 = R_2 = \sqrt{\rho^2 + (z - 1 + ia_1)^2}, \quad R_3 = R_4 = \sqrt{\rho^2 + (z + 1 + ia_2)^2}. \]

(43)

From (13) and (14)–(16) we find the form of the Ernst potential \( \mathcal{E} \):

\[
\mathcal{E} = 1 - 2l \left( \frac{m_1}{R_2} + \frac{m_2}{R_4} \right) + 2l^2 \left[ \left( \frac{m_1}{R_2} + \frac{m_2}{R_4} \right)^2 - i \nu \frac{1}{R_2 - R_4} \right].
\]

(44)

The corresponding potential \( \xi \) is defined by a particularly simple expression

\[
\xi = l \left( \frac{m_1}{R_2} + \frac{m_2}{R_4} \right) - i \nu l^2 \left( \frac{1}{R_2} - \frac{1}{R_4} \right).
\]

(45)

Remarkably, the formulae (44) and (45) do not contain the functions \( R_1 \) and \( R_3 \). However, all \( R_i, \ i = 1,4 \), enter the expressions for the metric.
functions $f$, $\omega$, $\gamma$ (the procedure of the calculation of these functions is fully analogous to the one used for scheme 1)

$$f = 1 - l \left( \frac{m_1}{R_1} + \frac{m_1}{R_2} + \frac{m_2}{R_3} + \frac{m_2}{R_4} \right)$$

$$+ l^2 \left[ \frac{m_1^2}{R_1^2} + \frac{m_1^2}{R_2^2} + \frac{m_2^2}{R_3^2} + \frac{m_2^2}{R_4^2} + \frac{2m_1m_2}{R_1R_3} + \frac{2m_1m_2}{R_2R_4} \right]$$

$$+ i\nu \left( \frac{1}{R_1} - \frac{1}{R_2} - \frac{1}{R_3} + \frac{1}{R_4} \right)$$

$$\omega = il \left( \frac{m_1(z - 1 + ia_1)}{R_1} + \frac{m_2(z + 1 + ia_2)}{R_3} - c.c. \right)$$

$$+ l^2 \left[ \frac{m_1^2}{a_1} + \frac{m_2^2}{a_2} + \frac{2m_1m_2}{a_1 + a_2 - 2i} + \frac{2m_1m_2}{a_1 + a_2 + 2i} \right]$$

$$+ \nu \left( z - 1 + ia_1 \right) - \frac{2m_1m_2}{a_1 + a_2 - 2i} + \frac{2m_1m_2}{a_1 + a_2 + 2i}$$

$$- \frac{m_2^2(z^2 + \rho^2 - 2z + a_2^2 + 1)}{a_2R_3R_4} - 2m_1m_2$$

$$\times \left( \frac{z^2 + \rho^2 + a_1a_2 - 1 - iz(a_1 - a_2) - i(a_1 + a_2)}{(a_1 + a_2 - 2i)R_2R_3} + c.c. \right)$$

$$\gamma = l^2 \left[ \frac{m_1^2}{4a_1^2} + \frac{m_2^2}{4a_2^2} + \frac{m_1m_2}{(a_1 + a_2 - 2i)^2} + \frac{m_1m_2}{(a_1 + a_2 + 2i)^2} \right]$$

$$- \frac{m_1^2[(z - 1)^2 + \rho^2 + a_1^2]}{4a_1^2R_1R_2} - \frac{m_2^2[(z + 1)^2 + \rho^2 + a_2^2]}{4a_2^2R_3R_4} - m_1m_2$$

$$\times \left( \frac{z^2 + \rho^2 + a_1a_2 - 1 + iz(a_1 - a_2) + i(a_1 + a_2)}{(a_1 + a_2 + 2i)^2R_1R_4} + c.c. \right)$$

(‘c.c.’ denotes complex conjugation of the terms in brackets). Interestingly, the constant $\nu$ does not enter the formula for $\gamma$.

