BINARY SYSTEMS OF NEUTRAL MESONS IN QUANTUM FIELD THEORY

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Abstract: Quasi-degenerate binary systems of neutral mesons of the kaon type are investigated in Quantum Field Theory (QFT). General constraints cast by analyticity and discrete symmetries $P$, $C$, $CP$, $TCP$ on the propagator (and on its spectral function) are deduced. Its poles are the physical masses; this unambiguously defines the propagating eigenstates. It is diagonalized and its spectrum thoroughly investigated. The role of “spurious” states, of zero norm at the poles, is emphasized, in particular for unitarity and for the realization of $TCP$ symmetry. The $K_L - K_S$ mass splitting triggers a tiny difference between their $CP$ violating parameters $\epsilon_L$ and $\epsilon_S$, without any violation of $TCP$. A constant mass matrix like used in Quantum Mechanics (QM) can only be introduced in a linear approximation to the inverse propagator, which respects its analyticity and positivity properties; it is however unable to faithfully describe all features of neutral mesons as we determine them in QFT, nor to provide any sensible parameterization of eventual effects of $TCP$ violation. The suitable way to diagonalize the propagator makes use of a bi-orthogonal basis; it is inequivalent to a bi-unitary transformation (unless the propagator is normal, which cannot occur here). Problems linked with the existence of different “in” and “out” eigenstates are smoothed out. We study phenomenological consequences of the differences between the QFT and QM treatments: the non-vanishing of the semi-leptonic asymmetry $\delta_S - \delta_L$, does not signal, unlike usually claimed, $TCP$ violation, while $A_{TCP}$ keeps vanishing when $TCP$ is realized. We provide expressions invariant by the rephasing of $K^0$ and $\bar{K}^0$.


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1 INTRODUCTION

Binary systems of quasi-degenerate neutral mesons are undoubtedly among the most interesting in particle physics, from both experimental and theoretical points of view. It is in particular thanks to them that the intriguing phenomenon of $CP$ violation \[17\] \[25\] has been discovered.

Such systems are beautiful test grounds for Quantum Mechanics (QM) and, indeed, most approaches to their peculiarities do not go beyond this level $^1$; it is only recently that the need arose of a treatment in the framework on Quantum Field Theory (QFT) \[12\] \[10\] (it was actually mainly motivated for the leptonic sector after the discovery of neutrino oscillations). However, conceptual problems still remained, in particular concerning the existence of two different sets of mass eigenstates, belonging respectively to the “in” and “out” spaces (see \[14\] \[4\] and references therein). General constraints cast by analyticity properties were never explicitly written, and the ones cast by discrete symmetries often written with conventions which forbade a full generality. The formalism of a mass matrix also seemed never to be cast in doubt, though its existence, as we shall see, can only be assumed in a certain approximation.

All these open questions, and the growing need for precise criteria to test discrete symmetries, made necessary an exhaustive investigation of these systems in QFT. This is what we propose here.

The plan of the paper is the following:

- In section 2 we give the general definitions and notations for the propagator of a binary system of neutral mesons, and deduce on the most general ground the constraints cast on it by analyticity, positivity, and the discrete symmetries $C$, $P$, $CP$ and $T CP$. All arbitrary phases have been kept in the definition of discrete symmetries, which make our formulæ more general than the ones in \[39\]; this has influence on particular on the role held by Lorentz invariance in the deduction of the symmetry properties of the propagator.

In subsection 2.3.2 we show how the introduction of a mass matrix can only be done in a linear approximation to the inverse propagator. This casts restrictions on it, which will be made explicit in subsection 4.

This section is completed by the long appendix A which explicitly gives all demonstrations concerning the role of discrete symmetries, and provides a detailed discussion of the special role of $T CP$. In particular, in the case of unstable particles under concern which necessitates the introduction of a non-hermitian Lagrangian, two ways of implementing the $T CP$ symmetry, that we call the conventions of Wightman and of Schwinger-Pauli are discussed in detail.

- Section 3 is dedicated to the diagonalization of the propagator, with a special emphasis on the property of normality.

  * $CP$ invariance entails that the propagator is a special type of normal matrix, and subsection 3.1 deals with normal propagators and $CP$ eigenstates; we show that, if one wants furthermore to implement the constraints set by $TCP$ invariance, the $CP$ violating parameter for a general normal propagator is constrained to be purely imaginary, which is in contradiction with experiments.

  * subsection 3.2 deals with non-normal propagators.

We first recall two different ways of diagonalizing a complex mass matrix: by a bi-unitary transformation and by using a bi-orthogonal basis. These procedures are inequivalent, as will be explicitly shown in subsection 3.2.7.

We then define, as they should be, the physical masses of the neutral kaons, as the poles of their full propagator.

Next we explicitly diagonalize the $TCP$ invariant propagator by using a bi-orthogonal basis and determine its physical (mass) eigenstates. We determine all $CP$ violating parameters and show $TCP$ invariance does not entail that the $CP$ parameter $\epsilon_L$ of $K_L$ is identical to the one $\epsilon_S$ of $K_S$. We study their dependence on an arbitrary rephasing of $K^0$ (and $\bar{K}^0$) and show that their real and imaginary parts depend on this phase; physically relevant quantities are of course, phase invariant. This smooths out conceptual difficulties linked

$^1$Ambiguities that appear in this treatment were recently outlined in \[28\].
to the existence of two sets of eigenstates, “in” and “out”. The study of the $CP$ violating parameters is completed in appendix $\mathbb{B}$. We then give the explicit form of all propagating mass eigenstates in terms of the $CP$ violating parameters.

We show the non-trivial way in which $TCP$ invariance is realized. At each of the two scales $q^2 = M^2_{K_L}$ and $q^2 = M^2_{K^0_S}$, the propagator has two sets of eigenstates: one corresponds to the propagating $K_L$ (“in” and “out”) or $K^0_S$ (“in” and “out”), and the other one does not propagate (we call it spurious). At any given $q^2$, $TCP$ invariance needs the two sets of eigenstates corresponding to this $q^2$, and, in particular, for $q^2$ equal to any of the two physical masses, both the propagating and the spurious states are essential. Both sets of states are also needed for the completeness relation at a given $q^2$.

We then show why bi-unitary transformations are not suitable to diagonalize the propagator of the neutral mesons: while they yield the correct physical masses and propagating eigenstates, their “spurious” eigenstates differ.

Last, we emphasize the role of the non-vanishing $\epsilon_L - \epsilon_S$ by depicting the simplified picture that arises when the two $CP$ violating parameters are assumed (like in $TCP$ invariant QM) to be identical.

- Section $\mathbb{A}$ deals with the eventual introduction of a mass matrix, like commonly done in QM.
  - First we recall the role of hermitian and normal mass matrices in QM, in relation with neutral mesons.
  - Then we show how a mass matrix in QFT cannot give consistent results for the systems of neutral mesons and cannot describe faithfully all its properties, in particular $TCP$ symmetry with different $CP$ violating parameters for $K_L$ and $K^0_S$ as was shown to occur. This yields restrictions on quantum mechanical treatments of such systems which, nevertheless, can provide a satisfying description of $CP$ violation.

- Section $\mathbb{S}$ is dedicated to calculating three semi-leptonic asymmetries. They are all unambiguously expressed in terms of the $CP$ violating parameters $\epsilon^0_L$ and $\epsilon^0_S$ of the mass eigenstates $| K_L \rangle_{in}$ and $| K^0_S \rangle_{in}$.
  - We suppose that the $\Delta S = \Delta Q$ rule is satisfied.
  - We first calculate the asymmetry $A_T$ measured in the CPLEAR experiment, and give a result which is independent of an arbitrary rephasing of $K^0$ (and $\overline{K^0}$), unlike the often quoted QM result $A_T \approx 4 \Re(\epsilon)$;
  - We next calculate the semi-leptonic charge asymmetries $\delta_L$ and $\delta_S$, and give, there too, a formula invariant by the rephasing of $K^0$, unlike a customary approximation often quoted in QM.

We show that $\delta_S - \delta_L$ measures the difference between the $CP$ violating parameters of the two mass eigenstates; this does not characterize $TCP$ violation, unlike in QM.

- We use the quark picture to find an estimate of $\epsilon_S - \epsilon_L$ and find $\epsilon_S - \epsilon_L \approx \epsilon_L \approx 10^{-17}$.
- Last, we calculate the so-called $TCP$ asymmetry $A_T^{TCP}$. This is achieved by evaluating the Feynman diagrams obtained from a $TCP$ invariant Lagrangian. It is shown to vanish, and, even when $\epsilon_L \neq \epsilon_S$, to be a test of $TCP$ invariance.

- Section $\mathbb{6}$ is a general conclusion.

The citations have been limited to a few number; it is impossible to quote all the literature devoted to so rich systems of particles, and it is fortunate that excellent textbooks are now available $[8][14][11]$. We refer the reader to these and all the references contained there.

Unless in subsection $[5.1.2]$ where estimates will be done with the help of the quark picture for mesons, we will work in the framework of a renormalizable local quantum field theory where the two neutral kaons are represented by a complex operator and its hermitian conjugate. We shall also consider that the instability of neutral mesons does not break the one-to-one relationship between fields and particles though, truly speaking, the only asymptotic fields are electrons, neutrinos and photons. Since we restrict ourselves to a binary system and, at the same time, want to account for the breaking of discrete symmetries which can only be observed through their decays, we have to allow for complex masses $[22]$ and a non-hermitian Lagrangian. The subtle interplay with, in particular, $TCP$ symmetry is discussed in the Appendix $\mathbb{A}$. 


2 THE PROPAGATOR; GENERAL CONSTRAINTS

2.1 DEFINITION AND NOTATIONS

Since we are dealing with complex (matricial) functions of a complex variable \( z \), it is essential to clearly set the notations and conventions which will be used throughout this work.

If \( z = x + iy \), \( x, y \in \mathbb{R} \) is a complex number, its conjugate is \( \bar{z} = x - iy \); its real part is noted \( \Re(z) \) and its imaginary part \( \Im(z) \).

If \( f(z) \) is a complex function of the complex variable \( z \), for example \( f(z) = az^2 + bz + c, a, b, c \in \mathbb{C} \), its complex conjugate is noted \( \bar{f}(z) \) or \( f(z) \), and, for the example proposed, one has \( \bar{f}(z) = \bar{a}z^2 + \bar{b}z + \bar{c} \). The notation \( \bar{f}(z) = \bar{a}z^2 + \bar{b}z + \bar{c} \) (for the given example) can also be useful.

If \( F(z) = \begin{pmatrix} f_1(z) & f_2(z) \\ f_3(z) & f_4(z) \end{pmatrix} \) is a complex \( 2 \times 2 \) matrix the elements of which are complex functions of the complex variable \( z \), its hermitian conjugate is noted \( F^\dagger(z) = \begin{pmatrix} \bar{f}_1(\bar{z}) & \bar{f}_2(\bar{z}) \\ \bar{f}_3(\bar{z}) & \bar{f}_4(\bar{z}) \end{pmatrix} \), and will also be noted \( [F(z)]^\dagger \). In the case when \( f_1, f_2, f_3 \) and \( f_4 \) are polynomial functions of \( z \), like in the example given above for \( f(z) \), it is also convenient to define \( F^\dagger(z) \) which is obtained from \( F(z) \) by:
- taking the transposed of \( F(z) \);
- changing all the coefficients of the \( z \) monomials into their complex conjugates;
- leaving \( z \) unchanged.

The transposed of an operator \( \mathcal{O} \) is noted \( \mathcal{O}^T \) and its hermitian conjugate is noted \( \mathcal{O}^\dagger \).

Unless specified, all propagators and mass matrices are written in the \( (K^0, \overline{K^0}) \) basis.

Let \( \varphi_{K^0}(x) \) be the Heisenberg operator for \( K^0 \) at space time point \( x = (\vec{x}, t) \) and \( \varphi_{\overline{K^0}}(\vec{x}) \) the corresponding Schrödinger operator (see also subsection A.3.1). Since other fields will be related to it, we shall often omit the corresponding subscript, writing instead \( \varphi \) when it is the only one appearing in a formula.

The Heisenberg field \( \varphi_{\overline{K^0}}(\vec{x}, t) \) associated with \( \overline{K^0} \) is defined in terms of \( \varphi_{K^0}(\vec{x}, t) \) in subsection A.1.1 of appendix A. This introduces two arbitrary phases \( \alpha \) and \( \delta \). \( C \) being the charge conjugation operator (operating on Schrödinger fields):

\[
C \varphi_{K^0}(\vec{x}) C^{-1} = e^{-i\alpha} \varphi_{\overline{K^0}}(\vec{x}),
\]

and

\[
\varphi_{\overline{K^0}}(\vec{x}) = e^{-i\delta} \varphi_{K^0}^\dagger(\vec{x}).
\]

lead to

\[
C \varphi_{K^0}(\vec{x}) C^{-1} = e^{-i(\alpha+\delta)} \varphi_{\overline{K^0}}^\dagger(\vec{x}).
\]

In \( x \) space, the Feynman propagator \( \Delta(x) \) is a \( 2 \times 2 \) matrix function which connects the \( K^0 \) and \( \overline{K^0} \) states to themselves and to each other; it is expressed in terms of vacuum expectation values of time-ordered products \( T\{\varphi(x)\varphi(y)\} \) of Heisenberg fields

\[
\Delta(x) = \begin{pmatrix} d(x) & -g(x) \\ -h(x) & f(x) \end{pmatrix},
\]

with, using (118) \(^3\)

\(^3\)At any value of \( z \), the elements of \( F^\dagger(z) \), which are complex numbers, coincide with the elements of the hermitian conjugate \([F(z)]^\dagger\) of \( F(z) \) at the same value of \( z \).

\(^3\)uses different conventions; though we work with the same type of approach, we found necessary to reinstall all phases that were canceled there, to take out the parity operation from the definition of \( \overline{K^0} \) from \( K^0 \) and to come back to the basic definition of antiparticles \( \overline{f(\bar{z})} \). This removes all ambiguities and fortuitous coincidences in our demonstrations and results.
\[ d(x) = \langle K^0 | \Delta(x) | K^0 \rangle = \langle 0 | T\{\varphi_{K^0}(\vec{x}, t) \varphi_{K^0}^\dagger(-\vec{x}, -t)\} | 0 \rangle, \]
\[ f(x) = \langle K^0 | \Delta(x) | K^0 \rangle = \langle 0 | T\{\varphi_{K^0}(\vec{x}, t) \varphi_{K^0}^\dagger(-\vec{x}, -t)\} | 0 \rangle, \]
\[ -g(x) = \langle K^0 | \Delta(x) | K^0 \rangle = \langle 0 | T\{\varphi_{K^0}(\vec{x}, t) \varphi_{K^0}^\dagger(-\vec{x}, -t)\} | 0 \rangle, \]
\[ -h(x) = \langle K^0 | \Delta(x) | K^0 \rangle = \langle 0 | T\{\varphi_{K^0}(\vec{x}, t) \varphi_{K^0}^\dagger(-\vec{x}, -t)\} | 0 \rangle. \]

(5)

A theory is called “Lorentz invariant” when it is invariant by the proper orthochronous Lorentz group \( L_+^1 \).

Now, and this will be important in the following, in particular in our discussion of \( CP \) transformation, because we are here dealing with scalar fields, \( \Delta(\vec{x}, t) \) can only be a function of \( (|\vec{x}|^2 - t^2) \), and, in particular, though space inversion, which has determinant \((-1)\) is not part of \( L_+^1 \),

\[ \Delta(\vec{x}, t) = \Delta(-\vec{x}, t). \]

(6)

### 2.2 THE PROPAGATOR IN FOURIER SPACE; RENORMALIZATION

In Fourier \((p^2 = z)\) space, the matrix elements of \( \Delta(z) \) are the Fourier transformed of the ones of \( \Delta(x) \) in (5), and we write

\[ \Delta(z) = \begin{pmatrix} d(z) & -g(z) \\ -h(z) & f(z) \end{pmatrix}. \]

(7)

We assume from now on-wards that we operate in the framework of a renormalizable quantum field theory for mesons. \( \varphi \) stands for the renormalized kaon field, and we note \( \varphi_0 \) the bare kaon field. All quantities occurring in (5) and (7) are the renormalized ones.

\( \varphi \) and \( \varphi_0 \) are connected as usual by

\[ \varphi_0 = \sqrt{Z_{K^0}} \varphi, \]

(8)

where \( Z_{K^0} \) is the renormalization constant of the kaon field. (5) and the definitions (5) entail that \( d(z), f(z), g(z) \) and \( h(z) \) occurring in (7) are connected to the bare \( d_0, f_0, g_0 \) and \( h_0 \) defined below

\[ d_0(x) = \langle 0 | T\{\varphi_0(\vec{x}, t) \varphi_0^\dagger(-\vec{x}, -t)\} | 0 \rangle, \]
\[ f_0(x) = \langle 0 | T\{\varphi_0^\dagger(\vec{x}, t) \varphi_0(-\vec{x}, -t)\} | 0 \rangle, \]
\[ -g_0(x) = e^{i\delta} \langle 0 | T\{\varphi_0(\vec{x}, t) \varphi_0(-\vec{x}, -t)\} | 0 \rangle, \]
\[ -h_0(x) = e^{-i\delta} \langle 0 | T\{\varphi_0^\dagger(\vec{x}, t) \varphi_0(-\vec{x}, -t)\} | 0 \rangle. \]

(9)

by

\[ d(z) = \frac{1}{\sqrt{Z_{K^0}}} d_0, \quad f(z) = \frac{1}{(\sqrt{Z_{K^0}})Z_{K^0}} f_0, \quad g(z) = \frac{1}{Z_{K^0}} g_0, \quad h(z) = \frac{1}{Z_{K^0}} h_0. \]

(10)
yields the following renormalized inverse propagator
\[
\Delta^{-1}(z) \equiv \frac{1}{d(z)f(z) - g(z)h(z)} \begin{pmatrix} f(z) & g(z) \\ h(z) & d(z) \end{pmatrix} \equiv \begin{pmatrix} a(z) & -b(z) \\ -c(z) & d(z) \end{pmatrix}.
\]
(11)

In (9) we have introduced the bare \( \phi_0 \) and \( \phi_0^\dagger \) fields and supposed that
\[
C\phi_0(\vec{x})C^{-1} = e^{-i(\alpha + \delta)}\phi_{0}^\dagger(\vec{x}).
\]
(12)

indeed, in a renormalizable theory, the counterterms (unlike the finite terms) are of the same nature as the operators present in the initial Lagrangian \[38\], and they respect in particular the way in which the fields transform by discrete symmetries and by complex (hermitian) conjugation; so, if the bare kaons fields are related to each other by charge conjugation in a certain way, the renormalized fields should be related to each other in the same way; likewise, since the counterterms control the renormalization constants \( Z_K^0 \) and \( Z_{K^0} \), the latter must satisfy (from (2), (12) and (119))
\[
Z_K^0 = Z_K^{0\dagger},
\]
(13)
such that, in all formulæ, \( \sqrt{Z_K^0} \) can be replaced with \( \sqrt{Z_K^{0\dagger}} \), furthermore, both are calculated from the evaluation of Green functions in the ultraviolet regime, that is far from any cut or physical singularity, which entails that they must be real and, accordingly
\[
Z_K^0 = Z_K^{0\dagger}.
\]
(14)

From (10) and (11), one gets the following relations between the renormalized and bare components of the inverse propagator
\[
a = \sqrt{Z_K^0}\sqrt{Z_K^{0\dagger}} \frac{d_0}{d_0f_0 - g_0h_0} = \sqrt{Z_K^0}\sqrt{Z_K^{0\dagger}} \hat{a}_0,
\]
\[
d = \sqrt{Z_K^0}\sqrt{Z_K^{0\dagger}} \frac{a_0}{d_0f_0 - g_0h_0} = \sqrt{Z_K^0}\sqrt{Z_K^{0\dagger}} \hat{d}_0,
\]
\[
b = Z_K^0 \frac{-g_0}{d_0f_0 - g_0h_0} = Z_K^0 b_0,
\]
\[
c = Z_K^{0\dagger} \frac{-h_0}{d_0f_0 - g_0h_0} = Z_K^{0\dagger} c_0.
\]
(15)

### 2.3 Analyticity and Positivity

#### 2.3.1 Källen-Lehmann representation [43]

The propagator can be demonstrated, with very general hypothesis \(^4\), to satisfy a Källen-Lehmann representation, which writes, in Fourier space
\[
\Delta(z) = \int_0^{\infty} d(k^2) \frac{\rho(k^2)}{k^2 - z},
\]
(16)

where, eventually, \( z \) gets close to the cut on the real axis by staying in the physical upper half-plane \( z \rightarrow (p^2 + i\varepsilon), p^2 \in \mathbb{R} \).

