Gaussian-random Ensembles of Pseudo-Hermitian Matrices *

Zafar Ahmed

Nuclear Physics Division, Bhabha Atomic Research Centre
Trombay, Bombay 400 085, India
zahmed@apsara.barc.ernet.in
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Abstract

Attention has been brought to the possibility that statistical fluctuation properties of several complex spectra, or, well-known number sequences may display strong signatures that the Hamiltonian yielding them as eigenvalues is PT-symmetric (Pseudo-Hermitian). We find that the random matrix theory of pseudo-Hermitian Hamiltonians gives rise to new universalities of level-spacing distributions other than those of GOE, GUE and GSE of Wigner and Dyson. We call the new proposals as Gaussian Pseudo-Orthogonal Ensemble and Gaussian Pseudo-Unitary Ensemble. We are also led to speculate that the enigmatic Riemann-zeros \((\frac{1}{2} \pm it_n)\) would rather correspond to some PT-symmetric (pseudo-Hermitian) Hamiltonian.

I. INTRODUCTION

A large body of spectra of the bound levels and resonances is available in nuclear physics wherein the nuclear interaction Hamiltonian is unknown. The well-known prime numbers 2,3,5,7,9,11,13,17,19,... do not have a representation so far. One may wonder if there is a Hermitian Hamiltonian that can yield them as its discrete eigenvalues. The zeta function,

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\( \zeta(z) \), which is real on real line as per the hypothesis of Riemann (1859) has all its non-real zeros as \( \frac{1}{2} \pm it_n \) (\( t_n \) being real) [1]. The marathon and accurate computation of more than \( 10^{20} \) zeros of \( \zeta(z) \) by Odlyzko [2] testifies RH the best, with not even a single exception so far. First few zeta-zeros (Riemann-zeros : RZs) are given by \( t_1 = 14.13, t_2 = 21.02, t_3 = 30.42, t_4 = 37.58 \). Hilbert and Polya have conjectured that \( t_n \) could be like the eigenvalues of a Hermitian Hamiltonian. Consequently the completeness of the spectra will lead to the proof of one of the most enigmatic and formidable problems called RH.

In nuclear spectroscopy, the pressing need was to identify the universality that underlies the nuclear levels. The Nearest Neighbour Level Spacing Distribution (NNLDS) is the statistical distribution of the fluctuations around the mean level spacing of a collection of spectra under a class of fixed parity or other quantum numbers. Random Matrix Theory (RMT) was discovered in the late 1950s to predict universalities of NNLSD in various situations. Among the notable names we have Wigner, Landau, Dyson, Gaudin, Mehta, Porter, Ginibre and Pandey to associate with RMT [3].

Since the interaction Hamiltonian is not known it would rather be taken as non-integrable, then there are three types of NNLSDs obtained by Wigner and Dyson. These are given as in Eq. (4) (see below) and called as GOE,GUE and GSE spacing-statistics [3]. Most remarkably the nuclear levels with same \((J,\pi)\) are well-known to follow GOE statistics. The energy levels of the chaotic Sinai-billiard are known to follow the same statistics. This has supported the idea that chaos may be there in nuclear dynamics too. Sinai-billiard refers to a particle in a square region with hard, reflecting edges along with a hard, reflecting circle in its center.

Well before the advent of RMT, Hilbert had prophesied that the real part of RZ are distributed as the eigenvalues of certain random Hermitian matrices. In fact, in terms of RMT it means that the real parts of RZs obey NNLSD corresponding to GUE. Over a hundred years old, this prophecy of Hilbert has been testified by Odlyzko [2] as late as 1989 using more than \( 10^{20} \) RZs. Montgomery in 1973 (see in Mehta [3]) analytically derived two-point correlation function of the real parts of RZs which turned out to be the same as that of GUE. Dyson had already expected this and it has also been confirmed numerically by Odlyzko [2]. These two remarkable affirmations have strengthened the Hilbert-Polya
conjecture to look for a Hermitian Hamiltonian for the RH.