Let us analyze now the possibility of balance of the Kerr particles in this approximation scheme. By construction, the functions $\omega$ and $\gamma$ vanish on the part $|z| > 1$ of the symmetry axis. Hence, to achieve the equilibrium of the particles, we only have to fulfil $\omega = 0$ and $\gamma = 0$ on the part $|z| < 1$ of the symmetry axis separating the particles. The condition $\omega = 0$ with $R_1 = \tilde{R}_2 = 1 - z - ia_1$, $R_3 = \tilde{R}_4 = 1 + z + ia_2$ yields

$$4l^2 \left( \frac{2m_1m_2(a_1 + a_2)}{(a_1 + a_2)^2 + 4} - \nu \right) = 0,$$
and the condition $\gamma = 0$ leads to

$$
\frac{4m_1m_2l^2[(a_1 + a_2)^2 - 4]}{[(a_1 + a_2)^2 + 4]^2} = 0.
$$

(48)

The solution of (47) is

$$
\nu = \frac{2m_1m_2(a_1 + a_2)}{(a_1 + a_2)^2 + 4},
$$

(49)

and it fixes the choice of $\nu$ in this approximation scheme. Equation (48) is fulfilled when $a_1 + a_2 = \pm 2$. Recalling that all the parameters are “measured” in the units of $b$, we obtain after introducing $b$ explicitly:

$$
2b = |a_1 + a_2|,
$$

(50)

and this is the relation between the distance at which equilibrium takes place and the angular momenta per unit mass of the particles.

**Scheme 3.** Let us consider now the remaining approximation scheme. We will show that it also permits equilibrium configurations, this time between a sub- and a superextreme Kerr particles, as it occurs in the exact double–Kerr solution.

The substitutions to be made in the axis data (22) are the following: $m_1 \rightarrow lm_1, m_2 \rightarrow lm_2, a_1 \rightarrow la_1, \nu \rightarrow l^2\nu$. This means that the particle with subindex 1 represents a subextreme constituent, while the particle with subindex 2 is a superextreme constituent ($a_2^2 > m_2^2$). Mention that choosing $\nu$ proportional to $l^2$ is a manner to introduce the inequality $\nu^2 < a_1^2$. As before, we shall develop this scheme in the second approximation.

Under the above assumptions equation (17) has the following four roots (the coefficients at $l^3$ are not given because of their complexity)

$$
\alpha_1 = 1 + l\sqrt{m_1^2 - a_1^2} + l^2 \frac{m_1m_2}{a_1^2} \left(2 + \frac{a_1a_2}{\sqrt{m_1^2 - a_1^2}}\right),
$$

$$
\alpha_2 = 1 - l\sqrt{m_1^2 - a_1^2} + l^2 \frac{m_1m_2}{a_1^2} \left(2 - \frac{a_1a_2}{\sqrt{m_1^2 - a_1^2}}\right),
$$

$$
\alpha_3 = \bar{\alpha}_4 = -1 - ia_2 + l^2 \frac{m_2[2m_2 + ia_2(2m_1 + m_2)]}{2a_2(a_2 - 2i)}.
$$

(51)
Then, after expanding $lX_i$ up to $l^2$, we get

$$lX_1 = -l \frac{(a_2 - 2i)(m_1 + ia_1 + \sqrt{m_1^2 - a_1^2})}{(a_2 + 2i)(m_1 - ia_1 + \sqrt{m_1^2 - a_1^2})} + \frac{l^2}{m_1^2(a_2 + 2i)^2}$$

$$\times \left[ im_1 a_2(2m_1^2 - 2a_1^2 + m_1 m_2) \left(1 + \frac{ia_1}{\sqrt{m_1^2 - a_1^2}}\right) + i\nu(a_2^2 + 4)(\sqrt{m_1^2 - a_1^2} + ia_1)\right],$$