Since the propagator is a matrix, so is the spectral function, the elements of which we shall call respectively \( \rho_d, \rho_f, -\rho_g, -\rho_h \). One has \( \rho_n \) being the momentum of the state \( n \)

\(^4\)Lorentz and translation invariance.
\[
\rho_d(k^2) = \sum_n \langle n | \varphi(\bar{0},0) | n \rangle \langle n | \varphi ^\dagger(0,0) | 0 \rangle >, \\
\rho_f(k^2) = \sum_n \langle n | \varphi^\dagger(\bar{0},0) | n \rangle \langle n | \varphi(\bar{0},0) | 0 \rangle >, \\
-\rho_g(k^2) = e^{i\delta} \sum_n \langle 0 | \varphi(\bar{0},0) | n \rangle \langle n | \varphi(\bar{0},0) | 0 \rangle >, \\
-\rho_h(k^2) = e^{-i\delta} \sum_n \langle 0 | \varphi^\dagger(\bar{0},0) | n \rangle \langle n | \varphi^\dagger(\bar{0},0) | 0 \rangle >, \\
p_n^2 = k^2, p_n^0 > 0. \tag{17}
\]

Since \( < 0 | \varphi(\bar{0},0) | n > = < n | \varphi^\dagger(\bar{0},0) | 0 > \), one gets the constraints
\[
\rho_d(k^2) = \overline{\rho_d(k^2)} = \sum_n \left| < 0 | \varphi(\bar{0},0) | n > \right|^2 \geq 0, \\
\rho_f(k^2) = \overline{\rho_f(k^2)} = \sum_n \left| < n | \varphi^\dagger(\bar{0},0) | 0 > \right|^2 \geq 0, \\
\rho_g(k^2) = \overline{\rho_g(k^2)}. \tag{18}
\]

The spectral function is accordingly a positive hermitian matrix \(^5\).

A consequence is that the propagator \( \Delta(z) \) is an holomorphic function in the complex \( z \) plane outside the cuts \(^6\), which satisfies \(^42\)
\[
\Delta(z) = [\Delta(\bar{z})]^\dagger. \tag{19}
\]

Indeed, \(^{19}\) writes, using the hermiticity of \( \rho \) \(^7\).
\[
\Delta(\bar{z}) = \int_0^\infty d(k^2) \frac{\rho(k^2)}{k^2 - \bar{z}}, \quad [\Delta(\bar{z})]^\dagger = \int_0^\infty d(k^2) \left[ \frac{\rho(k^2)}{k^2 - \bar{z}} \right]^\dagger = \int_0^\infty d(k^2) \frac{\rho^\dagger(k^2)}{k^2 - z}. \tag{20}
\]

This general property should be distinguished from the (Schwarz) reflection principle or its refined version called the “edge of the wedge” theorem \(^{42}\); indeed, as soon as complex coupling constants can enter the game, in particular to account for \( CP \) violation, the discontinuity on the cut is no longer the sole origin for the imaginary part of the propagator; it can be non-vanishing outside the cut (as can be checked in a quark model), which is likely to invalidate the principle of reflection.

2.3.2 The linear approximation: introducing a mass matrix \(^{43}\)

We show here how a mass matrix can be introduced, which can describe unstable particles and at the same time respect the positivity and analyticity properties of the propagator, and how it can only be considered as an approximation \(^8\).

The imaginary part of \( \Delta(z) \)
\[
\Im(\Delta(z)) = \frac{\Delta(z) - (\Delta(z))^\dagger}{2i} = \frac{\Delta(z) - \Delta^\dagger(\bar{z})}{2i} \tag{21}
\]
rewrites, using \(^{16}\),
\[
\Im(\Delta(z)) = \int_0^\infty d(k^2) \left( \frac{\rho(k^2)}{k^2 - z} - \frac{\rho^\dagger(k^2)}{k^2 - \bar{z}} \right); \tag{22}
\]

\(^5\)This cannot be carelessly transposed to the propagator since, in particular, \( z \) in \(^{16}\) spans the complex plane.
\(^6\)In the case under concern, two cuts on the real axis start respectively at \( z = (2m_n)^2 \) and \( z = (3m_n)^2 \).
\(^7\)Since \( k^2 \) spans the real axis, \( \rho(k^2) \) is a complex (matricial) function of a real variable, and the notation \( \rho^\dagger(k^2) \) is unambiguous.
\(^8\)This section has been written thanks to the help of R. Stora.
because of the constraints \( \{18\} \), one has for the four matrix elements of \( \Delta(z) \)

\[
\begin{align*}
\Im(d(z)) &= \int_0^\infty \frac{(z-\bar{z})|\rho_d(k^2)|}{|k^2-z|^2}, \\
\Im(f(z)) &= \int_0^\infty \frac{(z-\bar{z})|\rho_f(k^2)|}{|k^2-z|^2}, \\
\Im(-g(z)) &= \int_0^\infty \frac{(-\rho_g(k^2) + \rho_h(k^2)}{k^2-z} = \int_0^\infty \frac{-\rho_h(k^2)}{k^2-z}, \\
\Im(-h(z)) &= \int_0^\infty \frac{(-\rho_h(k^2) + \rho_g(k^2)}{k^2-z} = \int_0^\infty \frac{-\rho_h(k^2)}{k^2-z},
\end{align*}
\tag{23}
\]

such that, the imaginary part of the (matricial) Feynman propagator is \((z - \bar{z})\) times a positive hermitian matrix, and its sign is thus always the sign of \((z - \bar{z})\).

If this property is true for the propagator \( \Delta(z) \), it is also true for its inverse \( \Delta^{-1}(z) \). This is the property that we want to preserve when approximating the inverse propagator.

Close to the poles, a linear approximation of \( \Delta^{-1} \) should be suitable,

\[
\Delta^{-1}(z) \approx Az + B,
\tag{24}
\]

such that

\[
\Im(Az + B) = \frac{A - A^\dagger}{2i} z + \bar{z} + \frac{A + A^\dagger}{2i} z - \bar{z} + \frac{B - B^\dagger}{2i}.
\tag{25}
\]

When \( A = A^\dagger \) is a positive hermitian matrix, the sign of the first two terms is indeed the same as the sign of \( \Im(z) \). The property of positivity is true everywhere only if \( B = B^\dagger \); in this case, the mass matrix is hermitian, its eigenvalues are real and cannot describe unstable particles. However, if one only wants to preserve this property in the upper (physical) half plane \( \Im(z) \geq 0 \), it is enough to have \( \Im(B) \geq 0 \). If this is so, then, writing \( B = B_1 + iB_2, B_2 \geq 0 \), one has

\[
\Delta^{-1}(z) \approx \sqrt{A} \left( z + \frac{1}{\sqrt{A}}(B_1 + iB_2) \frac{1}{\sqrt{A}} \right) \sqrt{A} = \sqrt{A} \left( z - \left( m^{(2)} - \frac{i\Gamma^{(2)}}{2} \right) \right) \sqrt{A};
\tag{26}
\]

the mass matrix \( ^9 \) is accordingly

\[
M^{(2)} \equiv m^{(2)} - \frac{i\Gamma^{(2)}}{2} = -\frac{1}{\sqrt{A}}(B_1 + iB_2) \frac{1}{\sqrt{A}}, \text{ with } \Gamma^{(2)} \geq 0, A = A^\dagger.
\tag{27}
\]

It is no longer hermitian and can accordingly describe unstable kaons.

Since \( \Gamma^{(2)} \geq 0 \), the zeroes of the approximate inverse propagator (poles of the approximate propagator) are located in the lower (unphysical) half plane.

The hermitian matrix \( A \) normalizes the states.

\section*{2.4 DISCRETE SYMMETRIES AND LORENTZ INVARIANCE}

The first two paragraphs of this section summarizes the results obtained and demonstrated at length in Appendix \( \text{A} \) for the propagator.

The next paragraphs demonstrate which constraints can be obtained on the spectral function, using the two possible conventions for \( TCP \) transformations, the one of Wightman and the one of Schwinger-Pauli.

\footnote{The superscript \( \{2\} \) which appear in \( M\{2\}, m\{2\} \) and \( \Gamma\{2\} \) are to remind that these quantities have dimension \([\text{mass}]^2\).}
2.4.1 CP symmetry

CP symmetry constrains the two diagonal elements of the propagator to be identical, and the two antidiagonal elements to be related by (133). So, a CP invariant kaon propagator is in particular a (special type of) normal matrix; this leaves a priori for a general normal propagator the possibility to describe CP violating theories. We indeed investigate in subsection 3.1.2 the case of a general normal propagator, and show that, then, the CP violating parameter is non-vanishing but always lies on the imaginary axis.  

2.4.2 TCP symmetry

TCP symmetry constrains the two diagonal elements of the kaon propagator to be identical, and yields no constraint on the antidiagonal elements. Accordingly, a TCP invariant propagator can be normal or not.

2.4.3 Constraints on the spectral function [43]

One makes use of the notations and conventions explained in Appendix A.

- Constraints from TCP symmetry, using the convention of Wightman (see subsection A.4).

One uses (140) to express, in (17), \( \varphi^\dagger(0,0) = \Theta \varphi(0,0) \Theta^{-1} \) and, reciprocally (using \( \Theta = \Theta^{-1} \)) \( \varphi(0,0) = \Theta^{-1} \varphi^\dagger(0,0) \Theta = \Theta \varphi^\dagger(0,0) \Theta^{-1} \).

This yields

\[
\rho_d(k^2) = \sum_n < n | \Theta \varphi^\dagger(0,0) \Theta^{-1}| n > < n | \Theta \varphi(0,0) \Theta^{-1}| 0 >. 
\]  

(28)

The vacuum is invariant by TCP, \( | 0 >= \Theta| 0 > \), and one supposes furthermore that the spectrum is also TCP invariant \( \sum_n | n > < n | = \sum_n | \Theta n > < \Theta n | \), which yields

\[
\rho_d(k^2) = \sum_n < \Theta 0 | \Theta \varphi^\dagger(0,0) \Theta^{-1}| \Theta n > < \Theta n | \Theta \varphi(0,0) \Theta^{-1}| \Theta 0 >. 
\]  

(29)

One uses next the antiunitarity (138) of the \( \Theta \) operator to get

\[
\rho_d(k^2) = \sum_n < n | \varphi(0,0)| 0 > < 0 | \varphi^\dagger(0,0)| n > = \rho_f(k^2). 
\]  

(30)

The same procedure applied to the antidiagonal elements of \( \rho(k^2) \) only yields tautologies (like for the propagator) and thus no constraints.

- Constraints from TCP symmetry, using the Schwinger-Pauli convention (see subsection A.5).

Reading from right to left instead of from left to right, one gets

\[
\Theta \rho_d(k^2) = \rho_f(k^2), \quad \Theta(-\rho_g(k^2)) = -\overline{\rho_h}(k^2). 
\]  

(31)

\[\text{footnote}{\text{The phase of the CP violating parameter is not an observable \[57\}; in particular, asymmetries are proportional to the real part of the CP violating parameter (see subsection 5.3 for QFT). A purely imaginary \( \epsilon_L \) can nevertheless be considered to violate CP when it cannot be brought back to 0 by a (constant) rephasing of the neutral kaons, as shown in subsection 3.1.2. This is however incompatible with experiment. When direct CP violation is allowed, one gets, by quantum mechanical arguments \[57\] and considering that \( \epsilon_L = \epsilon_S, \eta_{+-} = \epsilon_L + \epsilon' + i \Im(A_0)/\Re(A_0) \) where \( A_0 \) is the amplitude for the decay of \( K^0 \) into two pions in the isospin 0 channel; \( \Im(A_0) \) and \( \Re(A_0) \) depend on the choice of phase for the neutral kaons; only the phase of the direct CP violating parameter \( \epsilon' \) is physically relevant. The phase of \( \epsilon = \epsilon_L + i \Im(A_0)/\Re(A_0) \) is measured to be close to 43.4 degrees while the phase \( \phi_{+-} \) is measured to be close to 43.5 degrees \[57\] and these two phases theoretically coincide in superweak models which do not allow for direct CP violation \( \epsilon' = 0 \) \[31\]. Suppose now that, as predicted from a normal TCP invariant propagator in our model, \( \epsilon_L \) is purely imaginary. Since we do not allow for direct CP violation, one expects, supposing that the relations obtained by QM arguments give results close to the one of QFT, \( \eta_{+-} = \epsilon_L + i \Im(A_0)/\Re(A_0) \); then \( \eta_{+-} \) should also be purely imaginary, which is in conflict with experiment.}
• Constraints arising from $CP$ symmetry.

Following the same lines as in subsections A.3.2 and A.3.1 if $CP$ invariance holds and if one supposes that the spectrum is $CP$ invariant ($\sum_n |n><n| = \sum_n |CP n><CP n|$) one gets

$$CP \rho_d(k^2) = \rho_f(k^2), \quad CP(-\rho_g(k^2)) = e^{-2i\alpha}(-\rho_h(k^2)).$$

(32)

The $CP$ constraints on the spectral function are the same as the ones for the propagator.

3 NORMAL VERSUS NON-NORMAL PROPAGATOR; DIAGONALIZATION

We recall the definition of a normal matrix:

$$M \text{ normal} \iff [M, M^\dagger] = 0. \quad (33)$$

Normality is a remarkable property of matrices: any matrix that commutes with its hermitian conjugate can be diagonalized by a single unitary transformation; its right and left eigenstates accordingly coincide; furthermore, unlike hermitian matrices, it admits complex eigenvalues [27], which makes it specially suited to describe unstable particles [22].

When $CP$ is conserved, we have shown that the propagator of neutral kaons must be normal. This will provides us with the most general $CP$ eigenstates in the $(K^0, \bar{K}^0)$ basis.

It is very tempting to have a normal propagator in any circumstance, since right eigenstates and left eigenstates coincide; we will show that this is impossible, since any normal matrix with equal diagonal elements (a $TCP$ invariant propagator must have equal diagonal elements) yields eigenstates with purely imaginary indirect $CP$ violating parameters $\epsilon_L$ and $\epsilon_S$. So, in particular on the cut(s), the propagator is non-normal, and there exist different right and left eigenstates. We will demonstrate that the appropriate way of diagonalizing the propagator is by using a bi-orthogonal basis, and that is is not fully equivalent to a bi-unitary transformation, like the one currently used for fermions. The “propagating states” are unambiguously determined to be the states which correspond to the poles of the full renormalized propagator. The $CP$ violating parameters of any pair of left and right propagating states can now be anywhere in the complex plane but have equal modulus, which is the physically relevant property. Conceptual problems linked with the non normality of the propagator on the cut and the subsequent existence of right and left eigenstates are thus wiped out. We will give the explicit form of all eigenstates and indirect $CP$ violating parameters.

The structure of the eigenstates of the full propagator will be investigated in details and will reveal in particular the subtle way $TCP$ symmetry is realized. We will exhibit the important role of states which correspond to a vanishing residue of the propagator (zero norm states), that we call spurious.

3.1 NORMAL MATRICES AND $CP$ EIGENSTATES

3.1.1 $CP$ eigenstates

$CP$ conserving propagators are special types of normal matrices with their two diagonal elements identical and their two antidiagonal elements related by [133]. Accordingly, we consider

$$\Delta_{CP}(z) = \begin{pmatrix} d(z) & e^{-i\alpha}l(z) \\ e^{i\alpha}l(z) & d(z) \end{pmatrix}. \quad (34)$$

The eigenvalues $\lambda_{CP}^\pm(z)$ are the two solutions of the characteristic equation of $\Delta_{CP}(z)$

$$\lambda_{CP}^\pm(z) = d(z) \pm l(z), \quad (35)$$
and the corresponding eigenvectors that we note \( \begin{pmatrix} u^C_P(z) \\ v^C_P(z) \end{pmatrix} \) satisfy

\[
    r^C_+(z) \equiv \frac{u^C_P(z)}{v^C_P(z)} = e^{i\alpha}, \quad r^C_-(z) \equiv \frac{v^C_P(z)}{u^C_P(z)} = -e^{i\alpha},
\]

which are quantities independent of \( z \): the \( CP \) eigenstates, which are, of course, function of the arbitrary phase \( \alpha \) introduced in (1), do not change with \( p^2 \); this is why we call them directly \( K^0_1 \) and \( K^0_2 \), explicitly:

\[
\begin{align*}
    | K^0_1 > &= \frac{1}{\sqrt{2}} (| K^0 > + e^{i\alpha} | \bar{K}^0 >), \quad | K^0_2 > &= \frac{1}{\sqrt{2}} (| K^0 > - e^{i\alpha} | \bar{K}^0 >).
\end{align*}
\]

3.1.2 Normal propagators

Let us now consider a general normal propagator

\[
    \Delta_N(z) = \begin{pmatrix} d(z) & -g(z) \\ -h(z) & f(z) \end{pmatrix}, \text{ with } |g| = |h| \text{ and } \bar{h}(d-f) = g(\bar{d}-\bar{f}).
\]

The condition at the right of (38) are the condition for the normality \([\Delta_N, \Delta_N^+] = 0\) of \( \Delta_N \).