Last few years have witnessed an interesting phenomenon whereby the real discrete eigenvalues need not necessarily be possessed by Hermitian Hamiltonians. Non-Hermitian, PT-symmetric [4] or pseudo-Hermitian [5,6] Hamiltonians too can possess real discrete spectrum. In RMT, the matrix ensembles GOE, GUE, GSE refer to Hamiltonians with TRI (Time Reversal Invariance), without TRI, and with TRI including Kramer’s degeneracy respectively. Recently, we have developed the Gaussian-random ensembles of pseudo-Hermitian matrices, that give rise to new “universalities” of NNLSD. We have called the new ensembles as GPUE [7,8] which are expected to represent the cases where Parity (P) and Time-reversal (T) symmetries are individually broken but preserved jointly.

In this paper, we would like to present GPUE with more refinements and reorientations. In Section 2, we briefly introduce RMT with the ensembles of Wigner and Dyson. We then find a natural scope to go in for new ensembles. In Section 3 and 4 the new ensembles are described. In Section 5, we report two interesting dichotomies where spacing statistics are like GOE and GUE, despite Hamiltonians being pseudo-Hermitian. Following one of these dichotomies, in Section 6, we speculate on the possible features of Hamiltonian corresponding to RH. We present a summary of conclusions in Section 6.

II. WIGNER-DYSON ENSEMBLES OF GAUSSIAN-RANDOM MATRICES:

GOE, GUE, GSE

An eigenspectrum is (practically) an outcome of the diagonalization of a Hamiltonian matrix. Since the analysis of NNLS requires at least two eigenlevels, in RMT, one begins with $2 \times 2$ Hamiltonian matrix for simplicity. The RMT takes an important note of the fact that for systems with TRI, the matrix Hamiltonians are real-symmetric ($H^R$); for systems without TRI, the Hamiltonians are Hermitian matrices ($H^H$) and for systems with TRI plus Kramer’s degeneracy, the Hamiltonians are even (at least 4) dimensional ($H^K$). These Hamiltonian matrices are given as [3,9]:
\[ H^R = \begin{bmatrix} a + b & c \\ c & a - b \end{bmatrix}, \quad H^H = \begin{bmatrix} \alpha & \gamma \\ \gamma^* & \beta \end{bmatrix}, \quad H^K = \begin{bmatrix} \alpha & 0 & \gamma^* & -\delta \\ 0 & \alpha & \delta^* & \gamma \\ \gamma & \delta & \beta & 0 \\ -\delta^* & \gamma^* & 0 & \beta \end{bmatrix}, \] \tag{1}

where \( \alpha = a + b, \beta = a - b, \gamma = c + id, \delta = e + if. \) In order to have a large collection of eigenvalue pairs, we assume the entries \( a, b, c, d, e, f \) are real and drawn independently from a Gaussian-random population. Notice that respective eigenvalues for (1) are

\[ E^R_{1,2} = a \pm \sqrt{b^2 + c^2}, \quad E^H_{1,2} = a \pm \sqrt{b^2 + c^2 + d^2}, \quad E^K_{1,2} = a \pm \sqrt{b^2 + c^2 + d^2 + e^2 + f^2}. \tag{2} \]

In RMT, to reemphasize, the energy eigenvalues for TRI systems turn out to have the form as \( E^R \), for non-TRI systems they are as \( E^H \) and for TRI systems with Kramer’s degeneracy are as \( E^K \)s. The level-spacings \( (s = |E_1 - E_2|) \) in various cases are

\[ s^R \sim \sqrt{b^2 + c^2}, \quad s^H \sim \sqrt{b^2 + c^2 + d^2}, \quad s^K \sim \sqrt{b^2 + c^2 + d^2 + e^2 + f^2}. \tag{3} \]

