Progress in the Small $x$ Resummation of the Singlet Anomalous Dimension

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ABSTRACT

We summarize our recent results on the small $x$ resummation of the singlet anomalous dimension. We recall the main features of our approach, and briefly describe some work in progress on the inclusion of subleading corrections to it.

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We summarize our recent results on the small $x$ resummation of the singlet anomalous dimension. We recall the main features of our approach, and briefly describe some work in progress on the inclusion of subleading corrections to it.

The understanding of scaling violations for deep inelastic structure functions at small $x$ has been characterized for some time by an apparent contradiction between theoretical expectations and experimental results. On the one hand, new effects beyond the fixed order perturbative approximation to anomalous dimensions are expected to become important at small $x$. On the other hand, no significant deviation from a standard next-to-leading order perturbative treatment of scaling violations of structure function data has been found. Thanks to a body of theoretical work by our group [1], and, along similar lines, by the authors of ref. [2] (see also [3]), the origin of this situation is now essentially understood.

Perturbative anomalous dimensions have been recently [4] computed up to NNLO:

$$\gamma(N,\alpha_s) = \alpha_s \gamma_0(N) + \alpha_s^2 \gamma_1(N) + \alpha_s^3 \gamma_2(N) \ldots (1)$$

This perturbative expansion is not reliable at small $x$, when $\alpha \ln \frac{1}{x} \sim 1$, as is already the case in the HERA region. The problem is how to use the information contained in the BFKL kernel to resum it in such a way that the improved splitting function remains a good approximation down to small values of $x$. This can be accomplished [1] by exploiting the fact that the solutions of the BFKL and GLAP equations coincide at leading twist if their respective evolution kernels are related by “duality” [5]. In the fixed coupling limit the duality relation is simply given by

$$\chi(\gamma(N,\alpha_s),\alpha_s) = N, \quad (2)$$

where $\chi(\alpha_s,M)$ is the BFKL kernel, which has been computed to NLO:

$$\chi(M,\alpha_s) = \alpha_s \chi_0(M) + \alpha_s^2 \chi_1(M) + \ldots (3)$$

The splitting function will then contain all terms of order $(\alpha_s \log 1/x)^n$, derived from $\chi_0(M)$, and of order $\alpha_s(\alpha_s \log 1/x)^n$, derived from $\chi_1(M)$.

The small $x$ behaviour of the recently computed NNLO anomalous dimensions displays two undesirable features which characterize small $x$ resummations. First, higher order corrections to the anomalous dimension are very large at small $x$: the NNLO/NLO ratio grows at small $x$. Second, the dominant small $x$ terms do not approximate well the anomalous dimension even at rather small $x$. In other words, resummation effects appear to be large, and the resummed expansion does not converge well.

We now understand how to treat both problems. The problem of the poor behaviour of the
resummed expansion of the anomalous dimension is cured if the small-$x$ resummation is combined with the standard resummation of collinear singularities, by constructing a ‘double-leading’ perturbative expansion. This double leading expansion resums the collinear poles at $M = 0$ in the expansion (3), enabling the imposition of the physical requirement of momentum conservation: $\gamma(1, \alpha_s) = 0$, whence

\[
\chi(0, \alpha_s) = 1. \tag{4}
\]

The problem of the excessive size of resummation corrections is solved by a full treatment of the running coupling: once running coupling effects are properly included in the improved anomalous dimension, the asymptotic behavior near $x = 0$ is much softened with respect to the Lipatov exponent which characterizes the fixed-coupling resummation. Hence, the corresponding dramatic rise of structure functions at small $x$, which is ruled out phenomenologically, is replaced by a milder rise.

We now briefly recall our improved splitting function, referring the reader to the original papers \[10\] for a full discussion. Assuming that one only knows $\gamma_0(N), \gamma_1(N)$ and $\chi_0(M)$, the improved anomalous dimension has the following expression:

\[
\gamma^N_0(\alpha_s, N) = \left[ \alpha_s \gamma_0(N) + \alpha_s^2 \gamma_1(N) \right] + \frac{1}{2} \beta_0 \alpha_s (1 + \frac{\alpha_s}{N} c_0) - \text{mom. sub.} \tag{5}
\]

The first group of terms on the right-hand side, within square brackets, is the sum of the leading order of the double leading expansion (DL-LO approximation), plus the next-to-leading correction $\gamma_1$ to the standard GLAP anomalous dimension: namely, it is the sum of the NLO perturbative term $\alpha_s \gamma_0(N) + \alpha_s^2 \gamma_1(N)$ plus the power series of terms $\left( \alpha_s / N \right)^n$ in $\gamma_s(\alpha_s / N)$, obtained from $\chi_0$ using eq. (2), with subtraction of the order $\alpha_s$ term to avoid double counting. In the second line, the “Airy” anomalous dimension $\gamma_A(c_0, \alpha_s, N)$ contains the running coupling resummation, and the terms in the third line subtract the contributions to $\gamma_A(c_0, \alpha_s, N)$ which are already included in $\gamma_s, \gamma_0$ and $\gamma_1$. The Airy anomalous dimension $\gamma_A(c_0, \alpha_s, N)$ is the exact solution of the running coupling BFKL equation which corresponds to a quadratic approximation to $\chi_0$ near $M = \frac{1}{2}$, $\chi_0 \approx \left[ c_0 + \frac{1}{2} c_0 (M - \frac{1}{2})^2 \right]$. Finally “mom. sub.” is a subleading subtraction that ensures exact momentum conservation $\gamma^N_0(1, \alpha_s) = 0$.