We introduce the phases \( \alpha_g \) and \( \alpha_h \) of \( -g \) and \( -h \) and the conditions of normality become

\[
    -g(z) = \rho(z)e^{i\alpha_g(z)}, \quad -h(z) = \rho(z)e^{i\alpha_h(z)}, \quad \rho(z) \in \mathbb{R},
\]

\[
    (d(z) - f(z)) - e^{i(\alpha_g(z) + \alpha_h(z))}(\bar{d}(z) - \bar{f}(z)) = 0 \iff d(z) - f(z) = |d(z) - f(z)|e^{i\alpha_g(z) + \alpha_h(z)}. \tag{39}
\]

It is convenient to introduce the following notations

\[
\begin{align*}
    \sigma(z) &= \frac{|d(z) - f(z)|}{2\rho(z)} = \frac{d(z) - f(z)}{2\rho(z)e^{i\Sigma\alpha(z)}}, \\
    \Sigma\alpha(z) &= \frac{1}{2}(\alpha_g(z) + \alpha_h(z)), \\
    \Delta\alpha(z) &= \frac{1}{2}(\alpha_g(z) - \alpha_h(z)).
\end{align*}
\]

The eigenvalues \( \lambda^N_\pm(z) \) are given by

\[
    \lambda^N_\pm = \frac{d(z) + f(z)}{2} \pm \rho(z)e^{i\Sigma\alpha(z)} \sqrt{1 + \sigma^2(z)}. \tag{41}
\]

Writing the eigenvectors \( \begin{pmatrix} u^N_\pm(z) \\ v^N_\pm(z) \end{pmatrix} \), and defining \( r^N_\pm(z) = \frac{u^N_\pm(z)}{v^N_\pm(z)} \), one gets

\[
    r^N_\pm(z) = e^{i\Delta\alpha(z)} \left( \sigma(z) \pm \sqrt{1 + \sigma^2(z)} \right). \tag{42}
\]

To determine the values of the indirect \( CP \) violating parameter \( \epsilon^N(z) \), one goes to the basis of \( CP \) eigenstates defined in subsection 3.1.1 above. This gives \(^{11}\)

\[
\begin{align*}
    \epsilon^N_\pm(z) &= 1 - e^{-i\alpha} r^N_\pm(z) \quad \text{with} \quad r^N_\pm(z) = \frac{1 - e^{i(-\alpha + \Delta\alpha(z))}}{1 + e^{i(-\alpha + \Delta\alpha(z))}} \left( \sigma(z) + \sqrt{1 + \sigma^2(z)} \right) \\
    \epsilon^N_+(z) &= 1 + e^{-i\alpha} r^N_+(z) \quad \text{with} \quad r^N_+(z) = \frac{1 - e^{i(-\alpha + \Delta\alpha(z))}}{1 + e^{i(-\alpha + \Delta\alpha(z))}} \left( \sigma(z) + \sqrt{1 + \sigma^2(z)} \right),
\end{align*}
\]

\(^{11}\)For the eigenstates will subscript “*+*”, \( \epsilon_+ \) is defined as the ratio of the \( K^0_1 \) component over the \( K^0_0 \) component, and for the eigenstate with subscript “*−*”, \( \epsilon_- \) is defined as the ration of the \( K^0_1 \) component over the \( K^0_2 \) component. So doing, we will match in the following the usual definitions of \( \epsilon_L \) (for “*+*” states) and \( \epsilon_S \) (for “*−*” states) for \( K_L \) and \( K_S \) mesons.
\[
\epsilon_N^-(z) = \frac{1 + e^{-i\alpha} r_N^-(z)}{1 - e^{-i\alpha} r_N^+(z)} = \frac{1 + e^{i(-\alpha + \Delta \alpha(z))}}{1 - e^{i(-\alpha + \Delta \alpha(z))}} \frac{\sigma(z) + \sqrt{1 + \sigma^2(z)}}{\sigma(z) + \sqrt{1 + \sigma^2(z)}},
\]

which is always purely imaginary when \(d(z) = f(z)\), i.e. when TCP is satisfied, since this entails \(\sigma(z) = 0\) and
\[
\epsilon_N^+(z) = \frac{1 - e^{i(-\alpha + \Delta \alpha(z))}}{1 + e^{i(-\alpha + \Delta \alpha(z))}} \frac{\sigma(z) - \sqrt{1 + \sigma^2(z)}}{\sigma(z) - \sqrt{1 + \sigma^2(z)}},
\]

so, if TCP is satisfied, for any value of \(z = p^2\) where the propagator is normal, its eigenstates have an indirect \(CP\) violating parameter which is purely imaginary; it cannot be brought back to 0 by a constant rephasing of the neutral kaons equivalent to choosing \(\alpha = 0\) since the difference of phases \(\Delta \alpha(z)\) between the antidiagonal elements of the propagator, which depend on \(z = q^2\), also enters \((44)\).

Normality of the propagator is consequently excluded; indeed, as will be emphasized in section 5 (see footnote 10), a purely imaginary \(CP\) violating parameter \(\epsilon_L\) is incompatible with experiment.

The other way to get \(\epsilon\) non purely imaginary for a normal propagator would be to keep \(\sigma(z) \neq 0\), that is to abandon the criteria of TCP invariance; this is certainly not desired.

### 3.2 NON-NORMAL MATRICES AND PROPAGATORS

When studying kaon decays, one has to deal with a non-normal propagator.

#### 3.2.1 Diagonalization

The diagonalization of a non-normal complex matrix is not unique, and this is why time has to be spent on this question \(^{12}\).

- The first way to diagonalize a complex matrix is via a bi-unitary transformation, that is two different unitary transformations, respectively acting on the left and on the right; this is always how the quark mass matrices are diagonalized \(^{13}\). Any given complex matrix \(C(z)\) can always be diagonalized by two unitary matrices \(U(z)\) and \(V(z)\) such that \((U(z))^\dagger C(z) V(z) = \text{diag}(\mu_1(z), \mu_2(z))\); \(U\) and \(V\) respectively diagonalize \(C(z)(C(z))^\dagger\) and \((C(z))^\dagger C(z)\) (now \(C\) and \(C^\dagger\) are supposed not to commute), and each of these two products are hermitian and have real positive eigenvalues; \(\mu_1\) and \(\mu_2\) can always be also chosen real and positive. The eigenvalues of \(C(z)\) determined in this way are not the roots of its characteristic equation; instead, the square of these eigenvalues are the roots of the characteristic equation of \(CC^\dagger\) or \(C^\dagger C\); this leads to a different result, though, as we shall see, the poles coincide.

- The second way to diagonalize a general complex mass matrix is by the standard procedure of determining its eigenvalues as the roots \(\lambda_{\pm}(z)\) of its characteristic equation \(^{14}\), and then determining the right and left eigenstates, respectively \(|R_{\pm}(z)\rangle = \frac{1}{n_{\pm}} \begin{pmatrix} u_{\pm}(z) \\ v_{\pm}(z) \end{pmatrix} \equiv \frac{1}{n_{\pm}} \begin{pmatrix} u_{\pm}(z) \mid K^0 > + v_{\pm}(z) \mid \bar{K}^0 > \end{pmatrix}\) and \(< L_{\pm}(z) |\n
\(^{12}\)The work \(^{31}\) is instructive, which emphasizes, in the framework of QM, the importance of using a “reciprocal” basis for the diagonalization of a non-normal effective mass matrix.

\(^{13}\)When one chirality of fermions does not participate in non-abelian weak interactions, like right-handed fermions in the Glashow-Salam-Weinberg model, one can use a single unitary transformation \(^{10}\).

\(^{14}\)The right and left eigenvalues always coincide.

\(^{15}\)There is no distinction between “in” and “out” states for the flavor eigenstates \(K^0\) and \(\bar{K}^0\) (see for example the discussion in the third section of \(^{33}\)). One has \(< K^0 \mid | = (| K^0 >)\), \(< \bar{K}^0 \mid | = (| \bar{K}^0 >)\), \(< K^0 \mid K^0 > = 1 = < \bar{K}^0 \mid \bar{K}^0 >\), \(< K^0 \mid \bar{K}^0 > = 0 = < \bar{K}^0 \mid K^0 >\).
and \( \langle L_\pm(z) | C(z) = \langle L_\pm(z) | \lambda_\pm(z) \rangle \). The normalization conditions are then written between the two spaces of “in” (left) and “out” (right) states

\[
\begin{align*}
\langle L_+(z) | R_+(z) \rangle & = 1 = \langle L_-(z) | R_-(z) \rangle, \\
\langle L_-(z) | R_+(z) \rangle & = 0 = \langle L_+(z) | R_-(z) \rangle,
\end{align*}
\]

which determines the normalization coefficients \( n_R^\pm(z) \) and \( n_L^\pm(z) \). In general

\[
\begin{align*}
\langle R_\pm(z) | R_\pm(z) \rangle & \neq 1, \\
\langle L_\pm(z) | L_\pm(z) \rangle & \neq 1, \\
\langle R_\pm(z) | R_\mp(z) \rangle & \neq 0, \\
\langle L_\pm(z) | L_\mp(z) \rangle & \neq 0,
\end{align*}
\]

where, for any vector, \( \langle | = | \rangle \)†: the “in” eigenvectors do not form a basis, nor the “out” eigenstates.

When dealing with constant matrices, the two procedures are different and non equivalent. The second procedure allows in particular complex eigenvalues, which is necessary for a mass matrix of unstable states. However, we will see in subsection 3.2.7 that, when dealing with the full renormalized propagator (with depend on \( p^2 \)), while the two procedures select the same physical masses and propagating eigenstates, they however differ as far as spurious states are concerned; the latter play an essential role in the realization of discrete symmetries, in particular \( TCP \).

Discriminating the two procedures by the reality or not of their eigenvalues is only valid for constant mass matrices; for \( p^2 \) dependent propagators, this does not provide a criterion for rejecting biunitary transformations.

### 3.2.2 Physical masses

In QFT, the physical masses are the values of \( z = p^2 \) which are poles of the the full renormalized propagator; accordingly, they are determined by the equation

\[
At \ z = \text{physical mass}, \det(1/\Delta(z)) = 0.
\]

We shall assume hereafter that there exist only two solutions to this equation, \( z_1 = M_L^2 \) and \( z_2 = M_S^2 \); they are both complex numbers.

For the sake of convenience, we shall work in the following with the \( TCP \) invariant inverse propagator (see (11))

\[
\Delta^{-1}(z) = \begin{pmatrix} a(z) & -b(z) \\ -c(z) & a(z) \end{pmatrix}
\]

where

\[
a(z) = \frac{f(z)}{f^2(z) - g(z)h(z)}, \quad b(z) = \frac{g(z)}{f^2(z) - g(z)h(z)}, \quad c(z) = \frac{h(z)}{f^2(z) - g(z)h(z)}.
\]

Eigenstates of \( \Delta^{-1} \) are of course the same of the ones of \( \Delta \), and the eigenvalues of the former are the inverse of the ones of the latter.

The physical masses are accordingly defined by

\[
a^2(z) = b(z)c(z),
\]

and we will choose, by convention

\[
a(z_1) = -\sqrt{b(z_1)c(z_1)}, \quad a(z_2) = +\sqrt{b(z_2)c(z_2)}.
\]
3.2.3 \textit{TCP} eigenstates

At any given \( z = p^2 \), \( \Delta^{-1}(z) \) has two eigenvalues \( \lambda_+(z) \) and \( \lambda_-(z) \)
\begin{equation}
\lambda_+(z) = a(z) + \sqrt{b(z)c(z)}, \quad \lambda_-(z) = a(z) - \sqrt{b(z)c(z)}.
\end{equation}

To each of them corresponds one right eigenstate \( | R_\pm(z) >_{in} \) and one left eigenstate \( out < L_\pm(z) | \)\(^{16} \); this occurs in particular at the two physical masses \( z = z_1 \) and \( z = z_2 \), such that we have to deal with a total of eight eigenvectors of \( \Delta^{-1}(z) \), which will all be important, for various reasons. They will be called
\begin{align}
| R_+(z_1) >_{in} &= | K_L >_{in}, \quad | R_-(z_2) >_{in} = | K_S >_{in}, \\
out < L_+(z_1) | &= out < K_L |, \quad out < L_-(z_2) | = out < K_S |,
| R_+(z_2) >_{in} &= | \tilde{K}_L >_{in}, \quad | R_-(z_1) >_{in} = | \tilde{K}_S >_{in}, \\
out < L_+(z_2) | &= out < \tilde{K}_L |, \quad out < L_-(z_1) | = out < \tilde{K}_S |.
\end{align}

As emphasized before
\begin{align}
out < K_L | \neq ( | K_L >_{in})^\dagger, \quad out < K_S | \neq ( | K_S >_{in})^\dagger, \\
out < \tilde{K}_L | \neq ( | \tilde{K}_L >_{in})^\dagger, \quad out < \tilde{K}_S | \neq ( | \tilde{K}_S >_{in})^\dagger.
\end{align}

The first four eigenstates of \( \ref{54} \) all have in common to correspond to a vanishing eigenvalue of \( \Delta^{-1}(z_1) \) or \( \Delta^{-1}(z_2) \), and the last four to a non-vanishing eigenvalue; indeed, one has, in virtue of \( \ref{52} \) and \( \ref{51} \)
\begin{equation}
\lambda_+(z_1) = 0 = \lambda_-(z_2), \quad \lambda_+(z_2) = 2a(z_2) = 2\sqrt{b(z_2)c(z_2)}, \quad \lambda_-(z_1) = 2a(z_1) = -2\sqrt{b(z_1)c(z_1)}.
\end{equation}

Only the first four eigenstates of \( \ref{56} \) are propagating eigenstates, and they correspond to the physical \( K_L \) and \( K_S \) mesons; we shall study below the difference between their “\( \text{in} \)” and “\( \text{out} \)” states. The four other eigenstates are non propagating in the sense that the corresponding residues of the propagators at respectively \( \lambda_+(z_2) \) and \( \lambda_-(z_1) \) are vanishing as can be easily checked by making an expansion of the propagator for \( z_2 \approx z \approx z_1 \); these states are zero norm states that we call “spurious”.

They are however important and should not be neglected; we shall come back at length on this point in subsection \( \ref{3.2.6} \) dealing with kaon decays, but the theoretical argument is the following: at any \( z \), the completeness relation writes\(^{17} \)
\begin{equation}
1 = | R_+(z) >_{in} out < L_+(z) | + | R_-(z) >_{in} out < L_-(z) |,
\end{equation}
and this should stay in particular true at the physical poles \( z = z_1 \) and \( z = z_2 \), in which case one of the two states appearing in the completeness relation becomes a spurious state: the space of eigenvectors shrinks to a one-dimensional space at the pole, and the propagator becomes a matrix of rank 1.

\(^{16}\)Since it will be used in subsection \( \ref{3.1.1} \) we give here the explicit form of the eigenstates at any \( z = q^2 \).
\begin{align}
| R_+(z) >_{in} &= \frac{1}{n(z)} \left( \sqrt{b(z)} | K^0 > - \sqrt{c(z)} | \bar{K}^0 > \right), \\
| R_-(z) >_{in} &= \frac{1}{n(z)} \left( \sqrt{b(z)} | K^0 > + \sqrt{c(z)} | \bar{K}^0 > \right), \\
out < L_+(z) | &= \frac{1}{n(z)} \left( \sqrt{c(z)} | K^0 > - \sqrt{b(z)} | \bar{K}^0 > \right), \\
out < L_-(z) | &= \frac{1}{n(z)} \left( \sqrt{c(z)} | K^0 > + \sqrt{b(z)} | \bar{K}^0 > \right), \\
n^2(z) &= 2\sqrt{b(z)c(z)}.
\end{align}

\(^{17}\)It is important to stress that the completeness relation cannot involve both propagating states and that, in particular \( | K_L >_{in} out < K_L | + | K_S >_{in} out < K_S | \neq 1 \).
The orthogonality relations that the eigenstates satisfy, which enable to fix their normalization, are the following:

\[
\Delta(z) = \left| R_+(z) > \right|_{\text{in}} \frac{1}{\lambda_+(z)} |_{\text{out}} < L_+(z) \right| + \left| R_-(z) > \right|_{\text{in}} \frac{1}{\lambda_-(z)} |_{\text{out}} < L_-(z) \right|, \tag{58}
\]

which selects only the propagating state at each pole, but that the inverse propagator (that is the quadratic renormalized Lagrangian) writes

\[
\Delta^{-1}(z) = \left| R_+(z) > \right|_{\text{in}} \lambda_+(z) |_{\text{out}} < L_+(z) \right| + \left| R_-(z) > \right|_{\text{in}} \lambda_-(z) |_{\text{out}} < L_-(z) \right|, \tag{59}
\]

which instead selects at each physical mass the spurious state.

The orthogonality relations that the eigenstates satisfy, which enable to fix their normalization, are the following:

\[
\begin{align*}
\text{out} < K_L | K_L > \text{in} &= 1, \\
\text{out} < K_S | K_S > \text{in} &= 1, \\
\text{out} < \tilde{K}_L | \tilde{K}_L > \text{in} &= 1, \\
\text{out} < \tilde{K}_S | \tilde{K}_S > \text{in} &= 1, \\
\text{out} < \tilde{K}_L | K_L > \text{in} &= 0 = \text{out} < K_L | \tilde{K}_L > \text{in}, \\
\text{out} < \tilde{K}_S | K_S > \text{in} &= 0 = \text{out} < K_S | \tilde{K}_S > \text{in}. \tag{60}
\end{align*}
\]

We now explicitly list all eigenstates of a TCP invariant propagator \(^{18}\):

\[^{18}\text{A remark is due here concerning the normalization of states in (54), (60) allows the multiplication of a given } | > \text{in} \text{ state by a constant } 1/N \text{ while the corresponding } \text{out} < \text{ state is multiplied by } N. \text{ Let us show that } N \text{ can only be a phase. Using the time evolution induced by the Schrödinger equation for unstable particles (see subsection 5.1.4), one gets then, for example for the } K_L \text{ meson:}
\]

\[
\begin{align*}
| K_L(t) > &_{\text{in}} = \frac{1}{N_L} e^{i m_L t - \frac{\Gamma_L t}{2}} \frac{i}{\sqrt{2 \alpha(z)}} \left( \sqrt{b(z)} | K^0 > - \sqrt{c(z)} | K^0 > \right), \\
| \text{out} < K_L(t) | & = N_L e^{i m_L t - \frac{\Gamma_L t}{2}} \frac{i}{\sqrt{2 \alpha(z)}} \left( \sqrt{c(z)} < K^0 | - \sqrt{b(z)} < K^0 | \right), \tag{61}
\end{align*}
\]

where the mass of \( K_L \) has been written \( M_L = m_L - i \frac{\Gamma_L}{2} \).

\[^{61}\text{yields in particular}
\]

\[
\begin{align*}
\text{in} < K_L(t) | K_L(t) > &_{\text{in}} = \frac{1}{|N_L|^2} e^{\Gamma_L t} \frac{1}{2 i \alpha(z)} (|b(z)| + |c(z)|), \\
\text{out} < K_L(t) | K_L(t) > &_{\text{out}} = \frac{1}{|N_L|^2} e^{-\Gamma_L t} \frac{1}{2 i \alpha(z)} (|b(z)| + |c(z)|). \tag{62}
\end{align*}
\]

However (see for example \(^{4}\)), \( \text{out} < K_L(t) | K_L(t) >_{\text{out}} \) is the time-reversed of \( \text{in} < K_L(t) | K_L(t) >_{\text{in}} \), such that \( N_L \) must satisfy

\[
\frac{1}{|N_L|^2} = T |N_L|^2 \Rightarrow |N_L|^2 = 1. \tag{63}
\]
\[ | K_L >_{in} = \frac{i}{\sqrt{2}a(z_1)} \left( \sqrt{b(z_1)} | K^0 > - \sqrt{c(z_1)} | \tilde{K}^0 > \right) = \frac{1}{n^{in}_L} (| K^0 > + \epsilon_L^{in} | K^0 >), \]
\[ | K_S >_{in} = \frac{1}{\sqrt{2}a(z_2)} \left( \sqrt{b(z_2)} | K^0 > + \sqrt{c(z_2)} | \tilde{K}^0 > \right) = \frac{1}{n^{in}_S} (| K^0 > + \epsilon_L^{in} | K^0 >), \]
\[ | \tilde{K}_L >_{in} = \frac{i}{\sqrt{2}a(z_1)} \left( \sqrt{b(z_1)} | K^0 > + \sqrt{c(z_1)} | \tilde{K}^0 > \right) = \frac{1}{n^{in}_S} (| K^0 > + \epsilon_L^{in} | K^0 >), \]
\[ | \tilde{K}_S >_{in} = \frac{1}{\sqrt{2}a(z_2)} \left( \sqrt{b(z_2)} | K^0 > - \sqrt{c(z_2)} | \tilde{K}^0 > \right) = \frac{1}{n^{in}_S} (| K^0 > + \epsilon_L^{in} | K^0 >), \]

where \( K^0_1 \) and \( K^0_2 \) can be found in (37) and where one can always take

\[ n^{in}_L = n^{out}_L = \sqrt{1 + \epsilon_L^{in} \epsilon_L^{out}}, \quad n^{in}_S = n^{out}_S = \sqrt{1 + \epsilon_S^{in} \epsilon_S^{out}}. \]  

(65)

Note that we have no a priori relations between states corresponding to different values of \( z \). One can check easily that “in” and “out” eigenstates match when the propagator is also normal, that is when \( |b_1 = |c_1 at z = z_1 \) and \( z = z_2 \); one writes the kets for the “out” eigenstates, for example \( | K_L >_{out} = - \frac{i}{\sqrt{2a(z_1)}} \left( \sqrt{c(z_1)} | K^0 > - \sqrt{b(z_1)} | \tilde{K}^0 > \right), \) from which the results immediately follows.