Now the question to be asked is: what is the probability distribution of the level-spacing \( s \), when \( a, b, ..., f \) are Gaussian-random variables? More importantly one wants to know whether the levels have tendency to repel or attract each other and then what the degree of repulsion/attraction is. The real-symmetric matrices have an Orthogonal symmetry which corresponds to \( \text{SO}(N) : \text{GOE} \). The Hermitian matrices have unitary symmetry which corresponds to \( \text{SU}(N) : \text{GUE} \). The matrices \( H^K \) have Symplectic symmetry corresponding to Symplectic group \( \text{Sp}(N) : \text{GSE} \). These facts are used to derive Wigner-Dyson universalities in NNLS as \( [3,9] \)

\[ P^{\text{GOE}}(x) = \frac{\pi}{2} x e^{-\frac{x^2}{4}}, \quad P^{\text{GUE}}(x) = \frac{32}{\pi^2} x^2 e^{-\frac{4x^2}{\pi}}, \quad P^{\text{GSE}}(x) = \frac{2^{18}}{3^6 \pi^3} x^4 e^{-\frac{64x^2}{9\pi}}. \tag{4} \]

Here \( x \) is a scaled spacing \( (s) \) with respect to its mean value \( (\langle s \rangle) \). For smaller values of spacing \( s \), notice the tendency of level repulsion as \( P(x \to 0) \to 0 \). The degree of repulsion is linear, quadratic and quartic respectively for GOE, GUE and GSE. Most importantly these distributions (4) turn out to be excellent approximants to \( P(x) \) for \( N \times N \) matrices \([3,9]\).

An alternative intuitive way of looking into these NNLS is to see that the three statistics (4) correspond to the following multidimensional integral
\[ P^{\text{GOE}}(s) \sim \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(a^2+b^2+c^2)} \delta(s - \sqrt{b^2 + c^2}) \, da \, db \, dc, \]  

(5)

and similarly others for GOE and GSE.

The common feature of the eigenvalues (2) or spacings (3) lies in their absolute reality. Let us ask the following questions: Firstly, can there be Hamiltonians possessing conditionally real eigenvalues or spacings e.g., \( s \sim \sqrt{b^2 - c^2} \) (real iff \( b^2 \geq c^2 \)) and \( s \sim \sqrt{b^2 + c^2 - d^2} \) (real iff \( b^2 + c^2 \geq d^2 \))? Secondly, what are the symmetries of such Hamiltonians? Thirdly, what are new universalities of NNLSD? Such questions have led us to think of new Gaussian-random ensembles of pseudo-Hermitian Hamiltonians [7,8]. By a Gaussian ensemble, we would mean that the probability distribution of Hamiltonian \( H \) is commonly given as

\[ P(H) = \mathcal{N} e^{-\frac{\text{Tr}(HH^\dagger)}{2\sigma^2}}. \]  

(6)

**III. GAUSSIAN PSEUDO-ORTHOGONAL ENSEMBLES (GPOE)**

**Pseudo-symmetric or complex-symmetric matrix Hamiltonians**

Let us consider a matrix Hamiltonian \( H \) given below which is pseudo-Hermitian as \( \eta H \eta^{-1} = H^\dagger \) [5] and pseudo-real i.e., \( \rho H \rho^{-1} = H^* \) [11]. It is self-pseudo-adjoint or symmetric as \( H' = H \). Here \( \eta \) are \( \rho \) are preferably involutary operators. It has got conditionally real eigenvalues iff \( b^2 \geq c^2 \) Here prime, asterisk (\( \mathcal{K}_0 \)) and dagger denote transpose, conjugate and transpose-conjugate, respectively.

\[ H = \begin{bmatrix} a + b & ic \\ ic & a - b \end{bmatrix}, \quad b^2 \geq c^2, \quad \eta = \rho = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad E_{1,2} = a \pm \sqrt{b^2 - c^2}. \]  

(7)