$$lX_2 = X_1(\sqrt{m_1^2 - a_1^2} \rightarrow -\sqrt{m_1^2 - a_1^2}),$$

$$lX_3 = \frac{l^2 m_2}{2a_2},$$

$$lX_4 = \frac{2ia_2}{m_2} + \frac{2a_2 l}{m_2^2} \left( \frac{im_1 a_2}{a_2 + 2i} - \nu \right) - il^2 \left( \frac{m_2}{2a_2} + \frac{2a_2 \nu}{m_3} \right)$$

$$- \frac{2a_2(m_1^2 + 2a_1^2)}{m_2(a_2 + 2i)^2} - \frac{4im_1 a_1 a_2 \nu}{m_2^2(a_2 + 2i)},$$

(52)

The expansion of the functions $r_i$ yields

$$r_1 = R_1 - l \left( \frac{(z - 1)\sqrt{m_1^2 - a_1^2}}{R_1} + \frac{l^2}{2R_1} \left( m_1^2 - a_1^2 \right) \left( 1 - \frac{(z - 1)^2}{R_1^2} \right) \right)$$

$$- \frac{2m_1 m_2(z - 1)}{a_2^2 + 4 \left( 2 + \frac{a_1 a_2}{\sqrt{m_1^2 - a_1^2}} \right)} \left[ 2 + \frac{a_1 a_2}{\sqrt{m_1^2 - a_1^2}} \right] \right],$$

$$r_2 = r_1(\sqrt{m_1^2 - a_1^2} \rightarrow -\sqrt{m_1^2 - a_1^2}),$$

$$r_3 = R_3 - l^2 \frac{m_2(z + 1 + ia_2)[2m_2 + ia_2(2m_1 + m_2)]}{2a_2(a_2 - 2i)R_3},$$

$$r_4 = R_4 - l^2 \frac{m_2(z + 1 - ia_2)[2m_2 - ia_2(2m_1 + m_2)]}{2a_2(a_2 + 2i)R_4},$$

(53)

where we have introduced

$$R_1 = \sqrt{\rho^2 + (z - 1)^2}, \quad R_3 = \bar{R}_4 = \sqrt{\rho^2 + (z + 1 + ia_2)^2}. \quad (54)$$

The potentials $\mathcal{E}$ and $\xi$ resulting from the formulae (51)-(54) and (15) have the form

$$\mathcal{E} = 1 - 2l \left( \frac{m_1}{R_1} + \frac{m_2}{R_4} \right) + 2l^2 \left[ \left( \frac{m_1}{R_1} + \frac{m_2}{R_4} \right)^2 \right.$$
\[
\xi = l \left( \frac{m_1}{R_1} + \frac{m_2}{R_4} \right) + il \left[ \frac{m_1 a_1 (z - 1)}{R_1^3} + \nu \left( \frac{1}{R_1} - \frac{1}{R_4} \right) \right].
\]

The set of the metric functions \( f, \omega, \gamma \) defined by (55) is given by the expressions

\[
f = 1 - l \left( \frac{2m_1}{R_1} + \frac{m_2}{R_3} + \frac{m_2}{R_4} \right) + l^2 \left[ \left( \frac{m_1}{R_1} + \frac{m_2}{R_3} \right) - i \nu \left( \frac{1}{R_3} - \frac{1}{R_4} \right) \right],
\]

\[
\omega = il \left( \frac{m_2(z + 1 + ia_2)}{R_3} - \text{c.c.} \right) + l^2 \left[ \frac{m_2[4m_2 + a_2^2(4m_1 + m_2)]}{a_2(a_2 + 4)} \right]
\]

\[
+ \nu \left( \frac{2(z - 1)}{R_1} - z + 1 + ia_2 \right) - \frac{m_2(z + 1 - ia_2)}{R_1} - \frac{2m_1a_1\rho^2}{R_3} + \text{c.c.},
\]

\[
\gamma = l^2 \left[ \frac{m_2^2(z + 1)^2 + \rho^2 + a_2^2}{a_2R_3R_4} \right] - \frac{2m_1m_2}{R_1} \left( \frac{z^2 + \rho^2 - 1 + ia_2(z - 1)}{(a_2 - 2i)R_3} + \text{c.c.} \right) - \frac{m_2^2(z + 1)^2 + \rho^2 + a_2^2}{4a_2^2R_3R_4} + \frac{m_2^2\rho^2}{2R_1^4}.
\]