### 3.2.4 \( CP \) violating parameters

To get the indirect \( CP \) violating parameters of all eigenstates in (64), it is enough to go to the basis of \( CP \) eigenstates (37). One defines the \( \epsilon^{out} \) parameters for the kets and not for the bras, which introduces complex conjugation of the coefficients \( b \) and \( c \) (see subsection 3.2.3 above). One gets

\[
\begin{align*}
\epsilon_L^{in} &= \frac{\sqrt{b(z_1)} - e^{-i\alpha} \sqrt{c(z_1)}}{\sqrt{b(z_1) + e^{-i\alpha} \sqrt{c(z_1)}},} & \epsilon_L^{in} &= \frac{\sqrt{b(z_2)} - e^{-i\alpha} \sqrt{c(z_2)}}{\sqrt{b(z_2) + e^{-i\alpha} \sqrt{c(z_2)}},} \\
\epsilon_S^{in} &= \frac{\sqrt{b(z_1)} + e^{-i\alpha} \sqrt{c(z_1)}}{\sqrt{b(z_1) - e^{-i\alpha} \sqrt{c(z_1)}},} & \epsilon_S^{in} &= \frac{\sqrt{b(z_2)} + e^{-i\alpha} \sqrt{c(z_2)}}{\sqrt{b(z_2) - e^{-i\alpha} \sqrt{c(z_2)}},} \\
\epsilon_L^{out} &= \frac{\sqrt{c(z_1)} - e^{-i\alpha} \sqrt{b(z_1)}}{\sqrt{c(z_1) + e^{-i\alpha} \sqrt{b(z_1)}},} & \epsilon_L^{out} &= \frac{\sqrt{c(z_2)} - e^{-i\alpha} \sqrt{b(z_2)}}{\sqrt{c(z_2) + e^{-i\alpha} \sqrt{b(z_2)}},} \\
\epsilon_S^{out} &= \frac{\sqrt{c(z_1)} + e^{-i\alpha} \sqrt{b(z_1)}}{\sqrt{c(z_1) - e^{-i\alpha} \sqrt{b(z_1)}},} & \epsilon_S^{out} &= \frac{\sqrt{c(z_2)} + e^{-i\alpha} \sqrt{b(z_2)}}{\sqrt{c(z_2) - e^{-i\alpha} \sqrt{b(z_2)}},}
\end{align*}
\]  

(66)

\(^{19}\)In particular, \( \text{out} < K_S > | K_L >_{in} \neq 0 \) and \( \text{out} < K_L > | K_S >_{in} \neq 0 \), unless \(^{81}\) is satisfied; this differs from \(^{4}\) (see in particular (13) and the end of section 4. We come back to CPLEAR in subsection 5.1.
and one has the relations

\[ \begin{align*}
\epsilon_{in}^L &= -\epsilon_{out}^L, & \epsilon_{in}^S &= -\epsilon_{out}^S, \\
\epsilon_{in}^S &= \frac{1}{\epsilon_{in}^L}, & \epsilon_{out}^S &= \frac{1}{\epsilon_{out}^L}; \\
\epsilon_{out}^L &= \frac{1}{\epsilon_{out}^S}, & \epsilon_{out}^L &= \frac{1}{\epsilon_{out}^S}.
\end{align*} \tag{67} \]

from the first line of which one gets in particular

\[ |\epsilon_{in}^L|^2 = |\epsilon_{out}^L|^2, |\epsilon_{in}^S|^2 = |\epsilon_{out}^S|^2, |\epsilon_{in}^S|^2 = |\epsilon_{out}^S|^2, |\epsilon_{in}^S|^2 = |\epsilon_{out}^S|^2. \tag{68} \]

It is important to determine explicitly the real and imaginary parts of the \( \epsilon \)'s, and to investigate how they change by a rephasing of the \( K^0 \) and \( \bar{K}^0 \) fields

\[ \varphi_{K^0} \to e^{i\omega} \varphi_{K^0}, \quad \varphi_{\bar{K}^0} \to e^{-i\omega} \varphi_{\bar{K}^0}. \tag{69} \]

We do this explicitly for \( \epsilon_{in}^L \) and \( \epsilon_{out}^L \). Since the operator \( \varphi_{K^0} \) annihilates the state \( | K^0 \rangle \) to give the vacuum, \( \epsilon \) entails that the states \( | K^0 \rangle \) and \( | \bar{K}^0 \rangle \) are re-phased by

\[ | K^0 \rangle \to e^{-i\omega} | K^0 \rangle, \quad | \bar{K}^0 \rangle \to e^{i\omega} | \bar{K}^0 \rangle. \tag{70} \]

The way \( \epsilon_{in}^L \) in \( \tag{66} \) is modified by \( \omega \) is obtained by considering the first line of \( \tag{66} \); it is equivalent to changing in the expression \( \tag{66} \) for \( \epsilon_{in}^L \sqrt{b(z)} \) into \( e^{-i\omega} \sqrt{b(z)} \) and \( \sqrt{c(z)} \) into \( \sqrt{c(z)} \); for \( \epsilon_{out}^L \), one finds that the same transformations are needed. \( \tag{66} \) for \( \epsilon_{in}^L \) and \( \epsilon_{out}^L \) are accordingly replaced by

\[ \begin{align*}
\epsilon_{in}^L &= \frac{\sqrt{b(z)} - e^{-i(\alpha-2\omega)} \sqrt{c(z)}}{\sqrt{b(z)} + e^{-i(\alpha-2\omega)} \sqrt{c(z)}}, & \epsilon_{out}^L &= \frac{\sqrt{c(z)} - e^{-i(\alpha-2\omega)} \sqrt{b(z)}}{\sqrt{c(z)} + e^{-i(\alpha-2\omega)} \sqrt{b(z)}}. \tag{71} \end{align*} \]

Writing

\[ b(z) = |b_1|e^{i\gamma_1}, \quad c(z) = |c_1|e^{i\gamma_1}, \quad \Omega_1 = \alpha - 2\omega + \frac{\beta_1 - \gamma_1}{2}, \tag{72} \]

one obtains

\[ \begin{align*}
\epsilon_{in}^L &= \frac{|b_1| - |c_1| + 2i \sqrt{|b_1||c_1|} \sin \Omega_1}{|b_1| + |c_1| + 2 \sqrt{|b_1||c_1|} \cos \Omega_1}, & \epsilon_{out}^L &= \frac{|c_1| - |b_1| + 2i \sqrt{|b_1||c_1|} \sin \Omega_1}{|b_1| + |c_1| + 2 \sqrt{|b_1||c_1|} \cos \Omega_1}. \tag{73} \end{align*} \]

and, for their moduli

\[ |\epsilon_{in}^L|^2 = |\epsilon_{out}^L|^2 = \frac{|b_1| + |c_1| - 2 \sqrt{|b_1||c_1|} \cos \Omega_1}{|b_1| + |c_1| + 2 \sqrt{|b_1||c_1|} \cos \Omega_1}. \tag{74} \]

which satisfy, when \( \omega \) varies from \( -\pi \) to \( +\pi \)

\[ \left| \frac{\sqrt{|b_1|} - \sqrt{|c_1|}}{\sqrt{|b_1|} + \sqrt{|c_1|}} \right| \leq |\epsilon_{in}^L| = |\epsilon_{out}^L| \leq \left| \frac{\sqrt{|b_1|} + \sqrt{|c_1|}}{\sqrt{|b_1|} - \sqrt{|c_1|}} \right|. \tag{75} \]

We observe that:
- the real and imaginary parts of \( \epsilon_{in}^L \) and \( \epsilon_{out}^L \) and their moduli depend on the arbitrary phases \( \omega \) and \( \alpha \);
- the imaginary parts of \( \epsilon_{in}^L \) and \( \epsilon_{out}^L \) can always both be turned to 0 by tuning \( \omega \) (or \( \alpha \));
- their real parts can never be cast to 0 by such rephasing;
- the real parts of \( \epsilon_{in}^L \) and \( \epsilon_{out}^L \) are opposite; their imaginary parts are identical;
- the modulus \( \frac{1}{\epsilon_{in}^L} \) is invariant by rephasing;
- a variation of \( \alpha \) can always compensate a variation of \( \omega \);
- when \( |b| = |c| \), \( \epsilon \) becomes purely imaginary, as already mentioned (see subsection 3.1.2).

The relations \( \tag{67} \) are unchanged by the rephasing (see also appendix B).
When the arbitrary phases are varied, \( \epsilon_L^{in} \) and \( \epsilon_L^{out} \) are located on two ellipsoids symmetric with respect to the imaginary axis, as described in Fig. 2 of Appendix B. The other \( CP \) violating parameters are also discussed there.

Accordingly, \textit{a priori}, neither the real, nor the imaginary part, nor the modulus of the \( \epsilon \)'s are physically relevant; the only physical quantities are the lower and upper bounds \([75]\) for the modulus of \( \epsilon \); the upper bound being much larger than \( 1 \) can reasonably be discarded. Nevertheless, as soon as \( b_1 \neq c_1 \), neither the real nor the modulus of \( \epsilon \) is vanishing; a non-zero measurement of these is accordingly a proof of \( CP \) (or \( T \)) violation (see subsection 5.1.1).

The identification of the physically relevant quantity smooths out the potential problems linked with the existence of two sets of physical eigenstates, “out” and “in” which only differ by the signs of the real parts of their \( CP \) violating parameters \(^{20}\).

Complements, in particular the comparison with the other \( \epsilon \)'s, can be found is Appendix B.

### 3.2.5 Expression of the eigenstates in terms of the \( CP \) violating parameters

In order to perform calculations of kaon decay amplitudes we will need the expressions for the propagating states \( K_L \) and \( K_S \) in terms of the states with definite strangeness, \( K_0 \) and \( \bar{K}_0 \). Using \([37, 64, 65, 57]\) we obtain \(^{21}\)

\[
|K_L >_{in} = \frac{1}{\sqrt{2}} \left[ \frac{1 + \epsilon_L^{in}}{1 - \epsilon_L^{in}} K_0 > -e^{i\alpha} \sqrt{\frac{1 - \epsilon_L^{in}}{1 + \epsilon_L^{in}}} \bar{K}_0 > \right] = \frac{1}{\sqrt{2}} \left[ \frac{1}{\xi_L} |K^0 > -e^{i\alpha} \xi_L |\bar{K}^0 > \right],
\]

\[
|K_S >_{in} = \frac{1}{\sqrt{2}} \left[ \frac{1 + \epsilon_S^{in}}{1 - \epsilon_S^{in}} K_0 > +e^{i\alpha} \sqrt{\frac{1 - \epsilon_S^{in}}{1 + \epsilon_S^{in}}} \bar{K}_0 > \right] = \frac{1}{\sqrt{2}} \left[ \frac{1}{\xi_S} |K^0 > +e^{i\alpha} \xi_S |\bar{K}^0 > \right],
\]

\[
\text{out} < |K_L| = \frac{1}{\sqrt{2}} \left[ \frac{1 - \epsilon_L^{in}}{1 + \epsilon_L^{in}} |K_0 > -e^{-i\alpha} \sqrt{\frac{1 + \epsilon_L^{in}}{1 - \epsilon_L^{in}}} \bar{K}_0 > \right] = \frac{1}{\sqrt{2}} \left[ \xi_L < K^0 > -e^{-i\alpha} \frac{1}{\xi_L} < \bar{K}^0 \right],
\]

\[
\text{out} < |K_S| = \frac{1}{\sqrt{2}} \left[ \frac{1 - \epsilon_S^{in}}{1 + \epsilon_S^{in}} |K_0 > +e^{-i\alpha} \sqrt{\frac{1 + \epsilon_S^{in}}{1 - \epsilon_S^{in}}} \bar{K}_0 > \right] = \frac{1}{\sqrt{2}} \left[ \xi_S < K^0 > +e^{-i\alpha} \frac{1}{\xi_S} < \bar{K}^0 \right],
\]

where we have introduced the notations

\[
\xi_S = \sqrt{\frac{1 - \epsilon_S^{in}}{1 + \epsilon_S^{in}}} \equiv \sqrt{\frac{c(z_2)}{b(z_2)}} e^{-i\alpha}, \quad \xi_L = \sqrt{\frac{1 - \epsilon_L^{in}}{1 + \epsilon_L^{in}}} \equiv \sqrt{\frac{c(z_1)}{b(z_1)}} e^{-i\alpha}.
\]

\(|\xi_L|, |\xi_S| \) and \( \frac{\xi_S}{\xi_L} \) are invariant by \( [70] \).

Inverting (76) one obtains \(^{22}\):

\(^{20}\)Section 5 will also show that no ambiguity arises in the calculation of semi-leptonic asymmetries, which write in terms of \( \epsilon_S^{in} \) and \( \epsilon_L^{in} \) only.

\(^{21}\)It is instructive to compare (76) with (15) and (16) in [4].

\(^{22}\)\( |K_L >_{in}, |K_L >_{out}, |K_S >_{in} \) and \( |K_S >_{out} \) are not linearly independent; since \( [28] \), unlike in [4], there is no distinction between \( |K^0 >_{in} \) and \( |K^0 >_{out} \). (78) determine \( |K_S >_{out} \) and \( |K_L >_{out} \) as linear combinations of \( |K_S >_{in} \) and \( |K_L >_{in} \).
correspond to two different values of the equalities are satisfied only when (see (66)) and spurious eigenstates to be realized. This means in particular that, at any of the two physical masses, |

\[ K^0 > = \frac{1}{D^{\frac{1}{2}}} \left[ \sqrt{\frac{1 - e^{in}_S}{1 + e^{in}_S}} | K_L >_{in} + \sqrt{\frac{1 - e^{in}_L}{1 + e^{in}_L}} | K_S >_{in} \right] = \frac{1}{D^{\frac{1}{2}}} \left[ \xi_S | K_L >_{in} + \xi_L | K_S >_{in} \right], \]

\[ \bar{K}^0 > = e^{-i\alpha} \frac{1}{D^{\frac{1}{2}}} \left[ \sqrt{\frac{1 + e^{in}_S}{1 - e^{in}_S}} | K_L >_{in} + \sqrt{\frac{1 + e^{in}_L}{1 - e^{in}_L}} | K_S >_{in} \right] = e^{-i\alpha} \frac{1}{D^{\frac{1}{2}}} \left[ -\xi_S | K_L >_{in} + \xi_L | K_S >_{in} \right], \]

\[ |K^0 > = \frac{1}{D^{\frac{1}{2}}} \left[ \sqrt{\frac{1 + e^{in}_S}{1 - e^{in}_S}} | K_L >_{out} + \sqrt{\frac{1 + e^{in}_L}{1 - e^{in}_L}} | K_S >_{out} \right] = \frac{1}{D^{\frac{1}{2}}} \left[ \xi_S | K_L >_{out} + \xi_L | K_S >_{out} \right], \]

\[ \bar{K}^0 > = e^{i\alpha} \frac{1}{D^{\frac{1}{2}}} \left[ \sqrt{\frac{1 - e^{in}_S}{1 + e^{in}_S}} | K_L >_{out} + \sqrt{\frac{1 - e^{in}_L}{1 + e^{in}_L}} | K_S >_{out} \right] = e^{i\alpha} \frac{1}{D^{\frac{1}{2}}} \left[ -\xi_S | K_L >_{out} + \xi_L | K_S >_{out} \right], \]

where

\[ D = \frac{1}{2} \left( \sqrt{\frac{(1 + e^{in}_L)(1 - e^{in}_S)}{(1 - e^{in}_S)(1 + e^{in}_S)}} + \sqrt{\frac{(1 - e^{in}_S)(1 - e^{in}_L)}{(1 + e^{in}_S)(1 + e^{in}_L)}} \right) = \frac{1 - e^{in}_Le^{in}_S}{\sqrt{(1 - (e^{in}_S)^2)(1 - (e^{in}_L)^2)}} \]

(79)

3.2.6 TCP invariance is realized in a non-trivial way

From (64) and the relations (67), one concludes that TCP symmetry is realized at each given \( z \), among the two corresponding “in” eigenstates (one physical and one “spurious”), and, likewise, among the two “out" eigenstates; \( |K_L >_{in} \) and \( |\tilde{K}_L >_{in} \) have CP-violating parameters satisfying \( e^{in}_L = 1/e^{in}_{L} \); the same type of relations occurs between \( |K_S >_{in} \) and \( |\tilde{K}_S >_{in} \), and between the two similar pairs of “out” states.

This means in particular that, at any of the two physical masses, TCP symmetry needs both the propagating and spurious eigenstates to be realized.

However this does not occur in general for the physical propagating \( K_L \) and \( K_S \) mesons, because they correspond to two different values of \( z \)

For \( z_1 \neq z_2 \), \( e^{in}_L \neq e^{in}_S \), \( e^{out}_L \neq e^{out}_S \); \hspace{3cm} (80)

the equalities are satisfied only when (see (66))

\[ \frac{b(z_1)}{c(z_1)} = \frac{b(z_2)}{c(z_2)} = \frac{1}{\zeta}. \]

(81)

transcribed for the elements of the propagator, instead of the inverse propagator writes

\[ \frac{g(z_1)}{h(z_1)} = \frac{g(z_2)}{h(z_2)}; \]

(82)

a particular cases when it is satisfied are when the two physical masses are identical \( z_1 = z_2 \) (a trivial one if of course when CP invariance holds, as (133) tells us, since the phase \( \alpha \) is a constant);

(81) (82) are, in general, not fulfilled, such that the physical mass eigenstates do not satisfy the most commonly used criterion of TCP invariance; of course the TCP symmetry is achieved and stays a fundamental property of the theory.

We shall investigate in sections 4 and 5 what are the consequences on the mass matrix, and if effects which would mimic TCP violating can be expected in experiments and wrongly interpreted as a violation of this fundamental symmetry.

\(^{23}\)The fact that these two CP-violating parameters are inverse of each other instead of being identical is only due to the convention that we have chosen, and it is indeed a consequence of TCP invariance.
3.2.7 Bi-unitary transformations

We should discriminate between the two ways of diagonalizing the propagator: using a biunitary transformation or a bi-orthogonal basis (see subsection 3.2.1).

We will compare below the two procedures at the poles; this will give us a criterion to reject bi-unitary transformations.

For this, we shall suppose that $\Delta^{-1}(z)$ is not normal at the poles, too, and we shall explicitly calculate the eigenvectors obtained by a bi-unitary transformation.

If $\det(\Delta^{-1}(z)) = 0$, the determinant of $\frac{1}{\Delta(z)}(\frac{1}{\Delta(z)})^\dagger$ vanishes, too, and so does the determinant of $\frac{1}{(\Delta(z))\Delta_z}$; so, these three sets of functions have poles at the same locations: the physical masses are the same in the two procedures.