One can construct an antilinear commutant \( \Theta = \rho^{-1} \mathcal{K}_0 \) [10] of \( H \) such that \([\Theta, H] = 0\) or \( \Theta H \Theta^{-1} = H \) and \( \Theta^2 = 1 \). We would like to assert that here \( P = \rho^{-1} \) and \( T = \mathcal{K}_0 \) and hence the antilinear symmetry \( \Theta = PT \). When eigenvalues are real \((b^2 > c^2)\), we have \( PT \Psi_n = (-1)^n \Psi_n \). When \( b^2 < c^2 \), the PT-symmetry is spontaneously broken. This Hamiltonian in our opinion is another realization of Hamiltonians with antilinear symmetry as visualized by Haake (page 217 in [9]) as \([A, D] = 0\) such that \( A^2 = 1 \).
We define pseudo-orthogonal transformation as \( pO' = \delta pO^{-1}\delta^{-1} \) such that for any two arbitrary vectors from a linear space the scalar product remains invariant i.e., \( \tilde{x}' \delta \tilde{y} = x'\delta y \), where \( \tilde{x} = pOx \) and \( \tilde{y} = pOy \). Let us represent \( pO \), energy-eigenvalue matrix \( E \) and a metric \( \delta \)

\[
\delta = \begin{bmatrix}
cosh \theta & i \sinh \theta \\
-i \sinh \theta & \cosh \theta
\end{bmatrix}, \quad E = \begin{bmatrix} E_1 & 0 \\
0 & E_2 \end{bmatrix}.
\]

where \(-\infty < \theta < \infty\). The single parameter matrix \( pO \), is expressible as \( \exp(2i\theta J_2) \) with \( J_2 = \frac{1}{2}i\sigma_y \), constitutes a subgroup of \( SU(1, 1) \) [12]. A very important consequence of the group connection is that we can generate all possible \( H \) in (7) as \( H = pO E pO^{-1} \). This provides us with a unique connection between \( (a, b, c) \) and \( (E_1, E_2, \theta) \) and the consequent Jacobian is \( J = \frac{\lvert \delta \rvert}{8} \). We have

\[
a = \frac{E_1 + E_2}{2}, \quad b = \frac{E_1 - E_2}{2} \cosh 2\theta, \quad c = -\frac{E_1 - E_2}{2} \sinh 2\theta.
\]

For brevity, we have used \( s = E_1 - E_2 \) and \( t = E_1 + E_2 \). We can write the probability distribution, \( P(H) \) (6) for the Hamiltonian in (7) as

\[
P(a, b, c) = N e^{-\frac{(a^2 + b^2 + c^2)}{\sigma^2}}.
\]

Using (9) we can transform (10) in terms of \( (t, s, \theta) \), further integration over \( \theta \) on \([-\infty, \infty]\) gives Joint Probability Distribution function of \( (E_1, E_2) \) as

\[
P(E_1, E_2) = N' s K_0 \left( \frac{s^2}{4\sigma^2} \right) e^{-\frac{t^2}{4\sigma^2}}.
\]

Next the integration over \( t \) on \([-\infty, \infty]\) yields the NNLSAD as

\[
P(s) = N'' s K_0 \left( \frac{s^2}{2\sigma^2} \right).
\]

By finding \( \langle s \rangle \) using (12) and introducing \( x = \frac{s}{\langle s \rangle} \), we eventually find the normalized NNLSAD and call it as \( P^{GPOE}(x) \)

\[
P^{GPOE}(x) = \frac{\Gamma^4(-\frac{1}{2})}{32\pi^3} x K_0 \left( \frac{2\Gamma^4(3/4)}{\pi^2} x^2 \right).
\]

If we write it as \( P^{GPOE}(x) = \alpha x K_0(\beta x^2) \), we have \( \alpha = 0.5818 \) and \( \beta = 0.4569 \). When \( 0 < x < 0.5 \), we have \( P^{GPOE}(x) \sim (0.5 - 1.2 \ln x) x \). For any other pseudo-symmetric or
complex symmetric matrix Hamiltonian that is composed of three independent Gaussian-random variables \((a, b, c)\) appearing linearly in \(H\), we claim that \(P^{GPOE}(x)\) is the universality. The new “universality” shows a distinctly different behaviour as compared to the usual ones (see Fig. 1(b)).