The metric (56) is asymptotically flat which means that on the symmetry axis \( \omega = \gamma = 0 \) when \( |z| > 1 \). The balance of two Kerr particles then requires \( \omega = 0 \) and \( \gamma = 0 \) for \( |z| < 1 \). The first of the latter two conditions yields, after setting \( R_1 = 1 - z, \ R_3 = 1 + ia_2 + z, \ R_4 = 1 - ia_2 + z, \)

\[
\frac{4l^2}{a_2^2 + 4} [2m_1m_2a_2 - \nu(a_2^2 + 4)] = 0,
\]

whereas the second condition leads to

\[
\frac{4m_1m_2^2(a_2^2 - 4)}{(a_2^2 + 4)^2} = 0.
\]

From (57) we get the value of \( \nu \) required for the equilibrium:

\[
\nu = \frac{2m_1m_2a_2}{a_2^2 + 4}.
\]
On the other hand, equation (58) is satisfied when $|a_2| = 2$. Introducing explicitly the separation parameter $b$, we finally obtain

$$2b = |a_2|. \quad (60)$$

Formula (60) defines, for a given value of the angular momentum per unit mass of the superextreme Kerr constituent, the coordinate distance at which the equilibrium of two particles takes place.

Therefore, this approximation scheme 3 too proves to be consistent with the exact double–Kerr solution.

5. The Komar masses and angular momenta in the approximate solutions

In this section we shall discuss the relation of the parameters $m_i$ and $a_i$ of our approximation procedure to the genuine individual masses $M_i$ and angular momenta $J_i$ of the particles defined by the Komar integrals [26]. For the metric [11] in the case of the cylindrical surface of integration we obtained the formulae [16]

$$4M_i = \int_{l_i}^{u_i} \left( \frac{\rho f \rho f - \omega \Omega_{\rho z}}{f} \right) \left| \begin{array}{c} \rho = \rho_0 \\ z = \rho_0 \end{array} \right. \left| \begin{array}{c} \rho = \rho_0 \\ z = u_i \end{array} \right. d\rho + \int_{0}^{\rho_0} \left( \frac{\rho f \rho f + \omega \Omega_{\rho \rho}}{f} \right) \left| \begin{array}{c} \rho = \rho_0 \\ z = u_i \end{array} \right. d\rho,$$

$$8J_i = -\int_{l_i}^{u_i} \left[ 2\omega - \frac{2\rho \omega}{f} f \rho f + \left( \frac{\rho^2}{f^2} + \omega^2 \right) \Omega_{\rho \rho} \right] \left| \begin{array}{c} \rho = \rho_0 \\ z = u_i \end{array} \right. d\rho + \int_{0}^{\rho_0} \left[ \frac{2\rho \omega}{f} f \rho f + \left( \frac{\rho^2}{f^2} + \omega^2 \right) \Omega_{\rho \rho} \right] \left| \begin{array}{c} \rho = \rho_0 \\ z = u_i \end{array} \right. d\rho,$$

where $u_i$ and $l_i$, $i = 1, 2$, are the points of the $z$–axis which are taken as centers of the upper and lower bases of the cylinder enclosing the $i$th particle, $\rho_0$ being the radius of the bases. In our calculations we shall assume that the upper cylinder extends from $z = 0$ to plus infinity (i.e., $l_1 = 0, u_1 = +\infty$),
and the other one from minus infinity to \( z = 0 \) \((l_2 = -\infty, u_2 = 0)\). The results of the evaluation of \( M_i \) and \( J_i \) can be summarized as follows.