At these poles $z = z_1$ and $z = z_2$, $\Delta^{-1}(z)$ can be written, using (81)

$$\Delta^{-1}(z) = \begin{pmatrix} \pm \sqrt{b(z)c(z)} & -b(z) \\ -c(z) & \pm \sqrt{b(z)c(z)} \end{pmatrix} \quad \text{at } z = z_1 \text{ or } z = z_2. \quad (83)$$

From (83) one gets immediately $(\Delta^{-1}(z))^\dagger$, and the roots of the characteristic equation of $\Delta^{-1}(z)(\Delta^{-1}(z))^\dagger$ or $(\Delta^{-1}(z))^\dagger\Delta^{-1}(z)$ are found to be

$$\delta = 0, \ \zeta = (|b(z)| + |c(z)|)^2, \quad (84)$$

where $z$ is to be considered to be equal to $z_1$ or to $z_2$.

The two unitary matrices $U$ and $V$ which are used to diagonalize $\Delta^{-1}$, and which respectively diagonalize the hermitian matrices $\Delta^{-1}(z)|\Delta^{-1}(z)|^\dagger$ and $[\Delta^{-1}(z)]^\dagger\Delta^{-1}(z)$ are accordingly constructed from the eigenvectors of $\Delta^{-1}(z)|\Delta^{-1}(z)|^\dagger$ and $[\Delta^{-1}(z)]^\dagger\Delta^{-1}(z)$, which form two different orthonormal basis.

The eigenvectors we characterize as usual by the ratio $v/u$ of their components in the $(K^0, \bar{K}^0)$ basis.

- The eigenvectors of $\Delta^{-1}(z)|\Delta^{-1}(z)|^\dagger$ are the following:
  * at $z = z_1$:
    - for the vanishing eigenvalue $\frac{v}{u}(z_1) = -\sqrt{\frac{b(z_1)}{c(z_1)}}$,
    - for the eigenvalue $(|b(z_1)| + |c(z_1)|)^2$, $\frac{v}{u}(z_1) = +\sqrt{\frac{c(z_1)}{b(z_1)}}$.
  * at $z = z_2$:
    - for the vanishing eigenvalue $\frac{v}{u}(z_2) = +\sqrt{\frac{b(z_2)}{c(z_2)}}$,
    - for the eigenvalue $(|b(z_2)| + |c(z_2)|)^2$, $\frac{v}{u}(z_2) = -\sqrt{\frac{c(z_2)}{b(z_2)}}$.

- The eigenvectors of $[\Delta^{-1}(z)]^\dagger(\Delta^{-1}(z))$ are the following:
  * at $z = z_1$:
    - for the vanishing eigenvalue $\frac{v}{u}(z_1) = -\sqrt{\frac{c(z_1)}{b(z_1)}}$,
    - for the eigenvalue $(|b(z_1)| + |c(z_1)|)^2$, $\frac{v}{u}(z_1) = +\sqrt{\frac{b(z_1)}{c(z_1)}}$.
  * at $z = z_2$:
    - for the vanishing eigenvalue $\frac{v}{u}(z_2) = +\sqrt{\frac{c(z_2)}{b(z_2)}}$,
    - for the eigenvalue $(|b(z_2)| + |c(z_2)|)^2$, $\frac{v}{u}(z_2) = -\sqrt{\frac{b(z_2)}{c(z_1)}}$.

Comparing the formulæ above with (64), we conclude that:

- the eigenvectors of $(\Delta^{-1}(z))^\dagger\Delta^{-1}(z)$ for the vanishing eigenvalues match the “in” propagating states of (64).
As shown in subsection 5.1.2 measuring the difference between the semi-leptonic asymmetries amounts to a test of the non-vanishing of \( \epsilon_K \) and \( \epsilon_L \). The set of \( \epsilon_K \) the inverse propagator vanishes) but \( z = z_2 \) at \( z = z_2 \) the corresponding orthonormal basis. If one evaluate flavor states \( K^0 \) and \( \bar{K}^0 \) in terms of the two physical” propagating states, the corresponding “mixing” matrix can no longer be unitary, because its columns are evaluated respectively at \( z = z_1 \) and \( z = z_2 \).

A deeper investigation of this phenomenon and its consequences for the mixing matrix will be performed in [29].

### 3.4 THE SPECIAL CASE \( \epsilon_L = \epsilon_S \)

[81] and the equality between \( \epsilon_L \) and \( \epsilon_S \) are not forced by the \( TCP \) symmetry; in the quark model, an estimate of their difference is presented in subsection 5.1.3.

It is nevertheless instructive to describe the simplifications that occur when their equality is assumed (this is the usual situation in QM).

[81] applied to (64) shows that the spurious eigenstates “disappear” by becoming identical to the mass eigenstates \( K_L \) and \( K_S \); the picture that arises is the following:

- at \( z = z_1 \), the eigenstates of the propagator (and of its inverse) are \( K_L \) and \( K_S \); \( K_L \) propagates (at \( z = z_1 \) the inverse propagator vanishes) but \( K_S \) does not, since it does not correspond to a pole of the propagator;
- at \( z = z_2 \) the reverse occurs: the eigenstates of \( \Delta \) and \( \Delta^{-1} \) are again \( K_L \) and \( K_S \) but, now, \( K_S \) propagates and \( K_L \) does not.

The set of \( K_L \) and \( K_S \) eigenstates satisfy \( \text{out} < K_L(t) \mid K_S(t) >_{\text{in}} = 0 = \text{out} < K_S(t) \mid K_L(t) >_{\text{in}} \) for all \( t \), which is not true in the general case.

As shown in subsection 5.1.2 measuring the difference between the semi-leptonic asymmetries \( \delta_L - \delta_S \) amounts to a test of the non-vanishing of \( \epsilon_L - \epsilon_S \).

---

24 Going from the “bras” of (64) to the corresponding “kets” yields the complex conjugation of coefficients, which provides the matching.

25 We thus disagree with footnote 1 of [4]; however, the matrices which connect the eigenstates of the Hamiltonian with the \( K^0 \) and \( \bar{K}^0 \) fields according to the bulk of the paper [4] are not unitary.
PROPERTIES AND LIMITATIONS OF MASS MATRICES

When, in textbooks of Quantum Mechanics or Quantum Field Theory a mass matrix is introduced for neutral kaons, it is a constant matrix\(^\text{26}\) (in the bare Lagrangian)\(^\text{27}\). We have seen in subsection \(^\text{2.3.2}\) that introducing a constant mass matrix can only be an approximation, valid when a linear expansion of the inverse propagator is suitable (most likely very close to the poles).

The link should nevertheless be made with such a matrix since, in particular, all experiments are analyzed and fitted with the corresponding parameters.

The question of its normality is rapidly settled: since the propagator cannot be normal, in particular on the cut, the mass matrix cannot either. Also, from our general discussion on normal matrices in subsection \(^\text{3.1.2}\) it is clear that the mass matrix \(M^{(2)}\) (see (27)) cannot be normal since this would lead to purely imaginary indirect \(CP\) violating parameters.

Non-normal mass matrices have different left and right eigenstates; the corresponding question of knowing which kind of eigenstate is detected, was up to now left unsolved, often qualified of “unavoidable mathematical necessity”. We showed in subsection \(^\text{3.2.4}\) that the \(CP\) violating parameters of the “in” and “out” states corresponding to the same pole of the propagator have equal imaginary parts and opposite real parts, but this distinction does not appear physically relevant since both quantities turn out to depend on an arbitrary rephasing of \(K^0\) (and \(\bar{K}^0\)). This question is finally wiped out in section \(^\text{5}\) where we show that semi-leptonic asymmetries can be unambiguously calculated in terms of the sole \(\epsilon^m_L\) and \(\epsilon^m_S\).

The determination of the mass matrix is the question that we address now. Can one define a unique constant mass matrix which has the correct eigenmasses and for eigenstates the correct propagating eigenstates that we have rigorously defined above?

4.1 THE MASS MATRIX IN QUANTUM MECHANICS

In QM, one introduces the complex mass matrix \(\mathcal{M}\)\(^\text{28}\) with dimension \([mass]\)

\[
\mathcal{M} = M - \frac{i}{2} \Gamma, \text{ with } M = M^\dagger, \Gamma = \Gamma^\dagger,
\]

\[
= \begin{pmatrix}
m_{11} - \frac{i}{2} \gamma_{11} & m_{12} - \frac{i}{2} \gamma_{12} \\
m_{12} - \frac{i}{2} \gamma_{12} & m_{22} - \frac{i}{2} \gamma_{22}
\end{pmatrix}, \text{ with } m_{11}, m_{22}, \gamma_{11}, \gamma_{22} \in \mathbb{R}.
\]

\(\mathcal{M}\) is normal if and only if \(M\) and \(\Gamma\) commute, \([M, \Gamma] = 0\).

The conditions of \(TCP\) invariance are \(M_{11} = M_{22}\)\(^\text{31}\)\(^\text{32}\)\(^\text{33}\), and \(CP\) invariance adds to it the condition \(M_{12} = e^{-2\alpha \gamma} M_{21}\), such that a \(CP\) invariant mass matrix is always of the form

\[
\mathcal{M}_{CP} = \begin{pmatrix}
m - \frac{i}{2} \gamma & e^{-i\alpha} (\omega - \frac{i}{2} \chi) \\
e^{i\alpha} (\omega - \frac{i}{2} \chi) & m - \frac{i}{2} \gamma
\end{pmatrix}, \text{ with } m, \gamma, \omega, \chi \in \mathbb{R},
\]

which is a normal matrix.

Since the mass matrix is supposed to describe unstable kaons, it cannot be hermitian, because its eigenvalues, which are the masses of the eigenstates, would then be real.

Experiments tell us that the mass matrix of neutral \(K\) mesons cannot even be normal\(^\text{29}\) and thus should be diagonalized either by a bi-unitary transformation, or by using a bi-orthogonal basis. Since bi-unitary

\(^{26}\)We limit our discussion to constant mass matrices. Results similar to ours (the non-equality of \(CP\) violating parameters for \(K_L\) and \(K_S\) despite \(TCP\) is satisfied) had previously been obtained in \([6]\), in the formalism of an energy-dependent Hamiltonian.

\(^{27}\)Its renormalization is still subject to many debates (see for example \([35]\) and references therein).

\(^{28}\)See also footnote \([12]\).

\(^{29}\)\(K^0 \to \bar{K}^0\) probability is different from \(\bar{K}^0 \to K^0\) probability and the corresponding \(CP\) violating parameter cannot be purely imaginary; this entails that the mass matrix cannot be normal (see also subsection \(^\text{3.1.2}\)).
transformations always yield real masses, they are excluded for the same reasons as mentioned above. As done in [4], one must use a bi-orthogonal basis \(^{30}\).

4.2 INCONSISTENCY OF A CONSTANT MASS MATRIX IN QFT

Let \( M^{(2)} \) be a constant complex matrix

\[
M^{(2)} = \begin{pmatrix} n & r \\ s & t \end{pmatrix}, n, r, s, t \in \mathbb{C}.
\]  (87)

We request that the exact propagating eigenstates \(| K_L >_\text{in}, | K_S >_\text{in, out} < K_L | \) and \( _\text{out} < K_S | \) determined in (64) be its eigenstates, and we forget about the spurious states at \( z = z_1 \) and \( z = z_2 \) which cannot be accounted for in this restricted formalism.

For the sake of simplicity we shall adopt the following notations \(^{31}\):

\[
b_1 = \sqrt{\frac{b(z_1)}{2a(z_1)}}, \quad b_2 = \sqrt{\frac{b(z_2)}{2a(z_2)}}, \quad c_1 = \sqrt{\frac{c(z_1)}{2a(z_1)}}, \quad c_2 = \sqrt{\frac{c(z_2)}{2a(z_2)}}.
\]  (88)

The eigenvalues of \( M^{(2)} \) are

\[
\mu_{\pm} = \frac{1}{2} \left( n + t \pm \sqrt{(n - t)^2 + 4rs} \right);
\]  (89)

the equations for the eigenstates and their identification with the true propagating states (64) write

\[
M^{(2)} \begin{pmatrix} u_+ \\ v_+ \end{pmatrix} = \mu_+ \begin{pmatrix} u_+ \\ v_+ \end{pmatrix}, \quad M^{(2)} \begin{pmatrix} u_- \\ v_- \end{pmatrix} = \mu_- \begin{pmatrix} u_- \\ v_- \end{pmatrix}
\]

\[
(x_+, y_+) M^{(2)} = (x_+, y_+) \mu_+ = (c_1, -b_1) i\mu_+,
\]

\[
(x_-, y_-) M^{(2)} = (x_-, y_-) \mu_- = (c_2, b_2) \mu_-.
\]  (90)

which, using (89), leads to

\[
v_+ = \frac{t - n + \sqrt{(t - n)^2 + 4rs}}{2r} = \frac{2s}{n - t + \sqrt{(t - n)^2 + 4rs}} = -\frac{c_1}{b_1}, \quad (a)
\]

\[
v_- = \frac{t - n - \sqrt{(t - n)^2 + 4rs}}{2r} = \frac{2s}{n - t - \sqrt{(t - n)^2 + 4rs}} = \frac{c_2}{b_2}, \quad (b)
\]

\[
y_+ = \frac{t - n + \sqrt{(t - n)^2 + 4rs}}{2r} = \frac{2r}{n - t + \sqrt{(t - n)^2 + 4rs}} = -\frac{b_1}{c_1}, \quad (c)
\]

\[
y_- = \frac{t - n - \sqrt{(t - n)^2 + 4rs}}{2r} = \frac{2r}{n - t - \sqrt{(t - n)^2 + 4rs}} = \frac{b_2}{c_2}. \quad (d) \]  (91)

Equations (91) are contradictory unless (89) is satisfied; indeed:

- for (a) and (c) to be inverse of each other, as their r.h.s. demand, one needs the equality of diagonal

\(^{30}\)The criterion of masses real or not is only valid for a mass matrix and not for the full propagator. Indeed, when \( \lambda_{\pm}(z) \) are the eigenvalues of the propagator, their reality does not prevent the physical masses, which are the solutions of \( \lambda_{\pm}(z) = 0 \) to be complex. Hence, we rejected bi-unitary transformations for the propagator with more involved arguments.

\(^{31}\)Do not confuse the present notations \( b_1 \) and \( c_1 \) with the ones used in [72].
elements \( n = t \); the same occurs for equations (b) and (d);
- they also show that the two diagonal elements \( n \) and \( t \) of \( M^{(2)} \) cannot be identical unless (81) is satisfied: indeed, if they were, one would get from (a) and (b) \( \frac{n}{n_i} = \frac{t}{t_i} = \sqrt{r} \), which, from their r.h.s. entails \( \frac{b_1}{c_1} = \frac{b_2}{c_2} \), which is condition (81); the same occurs between (c) and (d).

We conclude that one can introduce a constant mass matrix only with constant diagonal matrix elements, which then can only correspond to the case when \( TCP \) symmetry is achieved between the propagating eigenstates (the \( CP \) violating parameters for \( K_L \) and \( K_S \) are then identical).

### 4.3 CONSTANT MASS MATRIX AND DISCRETE SYMMETRIES

Since (81) has no reasons to be satisfied in general, we reach the following conclusion:

A constant mass matrix can never describe faithfully the correct propagating eigenstates of a quasi-degenerate system of neutral mesons;

indeed:
- choosing its diagonal elements equal is equivalent to imposing \( \epsilon_L = 1/\epsilon_S \), the equality of the \( CP \) violating parameters of \( K_L \) and \( K_L \), which we have seen to be untrue; setting different diagonal elements leads to contradictions between (a) and (c), and (b) and (d), which means that either the “in”, or the “out” eigenstates of the constant mass matrix match those of the propagator, but never both;
- it cannot include the spurious states, which means in particular that any completeness relation obtained from \( M^{(2)} \) is a priori incorrect (it does not build the appropriate Hilbert spaces of “in”; and “out states). A constant mass matrix is in particular inappropriate to provide a faithful description of \( TCP \) symmetry.

Can it describe faithfully \( CP \) violation?

Giving up a correct description of \( TCP \), let us choose \( M^{(2)} \) with equal diagonal elements \( t = n \), which is the “quantum mechanical condition” for \( TCP \) invariance. To be consistent, one must identify by brute force the indirect \( CP \) violating parameters of \( K_L \) and \( K_S \), which can only be done by assuming (see subsection 3.2.6) \( \frac{b_1}{c_1} = \frac{b_2}{c_2} \); (91) entails furthermore that this is automatically satisfied when \( n = t \). The number of parameters of \( M^{(2)} \) to be determined shrinks to three; to fix them we have at our disposal three equations, respectively for the two masses and for the (now unique) \( CP \) violating parameter \( \epsilon \) (the one for “out” eigenstates is easily deduced from the one for “in” eigenstates, see (66) (67) (68)).

We conclude that a constant mass matrix has enough parameters to provide a faithful parameterization of \( CP \) violation, or \( T \) violation, once \( TCP \) conservation is assumed and imposed. The numerical accuracy of the constant mass matrix approximation will be determined in subsection 5.1.3.

### 5 PHYSICAL PROCESSES AND ASYMMETRIES

The goal of this section is to complete the previous formal investigations by phenomenological considerations. In particular, we want to stress differences of interpretation between a description of neutral mesons by QM and by QFT.

Since the main difference that we outlined above concern discrete symmetries, in particular \( TCP \), and the difference of the \( CP \) violating parameters \( \epsilon_L \neq \epsilon_S \), this will be our principal topic below.

It is important to notice that all asymmetries are expressed in terms of the \( CP \) violating parameters \( \epsilon_L^{in} \) and \( \epsilon_S^{in} \); the parameters of the “out” states do not appear. The distinction between “in” and “out” states has no consequences for physical observables.
5.1 SEMI-LEPTONIC ASYMMETRIES

We shall calculate the following asymmetries [37]

\[ A_T = \frac{\left| \langle \pi^- \ell^+ \nu(t_f) | K^0(t_i) > \right|^2 - \left| \langle \pi^+ \ell^- \nu(t_f) | K^0(t_i) > \right|^2}{\left| \langle \pi^- \ell^+ \nu(t_f) | K^0(t_i) > \right|^2 + \left| \langle \pi^+ \ell^- \nu(t_f) | K^0(t_i) > \right|^2} \]

\[ \Delta S = \Delta Q = \frac{\left| \langle \pi^- \ell^+ \nu(t_f) | \bar{K}^0(t_i) > \right|^2 - \left| \langle \pi^+ \ell^- \nu(t_f) | \bar{K}^0(t_i) > \right|^2}{\left| \langle \pi^- \ell^+ \nu(t_f) | \bar{K}^0(t_i) > \right|^2 + \left| \langle \pi^+ \ell^- \nu(t_f) | \bar{K}^0(t_i) > \right|^2} \]

\[ \delta_{L,S} = \frac{| \langle \pi^- \ell^+ \nu | K_{L,S} >_{in} |^2 - | \langle \pi^+ \ell^- \nu | K_{L,S} >_{in} |^2}{| \langle \pi^- \ell^+ \nu | K_{L,S} >_{in} |^2 + | \langle \pi^+ \ell^- \nu | K_{L,S} >_{in} |^2} ; \]

\[ A_{TCP} = \frac{\left| \langle \pi^- \ell^+ \nu(t_f) | \bar{K}^0(t_i) > \right|^2 - \left| \langle \pi^+ \ell^- \nu(t_f) | \bar{K}^0(t_i) > \right|^2}{\left| \langle \pi^- \ell^+ \nu(t_f) | \bar{K}^0(t_i) > \right|^2 + \left| \langle \pi^+ \ell^- \nu(t_f) | \bar{K}^0(t_i) > \right|^2} \]

\[ \Delta S = \Delta Q \]

supposing that the rule \( \Delta S = \Delta Q \) holds [32] , which in particular only allows the semi-leptonic decays

\[ K^0 \rightarrow \pi^- \ell^+ \nu, \quad \bar{K}^0 \rightarrow \pi^+ \ell^- \nu. \]

The first and the third are the asymmetries tested in the CPLEAR experiment [13]. \( \delta_L \) has been accurately measured [37] and \( \delta_S \) should be measured with tagged \( K_S \) at \( \phi \) factories.