IV. GAUSSIAN PSEUDO-UNITARY ENSEMBLES

Pseudo-Hermitian matrix Hamiltonians

We now consider pseudo-Hermitian matrix Hamiltonians with four parameters \((a, b, c, d)\)

\[
H = \begin{bmatrix}
  a + b & d + ic \\
- d + ic & a - b
\end{bmatrix},
\]

\[
e^2 = b^2 - c^2 + d^2 \geq 0, \eta = \begin{bmatrix}
  1 & 0 \\
  0 & -1
\end{bmatrix}, E_{1,2} = a \pm e.
\]

(14)

Here we have \(\eta H \eta^{-1} = H^\dagger\), \(P\) and \(T\) operators can be constructed as prescribed in [10] and antilinear commutant, \(\Theta\), of \(H\) can be constructed as prescribed in [11]. Consider a transformation \(pU\) which preserves the pseudo-norm as \(\bar{x}^\dagger \eta \bar{y} = x^\dagger \eta y\), where \(\bar{x} = pUx, \bar{y} = pUy\).

In doing so, \(pU\) would satisfy an interesting condition i.e., \(pU^\dagger = \eta pU^{-1} \eta^{-1}\) which is called pseudo-unitarity. (see e.g., [7])

A general three parameter \((\theta, \psi, \phi)\) matrix, \(pU\), which is pseudo-Unitary under the same metric \(\eta\) (14) can be written as

\[
pU = \begin{bmatrix}
e^{i\psi} \cosh \theta & e^{i\phi} \sinh \theta \\
e^{i\psi} \sinh \theta & e^{-i\phi} \cosh \theta
\end{bmatrix}, 0 \leq \phi, \psi \leq 2\pi, 0 < \theta < \infty.
\]

(15)

This constitutes a Lie group \(SU(1, 1)\) [12] with generators as \(J_0 = \frac{1}{4}\sigma_z, J_1 = \frac{1}{4}i\sigma_y, J_3 = -\frac{1}{4}i\sigma_x\). However, in order to construct the pseudo-Hermitian matrix (14) we require only two parameters in \(pU\). The same situation arises [3,9] in case of GUE, where two out of three parameters suffice in writing the unitary matrix \(U\), nevertheless it requires three-parameters to have \(SU(2)\). Thus, we take \(\psi = 0\) in (15) and generate \(H\) in Eq. (14) as \(pUE_pU^{-1} = H\).

This is how we go over to \((E_1, E_2, \phi, \psi)\) from \((a, b, c, d)\). We find

\[
a = \frac{t}{2}, \quad b = \frac{s}{2} \cosh 2\theta, \quad c = -\frac{s}{2} \sinh 2\theta \cos \phi, \quad d = \frac{s}{2} \sinh 2\theta \sin \phi \quad \text{and} \quad J = \frac{s^2}{4} \sinh 2\theta.
\]

(16)
The probability function (6) for $H(14)$ works to $P(a,b,c,d) = \mathcal{N} \exp[-(a^2 + b^2 + c^2 + d^2)/\sigma^2]$. A similar procedure (yet more involved) as done in Section 3, from Eq. $(10)$ to $(13)$, leads us to a new NNLS as

$$P_{GPUE}(x) = \frac{B^2}{2(\sqrt{2} - 1)} x e^{\frac{B^2 x^2}{4}} \text{erfc} \left( \frac{Bx}{\sqrt{2}} \right), \quad B = \frac{2(\sqrt{2} - \log(1 + \sqrt{2}))}{\sqrt{\pi(\sqrt{2} - 1)}}.$$  \hspace{1cm} (17)

If we write as $P_{GPUE}(x) = \alpha x e^{\beta x^2} \text{erfc}(\gamma x)$ where $\alpha = 2.5433$, $\beta = 0.5267$, $\gamma = 1.0263$. Its linear dependence on $x$ is deceptive, its behaviour near small values of $x$ is actually curved (short dashed line in Fig. 1(b)) lying below the curve corresponding to $P_{GPOE}(x)$ (see solid line in Fig. 1(b)). For $0 < x < 0.5$, we have $P_{GPUE}(x) \sim 2.5x(1 - .95x)$.

V. INTERESTING DICHOTOMIES AND SPECULATIONS ON RH

Quasi-Hermitian matrix Hamiltonians

Pseudo-Hermitian Hamiltonians under a definite metric as given below are called quasi-Hermitian. Consider 3- and 4-parameter cases of such $2 \times 2$ matrix Hamiltonians.

$$H_3 = \begin{bmatrix} a & (b + ic)/\epsilon \\ (b - ic)/\epsilon & a \end{bmatrix}, \quad H_4 = \begin{bmatrix} \alpha & \gamma/\epsilon \\ \gamma^*/\epsilon & \beta \end{bmatrix}, \quad \eta = \begin{bmatrix} \epsilon & 0 \\ 0 & 1/\epsilon \end{bmatrix}.$$  \hspace{1cm} (18)

Here $\alpha = a + b, \beta = a - b, \gamma = c + id$. These matrices, despite being pseudo-Hermitian possess absolutely real eigenvalues. By constructing one and two parameter pseudo-unitary transformation matrices and by carrying out the procedure outlined in Sections, 3 and 4, the spacing distributions have been obtained. Interestingly, for $H_3$ the Wigner surmise, $P_{GPOE}(x)$ has been recovered identically [8]! Even more interestingly, by defining $\epsilon = e^{-\kappa}$, we find a new analytic expression [13] for $P(x)$ that hardly differs from $P_{GUE}(x)$ for $\kappa$ from 0 to 0.5. For other values of $\kappa$ the differences between are also not considerable.

For $\epsilon = 1$, the Hamiltonian is Hermitian, by changing $\epsilon$ it becomes non-Hermitian such that the spacing distribution does not change appreciably. We may therefore take such Hamiltonians and interpret them as a smooth perturbation of the Hermitian Hamiltonian.

We feel that the display of $P_{GUE}(x)$ by a certain class of spectra, though the underlying Hamiltonian is not Hermitian (instead it is quasi-Hermitian) is a very remarkable result.
Recall that RZs display $P_{GUE}(x)$ so the prospective Hamiltonian is expected to be both Hermitian and TRI-breaking. In the light of our dichotomous result, we speculate that the Hamiltonian relevant to RH could also be a PT-symmetric (Hamiltonian). Since complex-conjugate eigenvalues are found for a PT-symmetric Hamiltonian when PT-symmetry is spontaneously broken, so $\frac{1}{2} \pm i t_n$ would naturally follow from a PT-symmetric Hamiltonian. Furthermore, the vanishing of the norm (PT-norm) of the eigenstates in the domain where spontaneous breaking of symmetry occurs can be seen to be directly connected to a crucial criterion proposed by Alain Connes [1] for a Hamiltonian which could be relevant to RH. Recently, we have found [13] that for RH a few Hermitian Hamiltonians proposed so far [14] do not even possess a discrete spectrum.

VI. CONCLUSION

The present work tries to answer the basic question as to what could be the random matrix theory of currently researched pseudo-Hermitian Hamiltonians. The new “universalities” in Eqs. (13) and (17) as displayed in Fig. 1 are distinctly different from the usual ones, resulting in weaker repulsion among the energy-levels. Like the well established GOE,GUE and GSE, here it remains to be proved that the claimed results for $2 \times 2$ matrices would actually stay on at least as good approximants for the $N \times N$ case. This is an open challenge an answer to this would actually take us from “universalities” to universalities.

However, we feel that ‘the weaker level repulsion for small spacings’ is the essence of
pseudo-Hermiticity. And since the pseudo-Hermiticity has been re-cast in terms of more physical PT-symmetry, an observance of weaker level repulsion at small spacings would suggest an individual violation of P and T symmetries and joint invariance of PT.

PT-symmetric (pseudo-Hermitian) Hamiltonian as a physical model may now be far, nevertheless signature such as weaker level repulsion would give way to such Hamiltonians.

Two interesting dichotomies have also been presented - owing to one of them we have speculated that the prospective Hamiltonian for the Riemann Hypothesis would rather be PT-symmetric (pseudo-Hermitian).

Lastly, we would like to point out that in contrast to the Ginibre matrix ensembles and some more suggested by Haake [9], we consider such complex matrices (pseudo-Hermitian) which give conditionally real eigenvalues and we define level spacing as $(E_1 - E_2)$ and not as modulus of difference of two complex eigenvalues.

REFERENCES


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