**Scheme 1.** Although in this scheme there are no equilibrium configurations and the particles are kept apart by a supporting strut, it is important for our analysis of the Komar quantities that the strut be massless in which case we can be sure that \( M_i \) and \( J_i \) will not contain any contribution of the strut. Since the strut is massless when \( \omega = 0 \) on it, we have to put \( \nu = 0 \) in the formulae (33)–(37). The integration with the aid of the formulae (61) then yields

\[
M_i = l m_i, \quad J_i = l^2 m_i a_i. \tag{62}
\]

Therefore, within the framework of this approximation scheme with \( \nu = 0 \) the parameters \( m_i \) represent the masses of the particles, and the parameters \( a_i \) coincide with the corresponding angular momenta per unit mass.

**Scheme 2.** For the same reasons as in the previous case, in order to have correct estimations of \( M_i \) and \( J_i \) it is sufficient to consider a more general case of particles supported by a massless strut instead of considering configurations of balancing particles. So, in the formulae (44)–(46) we will take the constant \( \nu \) in the form (49) to assure \( \omega = 0 \) on the strut, and leave \( a_1 \) and \( a_2 \) as arbitrary parameters. The resulting expressions for \( m_i \) and \( J_i \) are

\[
M_i = l m_i, \quad J_i = J_1 = l^2 m_1 a_1, \quad J_2 = J_2 = l^2 m_2 a_2 + l^2 \frac{4 m_1 m_2 a_2}{a_2^2 + 4}, \tag{63}
\]

which means that in this scheme the parameters \( a_i \) do not fully coincide with the Komar angular momenta per unit mass of the particles.

**Scheme 3.** In the third approximation scheme the massless strut between the particles can be introduced by choosing \( \nu \) in the form (59). The evaluation of the Komar masses and angular momenta in (61) with the aid of (55)–(56) leads to the following result:

\[
M_i = l m_i, \quad J_1 = l^2 m_1 a_1, \quad J_2 = l m_2 a_2 + l^2 \frac{4 m_1 m_2 a_2}{a_2^2 + 4}. \tag{64}
\]

Hence, the parameters \( m_i \) in this scheme coincide with the Komar masses of the particles, and \( a_1 \) with the angular momentum per unit mass of the subextreme constituent. The angular momentum per unit mass of the superextreme constituent is

\[
\frac{J_2}{M_2} = a_2 \left( 1 + l^2 \frac{4 m_1}{a_2^2 + 4} \right), \tag{65}
\]
and it differs slightly from $a_2$ due to the presence of the second term in brackets.

6. Discussion of the results

Therefore, we have succeeded in working out correct approximate analogues of the exact double–Kerr solution by introducing a small parameter and expanding the Ernst complex potential in a series of powers of this parameter. It turns out that the approximation procedure depends drastically on the qualitative relations between the parameters; it is not unique since a concrete physical situation dictates the form of the corresponding approximation scheme. It is worthwhile mentioning that within all three schemes further variations of the procedure are possible. For instance, in scheme 1 it is possible to make the parameter $\nu$ proportional to $l^2$ or $l^3$. However, since these modifications cannot lead in principle to equilibrium configurations of two subextreme Kerr particles, we did not include them into this paper. At the same time, one should realize that by choosing $\nu$ proportional to $l^3$ he commits himself to the calculation of the Ernst potential and corresponding metric functions up to the third order in $l$ (the system of the balance conditions in this case takes the form

$$2l^3[m_1m_2(a_1 + a_2) - 2\nu] = 0, \quad -m_1m_2l^2 = 0,$$

and implies vanishing of one of the masses.)

A very important qualitative result with which the schemes 2 and 3 provide us is the following: the location of a superextreme constituent in the approximate solution is not a point on the symmetry axis but it is a cut determined by two complex conjugate square roots, say, $R_1 = \bar{R}_2$. One should be aware of this non-evident result when he wants to develop an approximation procedure for spinning particles without relating it to a concrete exact solution.