5.1.1 The CPLEAR asymmetry \( A_T \) [34]

At low energy \( \bar{p}p \) annihilation \( K^- K^0 (K^0 \bar{K}^0) \) are produced; \( K^0 (\bar{K}^0) \) is tagged by \( K^- (K^+) \) decay. The particle momenta are not measured with high accuracy (in the opposite case it would be known that \( K_L \) or \( K_S \) – was produced). As soon as an “averaging” over \( K^0 (\bar{K}^0) \) momentum is accepted, the contributions of both \( K_L \) and \( K_S \) in intermediate (propagating) states should be taken into account [33].

Let us calculate the amplitude for a produced \( \bar{K}^0 \) to decay into \( \ell^+ \nu \pi^- \) after time \((t_f - t_i)\):

\[ A(\bar{K}^0 \rightarrow \ell^+ \nu \pi^-)_{t_f - t_i} = \text{out} < \ell^+ \nu \pi^- | K_L(t_f) >_{in} \text{out} < K_L(t_f) | K_L(t_i) >_{in} \text{out} < K_L(t_i) | \bar{K}^0 >_{+} + \text{out} < \ell^+ \nu \pi^- | K_S(t_f) >_{in} \text{out} < K_S(t_f) | K_S(t_i) >_{in} \text{out} < K_S(t_i) | \bar{K}^0 >. \] (94)

Calculating [94] we will first neglect the tiny difference between \( \epsilon_L \) and \( \epsilon_S \), taking thus \( \epsilon^L_in = \epsilon^S_in = \epsilon^in \).

One uses the expressions for \( K_L \) and \( K_S \) in [76] together with the orthogonality relations between \( K^0 \) and \( \bar{K}^0 \); according to the \( \Delta Q = \Delta S \) rule, only the \( K^0 \) component of \( K_L \) and \( K_S \) can produce \( \ell^+ \); this yields [34]

\[ \text{It is expected from the standard model to be valid up to order } 10^{-14} [4] \]

\[ \text{As conspicuous below, the calculation amounts to inserting twice } | K_L >_{in} \ \text{out} < K_L | + | K_S >_{in} \ \text{out} < K_S |; \text{ this is justified in the approximation } \epsilon_L = \epsilon_S \text{ that we use, in which this expression becomes equal to one and where the crossed scalar products } \text{out} < K_L | K_S >_{in} \ \text{and out} < K_S | K_L >_{in} \text{ vanish.} \]

\[ \text{Time evolution: having introduced decaying particles fixes the direction of evolution of time. This is most easily seen by considering the time-dependent scalar product of “in” and “out” mass eigenstates, for example } \text{out} < K_L(t_f) | K_L(t_i) >_{in}; \text{ if one adopts, like in [4], the “usual” time evolution } | K_L(t_i) >_{in} = e^{-i M_L t_i} | K_L(0) >_{in}, \text{ out} < K_L(t_f) | = e^{i M_L t_f} \text{ out} < K_L(0) |, \text{ where } M_L \text{ and } M_S \text{ have been defined in [100], one gets out} < K_L(t_f) | K_L(t_i) >_{in} = e^{i M_L t_f} e^{-i M_L t_i} = e^{-i M_L(t_f-t_i)} e^{\Delta \theta(t_f-t_i)}, \text{ which leads to an exponential growth since } t_i < t_f. \]

The time evolution is arbitrary in ordinary QM with a hermitian Lagrangian (see for example [24] paragraphs 6 and 8), becomes here relevant: the Schrödinger equation has to be chosen here as \( H \psi = -i \hbar \frac{\partial}{\partial t} \psi \), and the time evolution of “in” and “out” states

\[ | K_L(t_i) >_{in} = e^{+i M_L t_i} | K_L(0) >_{in}, \text{ out} < K_L(t_f) | = e^{-i M_L t_f} \text{ out} < K_L(0) |, \]

and similar equations for \( K_S \).
\[
A(K^0 \to \ell^+\nu^-)_{\ell_f-\ell_i} = \frac{e^{-i\alpha}}{2D^2} \left[ \frac{1}{\xi_L} e^{-im_L(t_f-t_i)} - \frac{\Gamma_L}{2}(t_f-t_i) + \frac{1}{\xi_S} e^{-im_S(t_f-t_i)} - \frac{\Gamma_S}{2}(t_f-t_i) \right] \times A(K^0 \to \ell^+\nu^-)
\]

(96)

Analogously, the amplitude for a produced \( K^0 \) to decay into \( \ell^-\nu^+ \) after time \( (t_f-t_i) \) is given by:

\[
A(K^0 \to \ell^-\nu^+)_{\ell_f-\ell_i} = \frac{e^{i\alpha}}{2D^2} \left[ -\xi_L^2 e^{-im_L(t_f-t_i)} - \frac{\Gamma_L}{2}(t_f-t_i) + \xi_S^2 e^{-im_S(t_f-t_i)} - \frac{\Gamma_S}{2}(t_f-t_i) \right] \times A(K^0 \to \ell^-\nu^+)
\]

(97)

and \( A(K^0 \to \ell^-\nu^+) = A(K^0 \to \ell^+\nu^-) \).

In the approximation \( \epsilon_L^{in} = \epsilon_S^{in} = \epsilon^{in} \) at which we are working \( (77) \) become

\[
\xi_S = \xi_L = \sqrt{\frac{1 - \epsilon^{in}}{1 + \epsilon^{in}}},
\]

(98)

the modulus of which is invariant by the rephasing \( (70) \).

From \( (96) \), \( (97) \) and \( (98) \), the time dependence and the dependence on the arbitrary phase \( \alpha \) for the \( T \)-odd asymmetry defined in \( (92) \) cancel and we obtain:

\[
A_T = \frac{1 + \epsilon^{in}}{1 - \epsilon^{in}} - \frac{1 - \epsilon^{in}}{1 - \epsilon^{in}} = 4 \frac{\Re(\epsilon^{in})(1 + |\epsilon^{in}|^2)}{(1 + |\epsilon^{in}|^2)^2 + 4(\Re(\epsilon^{in}))^2}.
\]

(99)

This result is independent of the rephasing \( (70) \), unlike its approximation by \( 4 \Re(\epsilon) \) (see subsection \( 3.2.4 \) that one finds for example in \( [4] \); accordingly, it can now be evaluated for any value of \( \omega \) or \( \Omega_1 \) \( (72) \). However, when \( |\epsilon^{in}| \ll 1 \), it is well approximated by \( 4 \Re(\epsilon^{in}) \).

The corrections that would eventually arise from \( \epsilon_L \neq \epsilon_S \) where checked to vanish by calculating Feynman diagrams as we do later for \( A_{T_{CP}} \) in subsection \( 5.1.4 \) The calculation makes use of the explicit form of the non-diagonal \( K_L - K_S \) vertex \( V \) \( (114) \), and of the close approximation to the diagonal term \( a(q^2) \) in the inverse Lagrangian \( (48) \) \( a(q^2) \approx \frac{1}{2} \left( q^2 - \frac{M^2_{K_L} + M^2_{K_S}}{2} \right) \).

Another check that we did using the same technique is that each individual transition amplitude \( K^0(t_i) \to K^0(t_f) \) and \( \bar{K}^0(t_i) \to K^0(t_f) \) vanishes when \( t_f = t_i \) even when \( \epsilon_L \neq \epsilon_S \).

5.1.2 The semi-leptonic charge asymmetries for \( K_L \) and \( K_S \)

Supposing that the semi-leptonic decay rates \( (93) \) allowed by the \( \Delta S = \Delta Q \) rule are identical and using \( (76) \), one gets\(^{35}\)

\(^{35}\)For example, the transition \( K_L \to \pi^{-}\ell^+\nu \) can only occur by a first transition from \( K_L \) to \( K^0 \) and one writes \( \epsilon_{out} < \pi^{-}\ell^+\nu \mid K_L >_{in} = \epsilon_{out} < \pi^{-}\ell^+\nu \mid K^0 > < K^0 \mid K_L >_{in} \) etc
Using the notations of subsection 4.1 and according to the first line of (66) we have:

\[
\delta_L = \frac{1}{|\xi_L|^2 + |\xi_L|^2} = \frac{|b(z_1)| - |c(z_1)|}{|b(z_1)| + |c(z_1)|} = \frac{2 \mathfrak{R}(\epsilon_L^{\text{in}})}{1 + |\epsilon_L^{\text{in}}|^2},
\]

\[
\delta_S = \frac{1}{|\xi_S|^2 + |\xi_S|^2} = \frac{|b(z_2)| - |c(z_2)|}{|b(z_2)| + |c(z_2)|} = \frac{2 \mathfrak{R}(\epsilon_S^{\text{in}})}{1 + |\epsilon_S^{\text{in}}|^2},
\]

(100)

and

\[
\delta_S - \delta_L = 2 \left( \frac{\mathfrak{R}(\epsilon_S^{\text{in}})}{1 + |\epsilon_S^{\text{in}}|^2} - \frac{\mathfrak{R}(\epsilon_L^{\text{in}})}{1 + |\epsilon_L^{\text{in}}|^2} \right).
\]

(101)

When \(\epsilon_L = \epsilon_S = \epsilon^{\text{in}}\), (101) becomes zero; \(\delta_S - \delta_L\) accordingly measures the difference between the indirect \(CP\) violating parameters of the \(K_L\) and \(K_S\) mesons.

This result is to be compared with the one in [16] and the one in [37]. In [16], \(CP\) violation is simply parametrized by different diagonal elements in the mass matrix; since their \(CP\) violating parameter \(\delta\) is precisely defined as the ratio of the difference of the diagonal elements of the mass matrix and of the difference of the physical masses, the formula in [37], which comes from [16], is only consistent. In QFT, since \(\epsilon_L \neq \epsilon_S\), this is not a good way to parametrize \(CP\) violation.

\(\delta_S\) and \(\delta_L\) are both invariant by the rephasing (70); forgetting about their denominators or approximating them by 1 is illegitimate, since then real parts of the \(\epsilon\)'s is not invariant by this rephasing (see subsection 3.2.4). However, for \(|\epsilon_L^{\text{in}}| \ll 1, |\epsilon_S^{\text{in}}| \ll 1\), one can neglect the denominators in (100) (101) and obtain \(\delta_S - \delta_L \approx 2 \mathfrak{R}(\epsilon_S - \epsilon_L)\).

While \(\delta_L\) has already been measured with an average of \(\delta_L = (3.27 \pm 0.12) \times 10^{-3}\) [37], there are only preliminary results for \(\delta_S\) coming from the KLOE detector [23]: \(\delta_S = (-2 \pm 9_{\text{stat}} \pm 6_{\text{syst}}) \times 10^{-3}\). One is obviously very far from the precision of order \(10^{-17}\) (see subsection 5.1.3 below) requested to test the expected difference between \(\delta_L\) and \(\delta_S\).

### 5.1.3 An estimate of \((\epsilon_S^{\text{in}} - \epsilon_L^{\text{in}})\)

Using the notations of subsection 4.1 and according to the first line of (66) we have:

\[
\epsilon_L^{\text{in}} \approx \frac{1 - e^{-i \alpha}}{1 + e^{-i \alpha}} \sqrt{\frac{c(z_1, z_2)}{b(z_1, z_2)}}
\]

(102)

is expressed only in terms of the non-diagonal elements \(b(z)\) and \(c(z)\) of the inverse propagator (48).

The non-diagonal elements of the QM mass matrix (85) being expected to be close to the ones of the inverse propagator, and since we only need an order of magnitude estimate of the difference \(\epsilon_S - \epsilon_L\), we shall approximate (102) by

\[
\epsilon_L^{\text{in}} \approx \frac{1 - e^{-i \alpha}}{1 + e^{-i \alpha}} \sqrt{\frac{m_{12}(z_1, z_2) - \gamma_{12}(z_1, z_2)}{m_{12}(z_1, z_2) + \gamma_{12}(z_1, z_2)}};
\]

(103)

where \(m_{12}\) and \(\gamma_{12}\) should be taken at \(q_1^2 = z_1 = m_L^2\) and \(q_2^2 = z_2 = m_S^2\) for \(\epsilon_L\) and \(\epsilon_S\), correspondingly.

The standard choice \(\alpha = 0\) for the arbitrary phase \(\alpha\) corresponds to the condition \(\gamma_{12} = \gamma_{12}\) (in the quark mass matrix (85) of QM has dimension \([\text{mass}]^2\) while the inverse propagator (48) of QFT has dimension \([\text{mass}]^{-2}\). Nevertheless, in (103), the ratio of the non-diagonal elements \(b\) and \(c\) of the inverse propagator has been identified with the ratio of the non-diagonal elements of the quantum mechanical mass matrix. This is a good enough approximation for the order of magnitude estimate that we want to get as soon as the mass matrix in QFT is recognized to be the square of the mass matrix in QM and its non-diagonal elements are much smaller than its diagonal elements.
model this choice is equivalent to that of real \( V_{us} \) and \( V_{ud} \) CKM matrix elements:

\[
\epsilon_{L,S}^{\text{in}} = \frac{\sqrt{m_{12}^2 - \frac{1}{2} \gamma_{12}} - \sqrt{m_{12}^2 - \frac{1}{2} \gamma_{12}}}{\sqrt{m_{12}^2 - \frac{1}{2} \gamma_{12}} + \sqrt{m_{12}^2 - \frac{1}{2} \gamma_{12}}}.
\] (104)

To proceed with our estimate, we shall hereafter rely on the quark picture of neutral mesons, and the so-called “box diagrams” which generate \( K^0 - \bar{K}^0 \) transitions \[44\] \[14\] \[8\] \[11\].
m\( m_{12} \) is almost real; a nonzero phase is generated by the quark box diagram with \( t\bar{t} \) exchange which is highly suppressed by the smallness of the CKM matrix elements \( V_{ts} \) and \( V_{td} \) \[37\]. One has \[44\] \( m_{12} \approx \Re(m_{12}) \approx \gamma_{12} \gg \Im(m_{12}) \), and, expanding (104), one gets

\[
\epsilon_{L,S}^{\text{in}} \approx \frac{i \Im(m_{12})}{2(m_{12} - \frac{1}{2} \gamma_{12})} \approx \frac{i \Im(m_{12})}{m_S - m_L - \frac{1}{2}(\Gamma_S - \Gamma_L)} = -\frac{i \Im(m_{12})}{\Delta m_{LS} + \frac{1}{2} \Gamma_S},
\] (105)

where we have defined

\[
M_L = m_L - \frac{\Gamma_L}{2}, M_S = m_S - \frac{\Gamma_S}{2},
\] (106)

neglected \( \Gamma_L \) in comparison with \( \Gamma_S \) and substituted \( \Delta m_{LS} \equiv m_L - m_S \) \[37\].

There exists a dependence of \( m_{12} \) and \( \gamma_{12} \) on the momentum \( q^2 \), which leads to a tiny difference between \( \epsilon_L \) and \( \epsilon_S \).

The dominant contribution to \( \Im(m_{12}) \) is produced by the box diagram with two intermediate \( t \)-quarks \((G_F \text{ is the Fermi constant)}):

\[
[\Im(m_{12})]_{tt} \propto \lambda^{10} G_F^2 (m_t^2 + q^2),
\] (107)

where \( \lambda \) is the Cabibbo angle. However, the sub-dominant diagram with two intermediate \( c \)-quarks generates a larger \( q^2 \) dependence (it contributes to \( \Im(m_{12}) \) since \( \Re(V_{cd}) \sim \lambda^5 \)):

\[
[\Im(m_{12})]_{cc} \propto \lambda^6 G_F^2 (m_c^2 + q^2).
\] (108)

As a result, since \( \lambda^4 m_t^2 \gg m_c^2 \), one gets:

\[
\Im(m_{12})(q^2) \approx \Im(m_{12})(0) \left( 1 + \frac{q^2}{\lambda^4 m_c^2} \right).
\] (109)

Since \( \Delta m_{LS} \) is numerically close to \( \Gamma_S/2 \), the contributions of \( c \) and \( u \) quarks to \( m_{12} \) should be comparable. In this way, we get

\[
m_{12}(q^2) \sim \lambda^2 (m_c^2 + q^2) \approx m_{12}(0) \left( 1 + \mathcal{O}(\frac{q^2}{m_c^2}) \right),
\] (110)

where the contribution of the box diagrams with intermediate \( c \) and \( u \) quarks is taken into account.

Finally, the \( q^2 \) dependence of \( \epsilon_{L,S} \) is determined by that of \( \gamma_{12} \); the dependence of \( \gamma_{12} \) originates both from \( K \to \pi\pi \) matrix elements and two-pions phase space

\[
\gamma_{12}(q^2) \approx \gamma_{12}(0) \left( 1 + \mathcal{O}(\frac{q^2}{m_K^2}) \right).
\] (111)

Taking into account the dominant contributions \[110\] \[111\] we obtain

\[
\epsilon_{L}^{\text{in}} - \epsilon_{S}^{\text{in}} \sim e \frac{\Delta m_{LS}}{m_K} \sim 10^{-17}.
\] (112)

In this way we obtain that the \( q^2 \) dependence of the kaon self-energy leads to different values of \( \epsilon_L \) and \( \epsilon_S \): the statement that this difference signals TCP violation is seen to be wrong.

\[3\] The relation between the first and the second denominators of (105) is obtained from the expression of the eigenvalues of \[88\] by using \( \gamma_{12} \ll m_{11}, \gamma_{22} \ll m_{22}, m_{12} \) quasi real, and choosing the phase convention such that \( \gamma_{12} = \gamma_{22} \).
5.1.4 The asymmetry $A_{TCP}$

We have seen that finding a non vanishing difference of charge asymmetries $\delta_S - \delta_L$ cannot be a priori interpreted as a signal of $TCP$ non-invariance, because this difference is allowed to keep non-vanishing even when $TCP$ is achieved. If it exceeds an expected $10^{-17}$ (see [112]), one clearly has a problem.

The question arises whether some observable should identically vanish when $TCP$ symmetry holds. We show below that it is the case of $A_{TCP}$ asymmetry given by the third equation of (92). $A_{TCP}$ is analogous to $A_T$, but, this time, the asymmetry for "allowed" semi-leptonic decays is studied.

We explicitly calculate $K^0 \to K^0$ and $\bar{K}^0 \to \bar{K}^0$ transitions in QFT by calculating the corresponding Feynman diagrams, allowing $\epsilon_L$ to be different from $\epsilon_S$ and $K_L \leftrightarrow K_S$ transitions.

The diagrams that we evaluate are, for $K^0 \to K^0$ transitions, drawn in Fig. 1. The same type of diagrams occur, of course, for $\bar{K}^0 \to \bar{K}^0$ transitions. We have drawn diagrams only up to first order in the $K_L - K_S$ coupling $V$, but results have been checked to stay unchanged at second order in $V$.

\[ \begin{array}{cccc}
K^0 & K_L & K^0 & V \\
\hline
K^0 & K_S & K_L & K^0 \\
\end{array} \]

\[ \begin{array}{cccc}
K^0 & K^0 & K^0 & V \\
\hline
K^0 & K_S & K_S & K^0 \\
\end{array} \]

\[ \begin{array}{cccc}
K^0 & K_S & K_L & K^0 \\
\hline
K^0 & K_L & K_S & K^0 \\
\end{array} \]

\[ \begin{array}{cccc}
K^0 & K^0 & K^0 & V \\
\hline
K^0 & K_L & K_S & K^0 \\
\end{array} \]

*Fig. 1: Feynman diagrams for $K^0$ to $K^0$ transition up to first order in $V$.*

- The $K_L - K_S$ couplings.