The properties of the improved anomalous dimension in this approximation are the following. In the limit $\alpha_s \to 0$ with $N$ fixed, $\gamma_1(\alpha_s, N)$ reduces to $\alpha_s \gamma_0(N) + \alpha_s^2 \gamma_1(N) + O(\alpha_s^3)$. For $\alpha_s \to 0$ with $\alpha_s / N$ fixed, $\gamma_1(\alpha_s, N)$ reduces to $\gamma_s(\alpha_s / N)$, i.e. the leading term of the small $x$ expansion. Thus the Airy term is subleading in both limits. In spite of this, its role is very significant because of the singularity structure of the different terms in eq. (5). Indeed, $\gamma_0(N)$ has a pole at $N = 0$, $\gamma_A$ has a branch cut at $N = \alpha_s c_0$ and $\gamma_A$ has a pole at $N = N_0 < \alpha_s c_0$, where $N_0$ is the position of the rightmost zero of the Airy function. The importance of the Airy term is that the square root term subtracted from $\gamma_A$ cancels, within the relevant accuracy, the branch cut of $\gamma_s$ at $N = \alpha_s c_0$ and replaces the corresponding asymptotic behaviour at small $x$ with the much softer one from $\gamma_A$. Note that the quadratic approximation is sufficient to give the correct asymptotic behaviour up to terms which are of subleading order in comparison to those included in the double-leading expression in eq. (5).

The singlet splitting function obtained from eq. (5) (for $\alpha_s = 0.2$) is shown in Figure 1, compared with the NLO GLAP kernel and with the DL-LO approximation, which displays the sharp small-$x$ rise characteristic of the BFKL resummation. In the region of the HERA data, our improved splitting function, with no free parameters, closely follows the NLO GLAP evolution with a behaviour at small $x$ which is much softer than that of BFKL. It is interesting to note that the agreement between GLAP and resummed results is significantly improved by the inclusion in eq. (5) of $\gamma_1$ and the corresponding double-counting subtraction. This improvement was already shown in ref. \[11\] in the anomalous dimension, and is even more apparent in the splitting
Figure 1. The improved splitting function corresponding to $\gamma^N L(\alpha_s, N)$ eq. (5) with $\alpha_s = 0.2$ (dot-dashed), compared with those from the DL-LO approximation (dashed) and GLAP NLO (solid).

function from ref. [6] displayed in figure 1.

Given that also $\chi_1$ is in fact known [7], it might be worth pursuing a full next-to-leading order resummation. This requires the inclusion in the resummed anomalous dimension of the terms $\gamma_{ss}$ derived from the NLO BFKL kernel $\chi_1$, not included in eq. (5). However, the running-coupling resummation is based on a quadratic approximation of the BFKL kernel about its minimum, while the next-to-leading order double-leading anomalous dimension has no minimum. It is possible to include the next-to-leading corrections upon the assumption that the minimum is restored by unknown higher-order corrections, but the results are then affected by large ambiguities [1].

The lack of minimum in the next-to-leading order double-leading result is due to the fact that while the double-leading expansion resums the $M = 0$ poles of $\chi(M)$, the poles at $M = 1$ are not eliminated and in particular introduce a large negative contribution to $\chi_1(M)$ near $M = 1$. The resummation of these $M = 1$ poles, however, as emphasized by the authors of ref. [2], can be accomplished if we recall that the underlying BFKL kernel is symmetric under exchange of the two gluons of virtual masses $k_1^2$ and $k_2^2$ [7]. In Mellin space, this implies that the kernel must be symmetric upon $M \leftrightarrow 1 - M$. Nevertheless, the DIS kernel remains asymmetric, due (in the fixed coupling limit) to the change of the symmetric variable $s/k_1k_2$ into the corresponding variable $s/Q^2$ appropriate for DIS. Further asymmetric terms, to next-to-leading order, are due to the running of the coupling.

The kernel $\chi_{\text{DIS}}$, dual to the DIS anomalous dimension, is related to the symmetric one $\chi_\sigma$ through the implicit equation [7]

$$\chi_{\text{DIS}}(M + \frac{1}{2} \chi_\sigma(M)) = \chi_\sigma(M).$$

Hence, we can resum the $M = 1$ poles by performing the double-leading resummation of $M = 0$ poles of $\chi_{\text{DIS}}$, determining the associated $\chi_\sigma$ through eq. (6), then symmetrizing it (as $\chi_\sigma$ must be symmetric), and finally going back to DIS variables by using eq. (6) again in reverse. Using the momentum conservation eq. (4) and eq. (6), it is easy to show that $\chi_\sigma(M)$ is an entire function of $M$, with $\chi_\sigma(-\frac{1}{2}) = \chi_\sigma(\frac{3}{2}) = 1$. Since $c = \chi_\sigma(\frac{1}{2}) = O(\alpha_s)$, while $\chi_\sigma(M) \sim 2|M|$ as $|M| \to \infty$, $\chi_\sigma$ necessarily has a minimum at $M = \frac{1}{2}$. Going back to DIS variables, $\chi_{\text{DIS}}$ will also have a minimum, albeit slightly distorted.

In practice, at leading order this procedure leads to a kernel $\chi_{\text{DIS}}(M)$ defined as the solution of the implicit equation

$$\chi_{\text{DIS}}(M) = \chi_\sigma\left(\frac{\alpha}{M}\right) + \chi_\sigma\left(\frac{\alpha}{1 - M + \chi_{\text{DIS}}(M)}\right) +$$
$$+ \frac{n_c \alpha}{\pi} \left(\psi(1) + \psi(1 + \chi_{\text{DIS}}(M)) - \psi(M) - \psi(1 - M + \chi_{\text{DIS}}(M)) - \frac{1}{M} - \frac{1}{1 - M + \chi_{\text{DIS}}(M)}\right),$$

(7)

to be compared to the usual DL result

$$\chi_{\text{DL}}(M) = \chi_\sigma\left(\frac{\alpha}{M}\right) + \frac{\alpha}{\pi} \left(\chi_0(M) - \frac{n_c}{M}\right),$$

(8)
Figure 2. The $\chi_{DL}$ functions with and without symmetrisation. From top to bottom (on the right) the curves are: the LO DL without symmetrisation, the LO+NLO after symmetrisation, the LO after symmetrisation, the dual of GLAP LO+NLO and the LO+NLO DL without symmetrisation.

where $\chi_s(\alpha/M)$ is the kernel dual to the leading order GLAP anomalous dimension: $M = \alpha \gamma_0(\chi_s(\alpha/M))$. The kernel $\chi_{DIS}(M)$ defined by eq. (7) has the following two features: 1) it only differs from the double-leading kernel eq. (8) by terms which are subleading in the double-leading expansion; 2) the kernel $\chi_\sigma$ computed from it using eq. (6) is symmetric. As a consequence, $\chi_{DIS}(M)$ is free of singularities for all positive $M$, rising to unity at the symmetric momentum conservation point $M = 2$ (see Figure 2).

The procedure can be extended to next-to-leading order, where various technical complications arise, due to the need to treat consistently next-to-leading log $Q^2$ terms, specifically those related to the running of the coupling. In figure 2 we show the $\chi_{DL}$ kernels with and without symmetrisation compared to the dual of the GLAP LO+NLO anomalous dimension. The elimination of the bad behaviour of $\chi_1(M)$ near $M = 1$, which is obtained by symmetrisation, results in a much extended range of $N$ where the resummed $\chi$ follows GLAP, even before performing the running coupling resummation.

The presence of a minimum in the symmetrized $\chi$ makes it suitable for the running coupling resummation discussed above. In fact, one can easily prove that the parameters which characterize the running coupling resummation, namely the values $c$ and $\kappa$ of the function and the curvature at the minimum, are the same for the functions $\chi_{DIS}$ and $\chi_\sigma$ related by eq. (6). One may thus construct a fully resummed next-to-leading order anomalous dimension akin to eq. (5), but based on the symmetrized $\chi$. The main ambiguities in the result are related to the treatment of the running of the coupling.

The result, for a 'minimal' running coupling prescription, is displayed in figure 3. For clarity, we show an enlargement for the range $0 < N < 1$: beyond $N = 1$ the curves all essentially overlap. The upper (dotted) curve is the result from eq. (5), whose associate splitting function is displayed in figure 1. The dashed curve is the LO+NLO GLAP curve. Finally, the lowest (solid) curve and the central dot-dashed curve are respectively the LO and NLO results obtained.

Figure 3. Old and new results on the anomalous dimension (see text).
by applying the running coupling resummation to the symmetrized kernels. It is apparent that, even though at LO the new resummed curve somewhat undershoots the GLAP result, the NLO resummed curve is even closer to GLAP than the result in eq. (5). A detailed discussion of these results, and specifically of the ambiguities related to the symmetrization procedure and running of the coupling, will be presented elsewhere [8].

REFERENCES