In view of the above said and of our Section 2 it would be of interest to compare Bonnor’s solution (4) with our approximate solution from scheme 1. The approximation scheme employed by Bonnor in (4) treats the parameters $m_i$ and $j_i$ as small parameters of the same magnitude in $l$. This means in particular that the quantities $a_i = j_i/m_i$ in his approach are not small constants and, consequently, both spinning particles are superextreme because $a_i^2 > m_i^2$. The fact that the superextreme constituents of the solution
are placed at the points ±b of the symmetry axis as if they were subextreme constituents makes Bonnor’s approximation procedure contradictory from the point of view of the double–Kerr solution. For the solution (4) to represent two subextreme Kerr particles it is necessary to introduce the small parameters $a_i$ into (4) by additionally changing $j_i$ to $lm_i a_i$ (to make $j_i$ proportional to $l^2$) and eliminate all the arising terms of the third order in $l$. Then after such an extensive make–up the resulting solution will be a particular specialization ($\nu = 0$) of our approximate solution from scheme 1 (the lack of one arbitrary parameter in the solution (4) was discussed in [28]).

Mention that in the static limit the solutions (4) and (35)–(37) coincide and represent correctly an approximation to the double–Schwarzschild spacetime.

Turning back to our approximate solutions from schemes 2 and 3, below we would like to demonstrate that the condition (50) determining the balance of two superextreme Kerr particles and the condition (60) determining the balance of a sub- and a superextreme Kerr constituents agree with the exact results on the double–Kerr equilibrium problem not only qualitatively but also quantitatively. Indeed, in our paper [19] we obtained the general relation for balancing Kerr particles

$$J \pm (M + s)^2 + s(A_1 + A_2) = 0,$$  \hfill (67)

where $J = M_1 A_1 + M_2 A_2$, $M = M_1 + M_2$, $M_i$ are Komar masses, $J_i = M_i A_i$ are Komar angular momenta, and $s$ is the coordinate distance separating the particles ($s = 2b$ in the zeroth order in $l$).

First let us assume that both constituents in (67) are superextreme. Then, substituting $M_1 \rightarrow l M_1$, $M_2 \rightarrow l M_2$ into (67) and restricting ourselves to only zeroth order in $l$ we obtain

$$\pm s^2 + s(A_1 + A_2) = 0 \quad \Rightarrow \quad s = |A_1 + A_2|. \hfill (68)$$

This is exactly the condition (50).

Assuming now that the first particle is subextreme and the second one is superextreme, we substitute $M_1 \rightarrow l M_1$, $M_2 \rightarrow l M_2$, $A_1 \rightarrow l A_1$ into (67) and get, after retaining only the zero–order terms in $l$:

$$\pm s^2 + s A_2 = 0 \quad \Rightarrow \quad s = |A_2|, \hfill (69)$$

which is precisely the condition (60).
7. Conclusions

The formalism developed in this paper opens possibilities for the unified treatment of the exact and approximate solutions of Einstein’s equations. It is essentially based on Sibgatullin’s approach to the construction of exact solutions from the prescribed axis data that permits establishing very strict and clear connections between the parameters of the solution and the corresponding physical properties they determine uniquely through the relativistic multipole moments.

We have constructed approximate analogues of the well-known exact double–Kerr solution and demonstrated that they permit equilibrium configurations involving superextreme particles. In the case of the double–Kerr equilibrium problem the exact approach looks by far more attractive since the balance conditions can be solved analytically in a unified form, whereas a specific approximation procedure has to be worked out for each concrete type of the Kerr particles. The situation, however, may change in the case of more complex exact solutions describing the many–body configurations for which the balance conditions cannot be solved analytically in general. Then it is anticipated that the approximate procedure discussed in this paper may become the most efficient tool for obtaining the general relations for balancing particles, especially in the presence of the electromagnetic fields.

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Figure 1: Exact analogue of the approximate solution (4): a specific four-body system formed by two compound objects.
Figure 2: Three principle configurations of two Kerr particles treated by approximation schemes 1–3.