While transitions between $K_L$ and $\bar{K}_S$, or $K_S$ and $\bar{K}_L$, are forbidden, the ones between $K_L$ and $K_S$ are authorized, as shown by a very simple calculation.

\[ (48) \text{ is equivalent to writing the TCP invariant Lagrangian } \mathcal{L}_{TCP} \text{ in the } (K^0, \bar{K}^0) \text{ basis:} \]

\[ \mathcal{L}_{TCP}(z) = a(z) \left( | K^0 < K^0 | + | \bar{K}^0 | \bar{K}^0 | \right) - b(z) | K^0 < \bar{K}^0 | - c(z) | \bar{K}^0 > K^0 |. \]

(113)

Using then (78) to express $K^0$ and $\bar{K}^0$ in terms of $K_L$ and $K_S$, one obtains the $K_L - K_S$ couplings

\[ \mathcal{L}_{TCP}(z) \equiv \frac{1}{2D^2} \left[ a(z) \left( \frac{\xi_L}{\xi_S} - \frac{\xi_S}{\xi_L} \right) + b(z)e^{i\alpha}\xi_L\xi_S - c(z)e^{-i\alpha} \frac{1}{\xi_L\xi_S} \right] \left[ | K_S >_{in\ out} < K_L | - | K_L >_{in\ out} < K_S | \right]. \]

(114)

The important point for our concern is that, at any $z = q^2$, the $K_L$ to $K_S$ coupling $V(z)$ is the opposite of the $K_S$ to $K_L$ coupling.

- Explicit calculation

Let us calculate explicitly the diagram $c$ of Fig. 1, evaluated from right to left.

- The first vertex is the projection of $K^0$ on $K_L$, i.e. the scalar product $< K_L | K^0 >$; it is, according to (76), $\xi_L/\sqrt{2}$.

- The $K_L$ propagator $1/(q^2 - M_L^2)$ follows;

- The second vertex is the $K_L$ to $K_S$ vertex $V(q^2)$;

- The $K_S$ propagator $1/(q^2 - M_S^2)$ follows;

- The last vertex is the projection of $K_S$ on $K^0$, that is the scalar product $< K^0 | K_S >_{in}$; it is given by (76) and is equal to $1/(\sqrt{2}\xi_S)$. 

\[ 28 \]
Finally, the diagram c of Fig. 1 is given by
\[ \frac{V(q^2)}{2} \xi_L \frac{1}{q^2 - M_L^2} \frac{1}{q^2 - M_S^2}. \]
So doing for all diagrams, one gets for the amplitudes \( A_{K^0 \rightarrow K^0} \) and \( A_{\overline{K^0} \rightarrow \overline{K^0}} \)
\[ A(q^2)_{K^0 \rightarrow K^0} = \frac{1}{2} \left( \frac{1}{q^2 - M_L^2} + \frac{1}{q^2 - M_S^2} \right) \frac{V(q^2)}{2} \left( \xi_L \xi_S - \xi_S \xi_L \right) \frac{1}{q^2 - M_L^2} \frac{1}{q^2 - M_S^2} = A(q^2)_{\overline{K^0} \rightarrow \overline{K^0}}. \]
The Fourier transforms of these two amplitudes are identical, too. Their \( t \) dependence is just the dependence of the corresponding \( < K^0(t_f) | K^0(t_i) > \) and \( < \overline{K^0}(t_f) | \overline{K^0}(t_i) > \) on \( t_f - t_i \).

Then, according to the third equation in (92),
\[ A_{TCP} = 0. \]

This result is to be compared with (37), where \( A_{TCP} \) is mentioned to be equal to \( \delta_S - \delta_L \). We have shown that this is not the case: despite \( \epsilon_L \neq \epsilon_S \), \( A_{TCP} \) vanishes when \( TCP \) is realized. It is easy to trace the root of this mechanism in (114) valid for a TCP invariant Lagrangian.

5.2 Testing TCP?

Testing an eventual violation of \( TCP \) in binary systems of neutral mesons becomes a more and more important concern for both theorists [31] [32] [34] and experimentalists [15] [40] [3] [18] [5] [16].

The measurement of \( \delta_S - \delta_L \) was proved in this work not to be a test of \( TCP \). If it is detected to exceed the estimated value (112), questions would arise, but the last should most probably be whether \( TCP \) symmetry is broken. It is useful to recall that our calculations have been performed supposing the \( \Delta S = \Delta Q \) rule exactly satisfied (see footnote 32), assumption which could of course need a revision.

The non-vanishing of \( A_{TCP} \) stays, at the opposite, a clean test of a violation of \( TCP \).

Other experimental signals like the detection of a slight amount of longitudinal polarization for the emitted photons in \( \pi^0 \rightarrow \gamma \gamma \) decays [32] could provide a test of \( TCP \), according to their description in the usual framework of a local field theory.

The subject of investigating all possible tests of \( TCP \) violation goes anyhow beyond the scope of this work [32] and we shall not comment more on this subject here.

One should also always keep in mind that the logic of introducing explicit \( TCP \) violating parameters in a local field theory can appear questionable since the latter presupposes the former.

6 CONCLUSION

This work is a succession of elementary deductions from basic properties of propagators in QFT. The results are simple and unambiguous.

The main results of the paper can be summarized as follows. Taking the example of neutral kaons, we exhibited substantial differences between the treatments of binary systems of neutral mesons in QM and in QFT. The role of the \( TCP \) symmetry has been clarified, and QM has been shown to yield an improper characterization of this symmetry. An essential role is played by the definition of the physical masses as the poles of the full propagator; this smooths out conceptual problems linked with the existence of “in” and “out” eigenstates, and predicts a difference between the \( CP \) violating parameters of \( K_L \) and \( K_S \), which originates from their mass splitting and is not a characteristic signal of \( TCP \) violation. While the asymmetry \( A_{TCP} \) stays nevertheless a good test of the \( TCP \) symmetry, \( \delta_S - \delta_L \) has been shown to test the difference \( \epsilon_S - \epsilon_L \); the latter can be different from zero even when \( TCP \) is realized.

38 Remember that this estimate was done with a precise phase convention.
We have shown that QM formulæ often quoted in the literature depend on the arbitrary rephasing of $K^0$ and $\bar{K}^0$, and are, hence, not physically relevant; we have given the correct, phase independent, formulæ obtained in QFT.

The introduction of a (constant) unique mass matrix to describe these binary systems is inappropriate. The correct way to diagonalize a general complex propagator is by using a bi-orthogonal basis and not a bi-unitary transformation. Finally, local QFT, which presupposes $TCP$ symmetry, is not an appropriate framework to parameterize $TCP$ violation.

A similar study will be devoted to fermions [29] with a special emphasis on mixing matrices and unitarity.

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A CONSTRAINTS SET BY DISCRETE SYMMETRIES AND LORENTZ INVARIANCE ON THE PROPAGATOR OF NEUTRAL KAONS

We work in the \((K^0, \bar{K}^0)\) basis.

Let \(\varphi_{K^0}(x)\) be the Heisenberg operator for \(K^0\) at space time point \(x = (\vec{x}, t)\) and \(\varphi_{K^0}(\vec{x})\) the corresponding Schrödinger operator (see also subsection A.3.1). Since other fields will be related to it, we shall often omit the corresponding subscript, writing instead \(\varphi\) when it is the only one appearing in a formula.

A.1 DEFINITION OF \(C, P\) and \(CP\) TRANSFORMATIONS ON FIELD OPERATORS

The action of these symmetry transformations is defined for the Schrödinger fields.

A.1.1 Charge conjugation \(C\)

The charge conjugation operation \(C\) transforms a multiplet of an internal symmetry into a multiplet of the complex conjugate representation; we thus define, for field operators

\[ C\varphi_{K^0}(\vec{x}) C^{-1} = e^{-i\alpha} \varphi_{\bar{K}^0}(\vec{x}), \]

(117)

where we have introduced an arbitrary phase \(\alpha\); then, \(\overline{\varphi_{K^0}}\) and \(\varphi_{K^0}^\dagger\) are again connected by an arbitrary phase \(\delta\)

\[ \overline{\varphi_{K^0}(\vec{x})} = e^{-i\delta} \varphi_{K^0}^\dagger(\vec{x}). \]

(118)

such that

\[ C\varphi_{K^0}(\vec{x}) C^{-1} = e^{-i(\alpha+\delta)} \varphi_{K^0}^\dagger(\vec{x}). \]

(119)

\(C\) is a unitary operator \(CC^\dagger = 1 = C^\dagger C\), which entails that \(C^\dagger = C^{-1}\).

We find accordingly

\[ C\varphi_{K^0}^\dagger(\vec{x}) C^{-1} = C\varphi_{K^0}^\dagger(\vec{x}) C^\dagger = (C\varphi_{K^0}(\vec{x}) C^\dagger)^\dagger = e^{i(\alpha+\delta)} \varphi_{K^0}(\vec{x}), \]

(117) and

\[ C\overline{\varphi_{K^0}(\vec{x})} C^{-1} = Ce^{-i\delta} \overline{\varphi_{K^0}(\vec{x})} C^{-1} = e^{-i\delta} C\varphi_{K^0}(\vec{x}) C^{-1} = e^{-i\delta} e^{i(\alpha+\delta)} \varphi_{K^0}(\vec{x}) = e^{i\alpha} \varphi_{K^0}(\vec{x}). \]

It can then immediately be checked that

\[ C^2 = 1. \]

(120)

A.1.2 Parity \(P\)

The parity operator \(P\) is also a unitary operator \(PP^\dagger = 1 = P^\dagger P\); allowing again for an arbitrary phase \(\beta\), it acts on field operators according to

\[ P\varphi_{K^0}(\vec{x}) P^{-1} = e^{i\beta} \varphi_{K^0}(-\vec{x}). \]

(121)

One has then

\[ P\varphi_{K^0}^\dagger(\vec{x}) P^{-1} = e^{-i\beta} \varphi_{K^0}^\dagger(-\vec{x}); \]

(122)

indeed:

\[ P\varphi_{K^0}^\dagger(\vec{x}) P^{-1} = P\varphi_{K^0}^\dagger(\vec{x}) P^\dagger = (P\varphi_{K^0}(\vec{x}) P^\dagger)^\dagger. \]

(122)

In the second equality we have only replaced \(C^{-1}\) by \(C^\dagger\).

See footnote 39.

39In the second equality we have only replaced \(C^{-1}\) by \(C^\dagger\).

40See footnote 39.
Replacing in (122) $\varphi_{K^0}(\vec{x})$ by $e^{i\delta} \varphi_{K^0}(\vec{x})$ according to (117) yields

$$P \varphi_{K^0}(\vec{x}) P^{-1} = e^{-i\beta} \varphi_{K^0}(-\vec{x}).$$

Notice that the product of the phases occurring in (121) and (123) is +1: the relative intrinsic parity of $\varphi_{K^0}$ and $\varphi_{K^0}$ (or $\varphi_{K^0}^\dagger$) is +1.

The operation of parity transformation is a geometrical operation which, repeated twice, should give the identity; one can accordingly impose

$$P^2 = 1.$$  \hspace{1cm} (124)

This entails that $e^{2i\beta} = 1$; since kaons are pseudoscalar, we will choose hereafter

$$e^{i\beta} = -1.$$  \hspace{1cm} (125)

### A.1.3 CP transformation

We then find the laws of transformation by the combined symmetry $CP$; it is instructive that requesting that the two operators $C$ and $P$ commute (or anticommute), $[C, P] = 0$ ($\{C, P\} = 0$) yields the same condition on the phase $\beta$ of the parity transformation as the one given by $P^2 = 1$ \footnote{That $C$ and $P$ commute is indeed not the only possible choice. They can also anticommute, which leads in particular to the so-called Wigner bosons \cite{1}.} . We calculate in two ways $CP \varphi_{K^0}(\vec{x}) (CP)^{-1}$, using the linearity of both operators:

$$CP \varphi_{K^0}(\vec{x}) (CP)^{-1} = C (P \varphi_{K^0}(\vec{x}) P^{-1}) C^{-1} = C (e^{i\beta} \varphi_{K^0}(-\vec{x})) C^{-1} = e^{i\beta} C \varphi_{K^0}(-\vec{x}) C^{-1} = e^{i(\beta-\alpha)} \varphi_{K^0}^\dagger(-\vec{x});$$

and

$$CP \varphi_{K^0}(\vec{x}) (CP)^{-1} = P (C \varphi_{K^0}(\vec{x}) C^{-1}) P^{-1} = P (e^{-i\alpha} \varphi_{K^0}(\vec{x})) P^{-1} = e^{-i\alpha} P \varphi_{K^0}(\vec{x}) P^{-1} = e^{-i(\alpha+\beta)} \varphi_{K^0}^\dagger(-\vec{x}).$$

For these two expressions to be identical, one needs $e^{-i\beta} = e^{i\beta}$, which is the condition obtained in the previous subsection. With our choice (125), we get

$$CP \varphi_{K^0}(\vec{x}) (CP)^{-1} = -e^{-i\alpha} \varphi_{K^0}(-\vec{x}) = -e^{-i(\alpha+\delta)} \varphi_{K^0}^\dagger(-\vec{x}).$$

One then gets

$$CP \varphi_{K^0}^\dagger(\vec{x}) (CP)^{-1} = CP \varphi_{K^0}^\dagger(\vec{x}) (CP) = (CP \varphi_{K^0}(\vec{x}) (CP)^{-1})^\dagger = -e^{i(\alpha+\delta)} \varphi_{K^0}(-\vec{x}),$$

and

$$CP \varphi_{K^0}(\vec{x}) (CP)^{-1} = -e^{i\alpha} \varphi_{K^0}(-\vec{x}).$$

One has $(CP)^2 = 1$.

### A.2 THE NEUTRAL KAON PROPAGATOR

The notations and definition of the neutral kaon propagator have been given in subsection 2.1. We do not repeat them here.
A.3 CONSTRAINTS SET BY CP SYMMETRY ON THE PROPAGATOR OF NEUTRAL KAONS

A.3.1 Constraint set by CP symmetry on the anti-diagonal elements

Using (126), let us investigate the consequences of CP invariance on the propagator. Since CP is unitary, the following is an identity

\[- g(\vec{x}, t) = e^{i\delta} < CP 0 \mid \vartheta(t)CP\varphi(\vec{x}, t/2)(CP)^{-1}\varphi(\vec{x}, t/2) > CP 0 > . \]

The theory is invariant by CP if and only if \[ [CP, H] = 0, \] \( H \) being the Hamiltonian. The Heisenberg field \( \varphi(\vec{x}, t/2) \) can be expressed in term of the Schrödinger field \( \varphi(\vec{x}) \) by

\[ \varphi(\vec{x}, t/2) = e^{i\frac{t}{2}H} \varphi(\vec{x}) e^{-i\frac{t}{2}H}, \]

such that CP invariance yields

\[ CP\varphi_{K0}(\vec{x}, t/2)(CP)^{-1} = e^{i\frac{t}{2}H} CP\varphi_{K0}(\vec{x}) e^{-i\frac{t}{2}H}, \]

and, using (126),

\[ CP\varphi_{K0}(\vec{x}, t/2)(CP)^{-1} = -e^{-i(\alpha + \delta)} \varphi_{K0}^\dagger(\vec{x}, t/2); \]

the Heisenberg field transforms in the same way as the Schrödinger field.

Using the invariance of the vacuum and (131), the starting identity becomes

\[- g(\vec{x}, t) = e^{i\delta} < CP 0 \mid \vartheta(-t) e^{-i(\alpha + \delta)} \varphi(\vec{x}, t/2) e^{-i(\alpha + \delta)} \varphi_{K0}^\dagger(\vec{x}, t/2) > CP 0 > = e^{-2i\alpha}(-h(-\vec{x}, t)). \]

Lorentz invariance of the propagator (6), in the sense discussed in subsection 2.1, entails \( h(-\vec{x}, t) = h(\vec{x}, t) \), and one gets

\[ g(\vec{x}, t) = e^{-2i\alpha} h(\vec{x}, t). \]

A.3.2 Constraint set by CP symmetry on the diagonal elements

Going along similar lines, one gets:

\[ CP \varphi(\vec{x}, t) = < 0 \mid \vartheta(t) e^{-i(\alpha + \delta)} \varphi(\vec{x}, t/2)(CP)^{-1} e^{i(\alpha + \delta)} \varphi(\vec{x}, t/2) > | 0 >= f(-\vec{x}, t). \]

The phases cancel, and Lorentz invariance of the propagator (6), in the sense discussed in subsection 2.1, entails \( f(-\vec{x}, t) = f(\vec{x}, t) \); finally we obtain

\[ CP \varphi(\vec{x}, t) = f(\vec{x}, t). \]
A.3.3 Final remarks on CP

The diagonal elements of a CP invariant kaon propagator are identical, and its anti-diagonal elements are identical up to a phase $\alpha$.

Accordingly, a CP invariant kaon propagator is always normal; it is equivalent to saying that a non-normal propagator cannot describe a CP invariant theory; but this leaves the freedom for a normal mass matrix to also accommodate for CP violation. This is emphasized in the core of the paper.

Notice that proper Lorentz invariance enabled us to transform the $(-\vec{x}, t)$ dependence in the propagator into a $(\vec{x}, t)$ dependence without making any hypothesis concerning the link between $\varphi(-\vec{x}, t)$ and $\varphi(\vec{x}, t)$.

A.4 THE WIGHTMAN CONVENTION FOR TCP TRANSFORMATION

The TCP transformation $\Theta$ exists independently of the three individual transformations $P$, $C$ and $T$.

One defines it on Heisenberg fields because it also concerns time evolution for operators and eigenstates.

While phases can appear in the individual transformations $P$, $C$ (and $T$), there is no arbitrary phase in $\Theta$.

$\Theta$ satisfies

$$\Theta = \Theta^{-1}. \quad (136)$$

It is an antiunitary operator:

$$< \Theta A | \Theta B > = < A | B >^* = < B | A >. \quad (137)$$

One deduces in particular from (137), for any operator $\mathcal{O}(\vec{x}, t)$

$$< \Theta A | \Theta \mathcal{O}(\vec{x}, t) \Theta^{-1} | \Theta B > = < B | \mathcal{O}^\dagger(\vec{x}, t) | A >. \quad (138)$$

Indeed:

$$< \Theta A | \Theta \mathcal{O}(\vec{x}, t) \Theta^{-1} | \Theta B > = < \Theta A | \Theta \mathcal{O}(\vec{x}, t) | B > = < \Theta A | \Theta (\mathcal{O}(\vec{x}, t) B) >$$

$$= < \mathcal{O}(\vec{x}, t) B | A > = < B | \mathcal{O}^\dagger(\vec{x}, t) | A >. \quad (138)$$

$\Theta$ is an antilinear operator: it complex conjugates all c-numbers on its right

$$\Theta(a | A >) = a^* \Theta | A >. \quad (139)$$

(139) can easily be obtained from (138) by replacing the operator $\mathcal{O}$ by the c-number $a$.

By a TCP transformation, in addition to the 4-inversion $(\vec{x}, t) \to (-\vec{x}, -t)$, any operator should be changed into its hermitian conjugate.

$$\Theta \varphi(\vec{x}, t) \Theta^{-1} = \varphi^\dagger(-\vec{x}, -t). \quad (140)$$

In this work, we consider the existence of an antiunitary operator $\Theta$ satisfying (140) as the criterion for TCP invariance.

A.4.1 Constraint linked to TCP symmetry on the diagonal elements

By definition (see (5))

$$d(\vec{x}, t) = < K^0 | \Delta(\vec{x}, t) | K^0 > = < 0 | T\{\varphi(\vec{x}, t/2)\varphi^\dagger(-\vec{x}/2, -t/2)\} | 0 >$$

$$= < 0 | \varphi(\vec{x}, t/2)\varphi^\dagger(-\vec{x}/2, -t/2) + \varphi(\vec{x}, t/2)\varphi^\dagger(-\vec{x}/2, -t/2)\varphi(\vec{x}, t/2) | 0 >. \quad (141)$$

42This because $(A \Rightarrow B)$ entails $(B \Rightarrow A)$, but does not entail $B \Rightarrow A$, which is a wrong statement: if $B$ is true, $A$ can be either true or false.
The vacuum is supposed to be unique and invariant by TCP: \( \Theta |0> = |0> \); one replaces accordingly |0> by |\( \Theta 0> \); one then uses (140) to replace \( \varphi(\vec{x}, \frac{t}{2}) \) by \( \Theta^{-1}\varphi^\dagger(\vec{x}, -\frac{t}{2})\Theta \), which is identical to \( \Theta\varphi^\dagger(-\vec{x}, -\frac{t}{2})\Theta^{-1} \) because of (136), and one gets

\[
d(\vec{x}, t) = < \Theta 0 | \vartheta(t)\varphi^\dagger\Theta^\dagger(-\vec{x}, \frac{t}{2})\Theta^{-1} + \vartheta(-t)\varphi(-\vec{x}, \frac{t}{2})\varphi^\dagger(-\vec{x}, -\frac{t}{2})\Theta^{-1} | \Theta 0 >.
\]

(142)

One now uses the antiunitarity (138) of the \( \Theta \) operator to get, using (5)

\[
d(\vec{x}, t) = < 0 | \vartheta(t)\varphi^\dagger\Theta^\dagger(-\vec{x}, \frac{t}{2})\varphi(-\vec{x}, \frac{t}{2}) + \vartheta(-t)\varphi(-\vec{x}, \frac{t}{2})\varphi^\dagger(-\vec{x}, -\frac{t}{2})\Theta^{-1} | 0 > = f(\vec{x}, t).
\]

Accordingly, the sole existence of a antiunitary operator \( \Theta = \Theta^{-1} \) such that any complex scalar field transforms according to (140), and of the unicity and invariance of the vacuum by \( \Theta \) entail

\[
d(\vec{x}, t) = f(\vec{x}, t).
\]

(143)

A.4.2 TCP symmetry and the anti-diagonal elements

By definition (see (5))

\[
-h(\vec{x}, t) = e^{-i\delta} < 0 | \vartheta(t)\varphi^\dagger\Theta^\dagger(-\vec{x}, \frac{t}{2})\varphi(-\vec{x}, \frac{t}{2}) + \vartheta(-t)\varphi(-\vec{x}, \frac{t}{2})\varphi^\dagger(-\vec{x}, -\frac{t}{2})\Theta^{-1} | 0 >.
\]

(144)

The vacuum is supposed to be unique and invariant by TCP: \( \Theta |0> = |0> \); using (140), one gets by definition, along the same lines as for the diagonal elements, 44

\[
-h(\vec{x}, t) = e^{-i\delta} < \Theta 0 | \vartheta(t)\varphi(-\vec{x}, \frac{t}{2})\Theta^{-1}\varphi^\dagger\Theta^\dagger(-\vec{x}, \frac{t}{2})\Theta^{-1} + \vartheta(-t)\varphi(-\vec{x}, \frac{t}{2})\varphi^\dagger(-\vec{x}, -\frac{t}{2})\Theta^{-1} | \Theta 0 >.
\]

(145)

One now uses the antiunitarity of the \( \Theta \) operator (138) to get

\[
-h(\vec{x}, t) = e^{-i\delta} < 0 | \vartheta(t)\varphi^\dagger\Theta^\dagger(-\vec{x}, \frac{t}{2})\varphi(-\vec{x}, \frac{t}{2}) + \vartheta(-t)\varphi(-\vec{x}, \frac{t}{2})\varphi^\dagger(-\vec{x}, -\frac{t}{2})\Theta^{-1} | 0 > = -h(\vec{x}, t).
\]

(146)

We only get a tautology: TCP sets no constraint on the antidiagonal elements of the propagator.

A.5 THE SCHWINGER-PAULI CONVENTION FOR TCP TRANSFORMATION 2 36 24

In the Schwinger-Pauli convention, transforming a product of operators by TCP goes by performing the 4-inversion \((\vec{x}, t) \rightarrow (-\vec{x}, -t)\), not taking neither the hermitian nor the complex conjugate of operators, but reading all expressions from right to left instead of from left to right (this last prescription swaps in particular “in” and “out” states).

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41One does the same for the second operator

42In the equalities below, one cannot cancel the two \( \Theta \)'s in \(< \Theta 0 | \Theta \varphi \ldots \) since it is equal to \(< 0 | \Theta^\dagger \Theta \varphi \ldots \), and \( \Theta \) is not a unitary operator, \( \Theta^\dagger \Theta \neq 1 \).
When evaluating a scalar product, with no “sandwiched” operator, this convention is identical to the condition (137) of antiunitarity of $\Theta$; indeed, it yields $<\Theta A | \Theta B>^{Schwinger-Pauli} = < B | A > \equiv < A | B >$. However, when a string of operators is sandwiched between the two state vectors, one gets a result different from the Wightman convention; they only coincide for hermitian operators:

$$<\Theta A | O_1 O_2 \ldots O_n | \Theta B>^{Schwinger-Pauli} = < B | O_n \ldots O_2 O_1 | A >$$

while it would give $< B | O_1^{\dagger} O_2^{\dagger} \ldots O_n^{\dagger} | A >$ in the Wightman convention (see (138)). Since (138) is a direct consequence of the antiunitarity of the $\Theta$ operator, the differences between the two conventions are deep and one cannot even speak of a true antiunitary $\Theta$ for Schwinger.

The case of “sandwiched” scalars needs an investigation. If one applies the rule of simply inverting the order of all factors in the matrix element, one gets $\Theta ( < A | a | B > ) = < B | a | A > \equiv a < B | A >$.

If one instead sticks to the antiunitarity of $\Theta$, the c-number $a$ should be complex conjugated (see (139)). Since we already saw above that $\Theta$ cannot be considered here as an antiunitary operator, we shall not conjugate the c-numbers and show that this leads to a consistent result.

A caveat also exists: Pauli [36] always works with completely symmetrized strings of operators, which is not our case here.

### A.5.1 Constraint set by TCP symmetry on the diagonal elements

Since “in” and “out” states are both the vacuum, supposed invariant by TCP, the transformed by TCP of $d(x, t)$ (141) is

$$\Theta d(x, t) = f(x, t).$$

Indeed, $\Theta d(x, t) = < 0 | \vartheta(t) \varphi^{\dagger}(\frac{x}{2}, \frac{t}{2}) \varphi(-\frac{x}{2}, -\frac{t}{2}) + \vartheta(-t) \varphi(-\frac{x}{2}, -\frac{t}{2}) \varphi^{\dagger}(\frac{x}{2}, \frac{t}{2}) | 0 > = f(x, t)$.

A TCP transformation swaps the two diagonal elements; TCP invariance requires accordingly their equality.

### A.5.2 Constraint set by TCP symmetry on the anti-diagonal elements

The transformed by TCP of $-h(x, t)$ (144) is, when the phase is not transformed —see the discussion above—

$$\Theta h(x, t) = h(x, t).$$

Indeed, $\Theta(-h(x, t)) = e^{-i\delta} < 0 | \vartheta(t) \varphi^{\dagger}(\frac{x}{2}, \frac{t}{2}) \varphi(-\frac{x}{2}, -\frac{t}{2}) + \vartheta(-t) \varphi(-\frac{x}{2}, -\frac{t}{2}) \varphi^{\dagger}(\frac{x}{2}, \frac{t}{2}) | 0 > = -h(x, t)$, and TCP does not set any constraint on the antidiagonal elements of the propagator.

If one does conjugate the phase, the relation becomes $\Theta h(x, t) = e^{2i\delta} h(x, t)$, which is not consistent with what we obtained using the Wightman’s convention.

### A.5.3 Comments

In the Schwinger-Pauli convention for TCP-transforming a string of operators, TCP invariance constrains the two diagonal elements to be identical and gives no constraint on the antidiagonal elements. This is only achieved when c-numbers are left untouched by the transformation.

The operator $\Theta$ then does not appear as an antiunitary operator; nevertheless we get the same constraints as with the convention of Wightman.
A.6 FINAL REMARKS ON TCP

A.6.1 Constraints on the propagator

Both conventions lead us to the same constraints on the propagator: its diagonal elements must be identical, while no constraint exists on the antidiagonal elements. The propagator is not constrained to be normal: depending whether \(|g| = |h|\) or not, it can or cannot be so. In the core of the paper, we show that normality cannot be satisfied because it always leads to a purely imaginary \(CP\) violating parameter \(\epsilon\).

The Schwinger-Pauli’s convention has been claimed \(^{30}\) to be valid independently of the hermiticity of the Lagrangian, which is of concern to us here. It only coincides with the one of Wightman when dealing with hermitian operators, but they deeply differ for other cases: the \(TCP\) operator \(\Theta\) is in particular not truly antiunitary, nor truly antilinear in the Schwinger-Pauli’s convention \(^{45}\).

It is important to notice that the non-hermitian Lagrangian that one is led to introduce because of kaon instability can only be considered as an effective Lagrangian. A fundamental Lagrangian should include not only kaons but all its decay products, and should be hermitian; then Schwinger-Pauli and Wightman conventions coincide.

We develop the comparison between the two approaches in the next subsection.

A.6.2 Lagrangian versus Green’s functions

We have chosen to study the constraints set by \(TCP\) on the two-point Green function. It is a general theorem \(^{42}\) that one can reconstruct the \(S\) matrix of a theory from the (infinite) set of its Green functions, which goes beyond any perturbative approach based on a given Lagrangian.

We show below what our results mean in a Lagrangian approach. A general quadratic Lagrangian for \(K^0\) and \(\bar{K}^0\) writes

\[
L(\vec{x}, t) = a \varphi_{K^0}(\vec{x}, t)\varphi_{\bar{K}^0}(\vec{x}, t) + d \varphi_{K^0}(\vec{x}, t)\varphi_{\bar{K}^0}(\vec{x}, t) + b(\varphi_{K^0}(\vec{x}, t))^2 + c(\varphi_{\bar{K}^0}(\vec{x}, t))^2
\]

\[
= a \varphi_{K^0}(\vec{x}, t)e^{-i\delta}\varphi_{\bar{K}^0}(\vec{x}, t) + d e^{-i\delta} \varphi_{K^0}(\vec{x}, t)\varphi_{\bar{K}^0}(\vec{x}, t)
\]

\[+ b(\varphi_{K^0}(\vec{x}, t))^2 + c e^{-2i\delta}(\varphi_{\bar{K}^0}(\vec{x}, t))^2. \tag{150}\]

If one uses the Wightman convention for \(TCP\) transformation (which includes complex conjugating the c-numbers) one gets

\[
\Theta L(\vec{x}, t)\Theta^{-1} = a^* e^{i\delta} \varphi_{K^0}(-\vec{x}, -t)\varphi_{\bar{K}^0}(-\vec{x}, -t) + d^* e^{i\delta} \varphi_{K^0}(-\vec{x}, -t)\varphi_{\bar{K}^0}(-\vec{x}, -t)
\]

\[+ b^*(\varphi_{K^0}(-\vec{x}, -t))^2 + c^* e^{2i\delta}(\varphi_{\bar{K}^0}(-\vec{x}, -t))^2; \tag{151}\]

if one uses instead the Schwinger-Pauli convention (with no conjugation of c-numbers), one gets

\[
\Theta L(\vec{x}, t)\Theta^{-1} = ae^{-i\delta} \varphi_{K^0}(-\vec{x}, -t)\varphi_{\bar{K}^0}(-\vec{x}, -t) + de^{-i\delta} \varphi_{K^0}(-\vec{x}, -t)\varphi_{\bar{K}^0}(-\vec{x}, -t)
\]

\[+ b(\varphi_{K^0}(-\vec{x}, -t))^2 + c e^{-2i\delta}(\varphi_{\bar{K}^0}(-\vec{x}, -t))^2, \tag{152}\]

such that, supposing that the integration \(\int d^4x\) wipes out the change of \((\vec{x}, t)\) into \((-\vec{x}, -t)\), the conditions that \(L\) is invariant by \(TCP\) are:

- in the Wightman’s convention: \(ae^{-i\delta} = a^* e^{i\delta}\), \(de^{-i\delta} = d^* e^{i\delta}\), \(b = c^* e^{2i\delta}\);
- in the Schwinger-Pauli convention: \(a = d\), no condition on \(b\) and \(c\): these are the same results as the

\(^{30}\)It is to be noted that the demonstration that an operator should be either unitary or antiunitary rests (see \[^{43}\], vol.1, appendix A p.91) on the conservation of probabilities and on the existence of a complete orthogonal set of state-vectors for rays of the group of transformation under concern. In the case under study neutral kaons are unstable and we introduced a non-hermitian “effective” Lagrangian; hence, one may question the conservation of probabilities; furthermore, a complete set of eigenstates involves at most one true propagating state since, at each of the physical poles, a complete set of eigenstates involves the propagating (physical) state, and a spurious one; the two propagating states correspond to two different \(p^2\) and do not form, truly speaking, a complete orthogonal set. A detailed re-examination of the demonstration in our case is left for a further study.
ones that we have obtained for the propagator (which, at the lowest order, is the inverse of the quadratic Lagrangian)\textsuperscript{46}.

For the Lagrangian to be hermitian, \(a, b, c\) and \(d\) must satisfy \(ae^{-i\delta}\) real, \(de^{-i\delta}\) real, \(b^* = ce^{-2i\delta}\); these coincide with the TCP constraints that we obtained above by using the conventions of Wightman\textsuperscript{47}. What matters then is the sum \((a + d)\), as far as \(\varphi_{K^0}(x)\) and \(\varphi_{K^0}^\dagger(y)\) commute at the same space-time point \(x = y\)\textsuperscript{48}.

We conclude that:
- the Schwinger-Pauli convention always gives constraints for the propagator and for the Lagrangian which are compatible;
- in the case when the instability of particles forces us to use a non-hermitian Lagrangian, Wightman’s convention for the propagator and for the Lagrangian conflict.

So, as reported in Pauli’s paper\textsuperscript{36}, Schwinger’s convention for TCP is likely to apply to non-hermitian Lagrangians as well, and looks more general than Wightman’s.

When dealing with non-hermitian Lagrangians, it seems thus recommended:
- either to work with Green’s functions only; one can then keep a antiunitary \(\Theta\) operator and Wightman’s convention;
- or to use Schwinger-Pauli convention for TCP which, in particular, brings no conflict between Lagrangian and propagator; one has then to give up the antiunitarity of the \(\Theta\) operator.

\section{CP Violating Parameters and Arbitrary Phases}

This appendix is a complement to subsection\textsuperscript{3.2.4}.

First, on Fig. 2 are displayed the two ellipsoids of \(\epsilon_L^{\text{in}}\) and \(\epsilon_L^{\text{out}}\) when the arbitrary phase \(\Omega_1\) in (72) is varied.

\begin{center}
\includegraphics[width=0.5\textwidth]{ellipsoids.png}
\end{center}

\textit{Fig. 2:} \(\epsilon_L^{\text{in}}\) and \(\epsilon_L^{\text{out}}\) span symmetric ellipsoids when the phase \(\omega\) of \(K^0\) is varied.

\textsuperscript{46}When the arbitrary phase \(\delta\) is put to zero, these constraints become:
- in the Wightman’s convention: \(a\) and \(d\) real, \(b = c^*\);
- in the Schwinger-Pauli convention: \(a = d\) (equality of the masses of \(K^0\) and \(\bar{K}^0 \equiv (K^0)^\dagger\)) and no condition on \(b\) and \(c\).

\textsuperscript{47}Since the Lagrangian involves operators at the same point and no T-product is involved, the TCP transformation amounts, in addition to 4-inversion, to simple hermitian conjugation.

\textsuperscript{48}This gives back, at the Lagrangian level, the equality between the mass of a particle and the one of the corresponding antiparticle.
The coordinates of their intersections with the real axis have the same moduli as the lower and upper bounds \(|\epsilon_{\text{in}}|\) and \(|\epsilon_{\text{out}}|\). The length of their “real” axis of the ellipsoids is \(4\sqrt{|b_1c_1|}\) and the length of their “imaginary” axis is very close to 2. (Fig. 2 does not respect the scale: the lower bound of \(\epsilon\) is much smaller than the length of the “imaginary” axis, itself much smaller that the length of the “real” axis).

By the same inspection as done in subsection 3.2.4, one finds

\[
\begin{align*}
|\epsilon_{\text{in}}|^2 &= \left| \frac{b(z_1) - |c(z_1)|}{b(z_1) + |c(z_1)|} - 2i \sqrt{|b(z_1)c(z_1)|} \sin \Omega_1 \right|^2; \\
|\epsilon_{\text{out}}|^2 &= \left| \frac{c(z_1) - |b(z_1)|}{b(z_1) + |c(z_1)|} - 2i \sqrt{|b(z_1)c(z_1)|} \sin \Omega_1 \right|^2; \\
|\epsilon_{\text{in}}|^2 &= \left| \frac{|b_1| + |c_1| + 2 \sqrt{|b_1c_1|} \cos \Omega_1}{|b_1| + |c_1| - 2 \sqrt{|b_1c_1|} \cos \Omega_1} \right|^2.
\end{align*}
\]

Comparing (153) with (73), one sees that the \(CP\) violating parameters for the “spurious” states can be deduced from the ones of the propagating states by the change

\[
\Omega_1 \rightarrow \Omega_1 \pm \pi,
\]

that is, by

\[
\omega \rightarrow \omega \pm \frac{\pi}{2},
\]

which is also true for \(K_S\) and \(\bar{K}_S\).

So, the ellipsoids corresponding to the “spurious states” are globally the same as the ones of the propagating states. The value of \(\Omega_1\) corresponding to the lower bound of \(|\epsilon|\) in one case, \(\Omega_1 = 0\) or \(\Omega_1 = \pi\), corresponds to the upper bound in the other case.

The ellipsoids corresponding to \(K_S\) are shifted with respect to the ones for \(K_L\) according to the transformations \(b(z_1) \rightarrow b(z_2)\) and \(c(z_1) \rightarrow c(z_2)\), shift expected to be very small; they have the same symmetry properties.

Since by (155), \(\epsilon_{\text{in}}\) and \(\epsilon_{\text{out}}\) are turned into \(\epsilon_{\text{in}}\) and \(\epsilon_{\text{out}}\), the propagating \(K_L\) states are formally transformed by this rephasing into the “spurious” states \(\tilde{K}_L\), and \textit{vice versa}. The same remark applies to \(K_S\) and \(\bar{K}_S\).

This is an additional argument for the importance of both types of states, and that discarding \textit{a priori} the spurious states is unjustified.
List of Figures

Fig. 1: Feynman diagrams for the transition $K^0 \rightarrow K^0$;
Fig. 2: $\epsilon^\text{in}_L$ and $\epsilon^\text{out}_L$ span symmetric ellipsoids when the phase $\omega$ of $K^0$ is varied.
References


[5] B. AUBERT et al. (BaBar Collaboration): “Limits on the Decay-Rate of Neutral B Mesons and on CP, T and CPT Violation in B^0\bar{B}^0 Oscillations”, hep-ex/0311037.


[14] See for example:


ibidem: “Covariance of time-ordered products implies local commutativity of fields”, [hep-th/0405211]


[27] see for example:


[30] See for example:


[38] See for example:


