Complex Scaled Spectrum Completeness for Pedestrians

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The Complex Scaling Method (CSM) is rephrased in terms of a Hamiltonian containing an optical potential. Traditional theorems concerning Jost solutions, the Jost function, residues of a suitable Green's function at its poles and the resolution of the identity are extended. Elementary derivations of the completeness of the CSM spectrum are detailed for the one and the two channel cases.

I. INTRODUCTION

The CSM is a well-known method to convert the description of resonances by non-integrable Gamow states into one by square integrable states. This regularization does not modify the discrete spectrum [1], at the cost, however, of rotations of the cuts describing the continuum. Furthermore a *resolution of the identity* may become available, that sorts out the contribution of resonances [2]. This is of importance for many practical calculations, in nuclear physics for instance [3]. In any practical calculation such as a sum rule, a suitable subtraction of resonant contributions reduces the continuum ones to a smoother background, easier to estimate numerically or modelize [4].

The question of spectral completeness, however, remains partly open. The case of coupled channels with different thresholds, in particular, demands an investigation of specific integration contours of a suitable Green's function through a complicated Riemann surface [5]. For the simplest and most frequent case of multichannel resonant coupling, that of just *two* channels, the complication of the Riemann surface is not excessive, though. Such is the main motivation of this paper, and we will describe a very simple representation, which we call the *P*-representation, where the completeness can be proved, and operated practically.

Incidentally, before even considering several channels, the one channel case study may need distortions of integration contours and even the regularization of divergent integrals [2]. Furthermore, while the CSM is rather familiar to atomic physicists [6] [7], it is much less known to nuclear physicists, if only because the analytical continuation of nuclear potentials is much less easy than that of the Coulomb one. There is thus a need for a familiarization of nuclear physicists with the CSM. We also feel that, because of traps which are likely when non Hermitian Hamiltonians are involved, a “pedestrian” formulation of the theory is in order. A purpose of the present lectures is thus to visit again the CSM and reformulate it in terms as simple as possible. For this we rely heavily on the theorems proven by [1] and on the classic paper of [8]. The steps followed and proofs offered by [8] for the study of radial eigenfunctions are borrowed, to study spectral properties of that slightly modified CSM Hamiltonian,

$$H = p^2 + e^{2i\theta} V(e^{i\theta} r). \quad (1)$$

This operator H is just the usual CSM Hamiltonian multiplied by $e^{2i\theta}$. It describes the same physics, as far as resonances and bound states are concerned. But, as will be used at length in this paper, it has a serious advantage in terms of representations of scattering: the continuum is not rotated any more, scattering occurs at real, physical energies (added to a complex threshold if necessary), and H is actually quite similar to a familiar optical Hamiltonian. Conversely, however, according to [1], the discrete spectrum of H now rotates by $e^{2i\theta}$ in the complex energy plane, or, equivalently, by $e^{i\theta}$ in the complex momentum plane.

It is understood in the following that $V(r)$ is a local, real potential, with a fast decrease when $r \rightarrow \infty$. (This rate of decrease will be discussed in time in the following.) Also, naturally, we are investigating radial equations and wave functions. The coordinate r is a real number, with $0 \leq r$. And, for the sake of pedagogy, we find it convenient to first consider the *s*-wave radial equation only,

$$(k^2 + d^2/dr^2) f(k, r) = U(r) f(k, r), \quad (2)$$

where $U(r) \equiv e^{2i\theta} V(e^{i\theta}r)$, $0 < \theta < \pi/4$. We assume that the analytic continuation of $V(r)$ into $U(r)$ experiences no singularity and that again a “fast” decrease of $U(r)$ occurs when $r \rightarrow \infty$. For the sake of definiteness, and for applications to nuclear physics, we can for instance restrict V to be a finite sum of Gaussians (FSG). A simplest case combines an attractive, short range Gaussian with a repulsive, longer range one, $V = -v_s \exp[-(r/a_s)^2] + v_l \exp[-(r/a_l)^2]$. Obviously, upon complex scaling, any resulting exponential of the form $\exp[-r^2(\cos 2\theta + i \sin 2\theta)/a^2]$ still makes a (very!) fast decreasing potential as long as $\theta < \pi/4$.

For the sake of pedagogy we recall in Section II, for the one channel case, well known properties of the Jost solution, of the associated Jost function, of the residues of a suitable Green’s function at zeroes of the Jost function, of the associated normalized “projectors” (actually non Hermitian but symmetric and idempotent kernels) upon bound states and resonances. And finally Section II visits again a derivation of a resolution of the identity. The reader who is already familiar with the language and tools of Jost functions can skip directly to Section III (and consider the Appendixes). For the two channel case, Section III shows, with elementary arguments, how a resolution of the identity is still available. Finally the last Section, Section IV, discusses needed generalizations of the results of Secs.II-III.

II. ONE CHANNEL JOST SOLUTION AND ALL THAT; COMPLETENESS

A. Existence and derivabilities of the Jost solution

The definition of the Jost solution of Eq.(2) stems from the following integral equation,

$$f(k, r) = \exp(ikr) + k^{-1} \int_r^\infty dr' \sin[k(r' - r)] U(r') f(k, r'), \quad (3)$$

which specifies boundary conditions for a solution of Eq.(2). For the derivability and even analyticity of f with respect to r , one is also concerned with the equation for the derivative,

$$f'_r(k, r) = ik \exp(ikr) - \int_r^\infty dr' \cos[k(r' - r)] U(r') f(k, r'). \quad (4)$$

Existence of f with a suitable property such as continuity, existence of a solution f'_r of Eq.(4), proof that f'_r is indeed the derivative of f , all such properties derive from “uniform convergence properties” for both Eqs.(3,4). Thus one can take advantage of several fundamental theorems of calculus, which can be summarized as follows: i) if a series of continuous functions f_n converges uniformly, then the sum f of the series is continuous, and ii) if the series of the derivatives f'_n of the f_n ’s also converges uniformly, then the sum f' is indeed the derivative of f . (It will turn out that f is an analytical function of r , hence the existence of as many derivatives as needed, but we will not need to extend r to complex values.) We only sketch such a majoration argument for f , since all details can be adapted from [8], and the argument for f'_r is quite similar. The majoration for f obtains from iterating Eq.(3),

$$f = \sum_{n=0}^\infty f_n, \quad f_0(k, r) = \exp(ikr), \quad f_n(k, r) = k^{-1} \int_r^\infty dr' \sin[k(r' - r)] U(r') f_{n-1}(k, r'), \quad n > 0. \quad (5)$$

Accordingly one finds,

$$k^n f_n(k, r) = \int_r^\infty dr_1 \int_{r_1}^\infty dr_2 \dots \int_{r_{n-1}}^\infty dr_n \sin[k(r_1 - r)] U(r_1) \sin[k(r_2 - r_1)] U(r_2) \dots \sin[k(r_n - r_{n-1})] U(r_n) f_0(k, r_n). \quad (6)$$

Let $\eta \equiv \Im k$ be the imaginary part of k . The following obvious properties generate useful bounds for f (and f'_r),

$$|\sin[k(r' - r)]| \leq \exp[|\eta|(r' - r)] \text{ if } r' > r, \quad |\cos[k(r' - r)]| \leq \exp[|\eta|(r' - r)] \text{ if } r' > r, \quad |f_0(k, r)| = \exp(-\eta r). \quad (7)$$

When $\eta \geq 0$, hence $|\eta| = \eta$, one obtains from Eq.(6) the majoration,

$$|k^n f_n(k, r)| \leq \int_r^\infty dr_1 \int_{r_1}^\infty dr_2 \dots \int_{r_{n-1}}^\infty dr_n \exp[\eta(r_1 - r + r_2 - r_1 + \dots + r_n - r_{n-1})] |U(r_1)U(r_2)\dots U(r_n)| \exp(-\eta r_n), \quad (8)$$

The obvious chain of compensations between the exponentials reduces this into,

$$|f_n(k, r)| \leq \exp(-\eta r) |k|^{-n} \int_r^\infty dr_1 \int_{r_1}^\infty dr_2 \dots \int_{r_{n-1}}^\infty dr_n |U(r_1)| |U(r_2)| \dots |U(r_n)|. \quad (9)$$

Because of the $n!$ possible orderings of $r_1, r_2, \dots, r_n$ this also reads,

$$|f_n(k, r)| \leq \exp(-\eta r) (n! |k|^n)^{-1} \left[\int_r^\infty dr' |U(r')| \right]^n, \quad (10)$$

hence the majoration,

$$|f(k, r) - f_0(k, r)| \leq \exp(-\eta r) \left\{ \exp \left[|k|^{-1} \int_r^\infty dr' |U(r')| \right] - 1 \right\}. \quad (11)$$

This allows a uniform majoration of the left-hand side by a number, such as $\exp[\varepsilon^{-1} \int_0^\infty dr' |U(r')|]$, as long as, simultaneously, i) k stays out of a circle with any small radius $\varepsilon > 0$, and ii) there exists a finite integral $\int_0^\infty dr' |U(r')|$. In practice, for a large class of potentials, and in particular for the FSG potentials, the majoration, Eq.(11), proves the existence of f for every real $r \geq 0$ and every k such as $\Im k \geq 0$, except maybe for $k = 0$.

When $\eta < 0$, hence $|\eta| - \eta = 2|\eta|$, the exponentials giving an upper bound for the sines read $\exp[-\eta(r_{i+1} - r_i)]$ and, with still a factor $\exp(-\eta r_n)$ for $|f_0|$, the chain of compensations is slightly less complete. One still finds a majoration,

$$|f(k, r) - f_0(k, r)| \leq \exp(\eta r) \left\{ \exp \left[|k|^{-1} \int_r^\infty dr' |U(r')| \exp(2|\eta|r') \right] - 1 \right\}. \quad (12)$$

The very fast decay of a FSG potential thus allows the existence of f for any k , except maybe $k = 0$. In the following, we will not mention again that an exceptional accident may remain for $k = 0$. This exception will be understood.

It is clear from Eq.(11) and Eq.(12) that, when $|k|$ becomes infinite, a uniform limit $f \rightarrow f_0$ exists and is reached with an exponentially diminishing error. The same limit $f \rightarrow f_0$ holds when $r \rightarrow \infty$, if only because the integrals, $\int_r^\infty$, that drive our majorations, obviously vanish when $r \rightarrow \infty$.

Similar conclusions then hold for f'_r , namely domain of existence, continuity, limits when $r \rightarrow \infty$ or $|k| \rightarrow \infty$, exponentially diminishing errors in such limiting processes. Because of the uniform convergence of its constructing series $\sum f'_{rn}$, the function f'_r is indeed the derivative of f . Hence the existence of derivatives at all orders and analyticity of f . One can proceed to the second derivative, with derivatives under the integration symbol, and trivially show that f is that (unique) solution of the Schrödinger equation which behaves as $\exp(ikr)$ when $r \rightarrow \infty$.

B. Regular solutions

The following equation,

$$\phi(k, r) = \sin(kr) + k^{-1} \int_0^r dr' \sin[k(r - r')] U(r') \phi(k, r'), \quad (13)$$

defines another solution of the Schrödinger equation, with the following properties, i) ϕ is regular, as it behaves like kr when $r \rightarrow 0$, and ii) ϕ is an odd function of k . When $r \rightarrow 0$, it will be noticed that, provided $|U(r)|$ remains finite or increases more slowly than r^{-2} , then the modulus of the integral weight, $|k^{-1} \sin[k(r - r')] U(r')|$, vanishes or at worst increases more slowly than r^{-1} . Its multiplication by ϕ , which is assumed of order r , thus makes an integrand which vanishes at the origin. The integral itself must then vanish faster than r . This proves that the ansatz, $\phi \rightarrow \sin(kr)$ when $r \rightarrow 0$, is consistent.

Bounds to prove the uniform convergence of the corresponding series, $\phi = \sum_{n=0}^\infty \phi_n$, are even simpler to obtain than those for f . Indeed, a trivial majoration yields,

$$|\phi(k, r) - \sin(kr)| \leq \exp(|\eta|r) \left\{ \exp \left[|k|^{-1} \int_0^r dr' |U(r')| \right] - 1 \right\}. \quad (14)$$

The resulting uniform majoration and convergence hold whether $\eta \equiv \Im k$ is positive or negative. This defines ϕ as an entire function in the whole k -plane. Similar considerations are easy to prove for the derivative $\phi'_r(k, r)$ and, in

particular, the value $\phi'_r(k, 0) = k$. The asymptotic form $\phi(k, r) \rightarrow \sin(kr)$ when $|k| \rightarrow \infty$ also holds. Accordingly, the phase shift $\delta(k)$ also vanishes when $|k| \rightarrow \infty$.

For technical reasons, it is now convenient to use the regular solution $\varphi(k, r) \equiv \phi(k, r)/k$. It is an even function of k and has a fixed derivative at the origin, $\varphi'_r(k, 0) = 1$. Consider the Wronskian,

$$\mathcal{W}[f, \varphi](k, r) \equiv f(k, r) \varphi'_r(k, r) - f'_r(k, r) \varphi(k, r). \quad (15)$$

Since $\varphi(k, 0) = 0$ and $\varphi'_r(k, 0) = 1$, this Wronskian boils down to just $f(k, 0)$. We also know from the Schrödinger equation that, when $r \rightarrow \infty$, then φ becomes $C(k) \sin[kr + \delta(k)]/k$, where δ and C are, respectively, the phase shift and a dimensionless scalar, both being complex in general. (Even when k is real, a complex U will make δ and C complex.) Since in this asymptotic region the Jost solution and its derivative reduce to $\exp(ikr)$ and $ik \exp(ikr)$, respectively, the same Wronskian reads, asymptotically, $\mathcal{W}[f, \varphi](k, \infty) = C(k) \exp[-i\delta(k)]$. We know from the Schrödinger equation that the Wronskian does not depend on r , hence $f(k, 0) = C(k) \exp[-i\delta(k)]$,

$$f(k, 0) = f(k, r) \varphi'_r(k, r) - f'_r(k, r) \varphi(k, r) = C(k) \exp[-i\delta(k)]. \quad (16)$$

Of a special interest is the regular solution $\psi(k, r)$ that, when $r \rightarrow \infty$, asymptotically becomes $\exp(i\delta) \sin(kr + \delta)$, for its unit incoming flux is not phase shifted, according to its $\exp(-ikr)$ term. This solution thus reads, $\psi(k, r) = k \varphi(k, r)/f(k, 0)$. While φ , an entire function of k , exists in the whole k -plane, and so does f with FSG potentials, ψ diverges for those values of k which cancel $f(k, 0)$. Such poles, naturally, define bound states and resonances.

C. Derivative of the Jost solution with respect to the momentum

Consider now the equation,

$$f'_k(k, r) = f_k'^0(k, r) + k^{-1} \int_r^\infty dr' \sin[k(r' - r)] U(r') f'_k(k, r'), \quad (17)$$

with,

$$f_k'^0(k, r) = ir \exp(ikr) + k^{-1} \int_r^\infty dr' \{(r' - r) \cos[k(r' - r)] - k^{-1} \sin[k(r' - r)]\} U(r') f(k, r'). \quad (18)$$

A series, $f'_k = \sum_{n=0}^\infty f_k'^n$, upon iterating Eq.(17), is again in order, where we use superscripts for $f_k'^n$, because these are not the derivatives with respect to k of the f_n 's used to construct f . It will be noticed, however, that the result reads, formally, $f'_k = (1 - \sigma)^{-1} f_k'^0$, where σ is a short notation for the now familiar integral kernel $\sigma(r, r') = k^{-1} \sin[k(r' - r)] U(r') \Theta(r' - r)$, see Eqs.(3,4). (Here Θ is the usual Heaviside step function, enforcing $r' > r$.) We are now familiar with the manipulations needed for that inversion, $(1 - \sigma)^{-1}$. As we will presently be interested in $f'_k(k, 0)$ only, this reminder does not elaborate further on bounds for $f_k'^0(k, r)$ and final bounds for $f'_k(k, r)$. The Jost function $f(k, 0)$ and special properties of its derivative $f'_k(k, 0)$ are the subject of the next two subsections.

D. Zeroes of the Jost function; bound states and resonances

In the following, we restrict k to its upper complex half-plane. Accordingly, the Jost solution asymptotically decreases exponentially, $|f(k, r)| \rightarrow |\exp(ikr)| = \exp(-\eta r)$ when $r \rightarrow \infty$. In this upper half-plane, let κ be a root of the Jost function $j(k) \equiv f(k, 0)$. The Wronskian of f and φ , see Eq.(16), then vanishes. This indicates that f and φ are not independent solutions of the Schrödinger equation in this special case. There exists a constant c such as $f(\kappa, r) = c \varphi(\kappa, r)$. Since φ is regular at the origin and f decreases exponentially at infinity, the state described by the degenerate pair $\{f, \varphi\}$ is square integrable. Two questions arise, namely, i) is this a bound state or a resonance, and ii) is this an isolated state, member of a discrete spectrum?

According to [1], the discrete (both bound and resonant) spectrum of H is the same when θ is finite as that for $\theta = 0$, except for an obvious rotation of amount 2θ , because of our multiplication of the usual CSM Hamiltonian by $\exp(2i\theta)$. Since $E = k^2$, a 2θ rotation for energies translates into a θ rotation for k . Hence two cases are expected, - either κ derives from the purely imaginary axis, namely $\kappa = \exp(i\theta) i |\kappa|$, and this corresponds to a bound state, with a signature, $\Re \kappa < 0$, $\text{Arg}(\kappa) = \theta + \pi/2$, with an alignment if more than one bound state occur,

- or, for a resonance, κ “just emerged” from the lower half k -plane, namely $\kappa = \exp(i\theta)(\beta - i\gamma)$, where $\beta > 0$ and $\gamma > 0$ parameterized the initial position of the resonance. The signature is now, $\Re \kappa > 0$. No alignment is expected. It will be noticed here that, since $\theta < \pi/4$, a bound state is protected against being rotated into the lower half of the k -plane. Conversely, resonances such that $\gamma > \beta$ cannot be rotated into the upper half of the k -plane, but this is not important physically, since these are “broad” resonances, of a lesser importance for the structure of excitation functions. Finally, should a “narrow” resonance occur such as $\tan \theta = \gamma/\beta$, the θ -rotation makes it entangled with the continuum, but it is always possible to slightly increase θ and avoid this embedding.

Such considerations answer question i), namely the sorting out of bound states and resonances, both square integrable. There remains to answer question ii), namely prove, in the next subsection, that these are isolated states.

E. Derivative of the Jost function at roots of this Jost function

If the derivative $j'_k(\kappa)$ of the Jost function $j(k) \equiv f(k, 0)$ does not vanish, then the root κ is an isolated, simple root, and question ii) is answered positively. Consider then the derivative of Eq.(16) with respect to k ,

$$j'_k(\kappa) = f'_k(\kappa, r) \varphi'_r(\kappa, r) - f''_{kr}(\kappa, r) \varphi(\kappa, r) + f(\kappa, r) \varphi''_{kr}(\kappa, r) - f'_r(\kappa, r) \varphi'_k(\kappa, r). \quad (19)$$

Since for such a root κ there exists a number c such that $f = c\varphi$, and, naturally, $f'_r = c\varphi'_r$, this becomes,

$$j'_k(\kappa) = c^{-1} [f'_k(\kappa, r) f'_r(\kappa, r) - f''_{kr}(\kappa, r) f(\kappa, r)] + c [\varphi(\kappa, r) \varphi''_{kr}(\kappa, r) - \varphi'_r(\kappa, r) \varphi'_k(\kappa, r)]. \quad (20)$$

The two brackets [] are Wronskians of f'_k , f and φ , φ'_k , respectively. But, since f'_k , resp. φ'_k , is not driven by the Schrödinger equation of f , resp. φ , such Wronskians are functions of r . As a preliminary step for their calculation, take the derivative of the Schrödinger equation with respect to k ,

$$[k^2 + d^2/dr^2 - U(r)] f'_k(k, r) + 2k f(k, r) = 0, \quad [k^2 + d^2/dr^2 - U(r)] \varphi'_k(k, r) + 2k \varphi(k, r) = 0, \quad (21)$$

then left multiply this by $f(k, r)$, resp. $\varphi(k, r)$, and replace $f(k, r) [k^2 - U(r)]$, resp. $\varphi(k, r) [k^2 - U(r)]$, by $-d^2/dr^2 f(k, r)$, resp. $-d^2/dr^2 \varphi(k, r)$. This gives, in a transparent condensed notation,

$$f''_{r^2} f'_k - f f'''_{kr^2} = 2k f^2, \quad \varphi \varphi'''_{kr^2} - \varphi''_{r^2} \varphi'_k = -2k \varphi^2. \quad (22)$$

Now, obviously, the left-hand sides of Eqs.(22) are the derivatives, with respect to r , of the brackets of Eq.(20),

$$f'_k f''_{r^2} - f'''_{kr^2} f = d/dr (f'_k f'_r - f''_{kr} f), \quad \varphi \varphi'''_{kr^2} - \varphi''_{r^2} \varphi'_k = d/dr (\varphi \varphi''_{kr} - \varphi'_r \varphi'_k). \quad (23)$$

Since $\Im k > 0$, both asymptotic forms, $\exp(ikr)$ and $ir \exp(ikr)$, of f and f'_k vanish exponentially when $r \rightarrow \infty$. Hence, upon integrating between r and ∞ the first of Eqs.(23), one finds,

$$f'_r(\kappa, r) f'_k(\kappa, r) - f(\kappa, r) f''_{kr}(\kappa, r) = -2\kappa \int_r^\infty dr' f(\kappa, r')^2. \quad (24)$$

Furthermore, both $\varphi(k, r)$ and $\varphi'_{kr}(k, r)$ vanish when $r = 0$. For the former, this is because of the very definition of φ by such a boundary condition. For the latter, this is because, again by the definition of φ , the derivative with respect to r at the origin, $\varphi'_r(k, 0) = 1$, does not depend on k . Accordingly,

$$\varphi(\kappa, r) \varphi''_{kr}(\kappa, r) - \varphi'_r(\kappa, r) \varphi'_k(\kappa, r) = -2\kappa \int_0^r dr' \varphi(\kappa, r')^2. \quad (25)$$

The sum of the two bracket contributions to the right-hand side of Eq.(20) finally gives,

$$j'_k(\kappa) = -2\kappa c \int_0^\infty dr' \varphi(\kappa, r')^2. \quad (26)$$

We know from [8] that, for $\theta = 0$, only simple poles occur. We also know from [1] and our $e^{2i\theta}$ factor in H , that for $\theta \neq 0$, only a rotation of those same poles by θ is needed to locate the CSM discrete singularities in the momentum plane. Isolated singularities remain isolated and our whole pattern of pole trajectories, when θ increases from 0 to a finite value, is made of concentric circular arcs. Could it happen that, for some special value θ_s , a double pole κ_s

occurs inside such an arc? Actually, since κ is a solution of the condition, $j(k; \theta) = 0$, the tangent to its trajectory reads, $d\kappa/d\theta = -(\partial j/\partial\theta)/(\partial j/\partial k)$. It also reads, because of the circular shape of this trajectory, $d\kappa/d\theta = i\kappa$, a finite number indeed. Hence $j'_k \equiv \partial j/\partial k$ cannot vanish, unless $\partial j/\partial\theta$ vanishes simultaneously. But this simultaneity would indicate a cusp at κ_s on the trajectory, contradictory with the existence of an arc. It can be concluded that the integral, $\int_0^\infty dr' \varphi(\kappa, r')^2$, does not vanish for any of the roots κ . Since this derivative $j'_k(\kappa)$ of the Jost function $j(k)$ does not vanish when j itself vanishes, the roots of j are generically isolated and a discrete spectrum is found.

The well known CSM unifying picture of bound states and resonances is thus confirmed with the “optical” Hamiltonian, Eq.(1). Those roots of the Jost function in the “upper-right quarter” of the complex k -plane generate resonances, while those in the “upper-left quarter” generate bound states. Both show square integrability, $\int_0^\infty dr' |\varphi(\kappa, r')|^2 < \infty$.

F. Residues of the Green's function G at its poles

The Green's function G (a resolvent kernel, actually) is defined by the usual formula,

$$G(k, r, r') = k^{-1} \psi(k, r_<) f(k, r_>) = \frac{\varphi(k, r_<) f(k, r_>)}{j(k)}. \quad (27)$$

In the analyticity domain of f with respect to k and for any pair $\{r, r'\}$, this G is meromorphic, with poles when $j(k) = 0$. At $r = r'$, this G is continuous. Its derivative G'_r reads $G'_r(k, r, r') = \varphi(k, r') f'_r(k, r)/j(k)$ for $r > r'$ and $G'_r(k, r, r') = \varphi'_r(k, r) f(k, r')/j(k)$ for $r < r'$, hence a discontinuity $[-f(k, r) \varphi'_r(k, r) + f'_r(k, r) \varphi(k, r)]/j(k) = -1$. This gives the usual property, $[k^2 + d^2/dr^2 - U(r)] G(k, r, r') = -\Delta(r - r')$, where we use the symbol Δ for the Dirac representation of the identity operator.

As already discussed, for each root κ of $j(k)$, two events happen, namely i) f and φ become proportional, $f = c\varphi$, and ii) G has a pole. At such a pole, it makes no difference whether $r < r'$ or $r > r'$ when we need to calculate the residue, since anyhow the product φf occurs in the calculation; it reduces to $\varphi c\varphi$, a symmetric form with respect to $\{r, r'\}'$. It is trivial to take advantage of Eq.(26) and find the residue,

$$\lim_{k \rightarrow \kappa} (k - \kappa) G(k, r, r') = -\frac{\varphi(\kappa, r) \varphi(\kappa, r')}{2\kappa \int_0^\infty dr'' \varphi(\kappa, r'')^2}. \quad (28)$$

Multiply this residue by (-2κ) . This makes an obviously idempotent, symmetric kernel, which may be viewed, although non Hermitian, as a “normalized projector” upon the state φ , whether a bound state or a resonance. Notice that the normalization integral in the denominator is a complex number and has an “Euclidean” form, for one integrates the square of a complex wave function rather than the square of its modulus.

G. “Projectors” upon the continuum

In the next subsection we will meet an integrand of the form,

$$\mathcal{P}(k, r, r') = \frac{\varphi(k, r_<) f(k, r_>)}{j(k)} - \frac{\varphi(-k, r_<) f(-k, r_>)}{j(-k)}. \quad (29)$$

Since φ is an even function of k this also reads, from the very definition of j ,

$$\mathcal{P}(k, r, r') = \varphi(k, r_<) \frac{f(-k, 0) f(k, r_>) - f(k, 0) f(-k, r_>)}{j(k) j(-k)}. \quad (30)$$

The numerator, $\Phi(k, r_>) \equiv f(k, r_>) f(-k, 0) - f(-k, r_>) f(k, 0)$, a linear combination of $f(k, r_>)$ and $f(-k, r_>)$, vanishes when $r_> \rightarrow 0$. It must therefore be proportional to $\varphi(k, r_>)$. From the asymptotic form, $f(k, r_>) \rightarrow \exp(ikr_>)$, the Wronskian, $\mathcal{W}[f, \Phi]$, reads,

$$f(k, r_>) \Phi'_{r_>}(k, r_>) - f'_{r_>}(k, r_>) \Phi(k, r_>) = -f(k, 0) [f(k, r_>) f'_{r_>}(-k, r_>) - f'_{r_>}(k, r_>) f(-k, r_>)] = 2ik f(k, 0). \quad (31)$$

This shows that $\Phi = 2ik\varphi$ and that $\mathcal{P}$, a symmetric kernel which would could be made idempotent if φ could be square normalized, plays the role of a “projector upon a continuum state”,

$$\mathcal{P}(k, r, r') = 2ik \frac{\varphi(k, r) \varphi(k, r')}{j(k) j(-k)}. \quad (32)$$

Notice again that there is no need to specify whether r is larger or smaller than r' , because the formula is symmetric. Naturally, the denominator, $j(k) j(-k)$, is not strictly the square of a norm and rather may replace a “level density”, since we will be dealing with an integral upon a continuum. It will be noticed that this “density” is here complex, and that φ makes its own dual state for the building of the (pseudo) “projector” $\mathcal{P}$.

H. One channel resolution of the identity

The previous two subsections have generated two structures which play the role of pseudo projectors, one for bound states and resonances, the other for continuum wave functions. Such pseudo projectors are orthogonal to each other, because they correspond to distinct eigenvalues. The following integral,

$$\mathcal{I}(r, r') \equiv \int_{\mathcal{C}} k dk G(k, r, r') = \int_{\mathcal{C}} k dk \frac{\varphi(k, r_{<}) f(k, r_{>})}{j(k)}, \quad (33)$$

now shows how such “projectors” construct a resolution of the identity, proving the completeness of the CSM spectrum. Here $\mathcal{C}$ is a contour, oriented anti-clockwise, containing the real axis and a half-circle at infinity in the upper half k -plane. On the one hand, from all the residues at roots κ_n of $j(k)$ in this upper half-plane, one trivially finds,

$$\mathcal{I}(r, r') = -i\pi \sum_n \frac{\varphi(\kappa_n, r) \varphi(\kappa_n, r')}{\int_0^\infty dr'' \varphi(\kappa_n, r'')^2}. \quad (34)$$

On the other hand, a direct calculation of the contribution $\mathcal{I}_{ax}$ of the real axis to $\mathcal{I}$ yields, obviously,

$$\mathcal{I}_{ax}(r, r') \equiv \int_{-\infty}^{\infty} k dk \frac{\varphi(k, r_{<}) f(k, r_{>})}{j(k)} = \int_0^{\infty} k dk \left[\frac{\varphi(k, r_{<}) f(k, r_{>})}{j(k)} - \frac{\varphi(-k, r_{<}) f(-k, r_{>})}{j(-k)} \right]. \quad (35)$$

According to Eq.(32) this gives,

$$\mathcal{I}_{ax}(r, r') = 2i \int_0^{\infty} k^2 dk \frac{\varphi(k, r) \varphi(k, r')}{j(k) j(-k)}. \quad (36)$$

An infinitesimal neighborhood of $k = 0$ might raise problem for this integral. For this technicality, we refer to [8]. The result is, no singularity of the integral occurs unless $j(0) = 0$, a quite exceptional case.

There remains the contribution $\mathcal{I}_{hc}$ of the half-circle. This obtains from the asymptotic forms, $\varphi(k, r) \rightarrow \sin(kr)/k$ and $f(k, r) \rightarrow \exp(ikr)$, when $|k| \rightarrow \infty$. Moreover, the phase shift also vanishes if $|k| \rightarrow \infty$. The result is thus simple,

$$\mathcal{I}_{hc}(r, r') = (2i)^{-1} \int_{hc} dk \{ \exp[ik(r_{>} + r_{<})] - \exp[ik(r_{>} - r_{<})] \}. \quad (37)$$

With such an integrand, an entire function, the half-circle can be deformed back to the real axis,

$$\mathcal{I}_{hc}(r, r') = (2i)^{-1} \int_{-\infty}^{\infty} dk \{ \exp[ik(r + r')] - \exp[ik(r_{>} - r_{<})] \}, \quad (38)$$

where $r_{>} + r_{<} = r + r'$. Whether r is larger or smaller than r' , the term $\exp[ik(r_{>} - r_{<})]$ integrates at once into a contribution $-i\pi \Delta(r - r')$, where we again denote Δ the Dirac representation of the identity operator. And the term $\exp[ik(r + r')]$, which integrates into an $i\pi \Delta(r + r')$, can be discarded since both r and r' are non-negative. (The limit case, $r = r' = 0$, is of no avail for a space of test functions which vanish at the origin.). The final result is,

$$\mathcal{I}_{hc}(r, r') = -i\pi \Delta(r - r'). \quad (39)$$

Add Eq.(39) and Eq.(36), equate the result with Eq.(34), transfer $\mathcal{I}_{ax}$ to the right-hand side and multiply every term by i/π . The resolution of the identity thus reads,

$$\Delta(r-r') = \int_0^\infty dk \frac{2\phi(k,r)\phi(k,r')}{\pi j(k)j(-k)} + \sum_n \frac{\varphi(\kappa_n,r)\varphi(\kappa_n,r')}{\int_0^\infty dr'' \varphi(\kappa_n,r'')^2}. \quad (40)$$

To summarize the one channel case, a contour integral of its Green's function gives, if performed explicitly, the difference between the identity and the pseudo projector upon the CSM continuum. The same integral, from Cauchy's theorem, equals a pseudo projector upon the bound states and those resonances captured by the contour. The identity is thus a sum of "CSM projectors", upon the continuum, the bound states and those resonances made square integrable.

III. COUPLED CHANNELS WITH THRESHOLDS

The case of coupled channels without thresholds demands vector and matrix notations and manipulations. It also may demand technical precautions because of frequent couplings between different angular momenta. None of such complications raises major problems, however, and, anyhow, the present Section will as well illustrate the necessary vector and matrix algebra. The most frequent case of interest in nuclear physics is that of two coupled channels *with* a threshold. Hence we now study two *s*-waves, (*s* for simplicity again), where the second channel starts at a real, positive excitation energy E^* . Such coupled radial equations read, with naturally $V_{21} = V_{12}$,

$$(k^2 + d^2/dr^2) f_1(k, r) = V_{11}(r) f_1(k, r) + V_{12}(r) f_2(k, r), \quad (41a)$$

$$(k^2 - E^* + d^2/dr^2) f_2(k, r) = V_{21}(r) f_1(k, r) + V_{22}(r) f_2(k, r). \quad (41b)$$

Note that, as a matrix of operators, the Hamiltonian reads, $\begin{bmatrix} T + V_{11} & V_{12} \\ V_{21} & E^* + T + V_{22} \end{bmatrix}$, with an explicit presence of E^* . Define $\tilde{E} = \exp(2i\theta) E^*$. Under complex scaling and multiplication by $\exp 2i\theta$ the Hamiltonian becomes, in an obvious notation, $\mathbf{H} = \begin{bmatrix} T + U_{11} & U_{12} \\ U_{21} & \tilde{E} + T + U_{22} \end{bmatrix}$, with $U_{ij}(r) = \exp(2i\theta) V_{ij}[r \exp(i\theta)]$. Hence the equations,

$$(k^2 + d^2/dr^2) f_1(k, r) = U_{11}(r) f_1(k, r) + U_{12}(r) f_2(k, r), \quad (K^2 + d^2/dr^2) f_2(k, r) = U_{21}(r) f_1(k, r) + U_{22}(r) f_2(k, r), \quad (42)$$

with $K^2 \equiv k^2 - \tilde{E}$. With a column vector notation $f \equiv \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and matrix notations for the kinetic and potential operators, this looks like Eq.(2), provided $\tilde{E}$ is absorbed by T or by U_{22} in the equation for the excited channel. Such vector and matrix notations are also useful in defining a generalization of the Wronskian. Namely, if $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ and $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ is any pair of solutions of Eqs.(42), then the function of r ,

$$\mathcal{W}[f, g] \equiv \tilde{f}g' - \tilde{f}'g = f_1(k, r) g_1'(k, r) + f_2(k, r) g_2'(k, r) - f_1'(k, r) g_1(k, r) - f_2'(k, r) g_2(k, r), \quad (43)$$

actually does not depend on r . The tilde sign is used here for transposition and the prime means d/dr . The proof of this constancy of W is trivial along Eqs.(42). It relies on the locality of the potentials and the symmetry $U_{12}(r) = U_{21}(r)$.

There are now two distinct Jost solutions and two distinct regular solutions, along the two pairs of equations,

$$f_{1\mu}(k, r) = \exp(ikr) \delta_{1\mu} + k^{-1} \int_r^\infty dr' \sin[k(r' - r)] [U_{11}(r') f_{1\mu}(k, r') + U_{12}(r') f_{2\mu}(k, r')], \quad (44a)$$

$$f_{2\mu}(k, r) = \exp(iKr) \delta_{2\mu} + K^{-1} \int_r^\infty dr' \sin[K(r' - r)] [U_{21}(r') f_{1\mu}(k, r') + U_{22}(r') f_{2\mu}(k, r')], \quad (44b)$$

$$\varphi_{1\mu}(k, r) = k^{-1} \sin(kr) \delta_{1\mu} + k^{-1} \int_0^r dr' \sin[k(r - r')] [U_{11}(r') \varphi_{1\mu}(k, r') + U_{12}(r') \varphi_{2\mu}(k, r')], \quad (44c)$$

$$\varphi_{2\mu}(k, r) = K^{-1} \sin(Kr) \delta_{2\mu} + K^{-1} \int_0^r dr' \sin[K(r - r')] [U_{21}(r') \varphi_{1\mu}(k, r') + U_{22}(r') \varphi_{2\mu}(k, r')]. \quad (44d)$$

We use here δ as the Kronecker symbol and the subscript $\mu = 1, 2$ tells which channel contains a source term, namely $\exp(ikr)$, $\exp(iKr)$, $\sin(kr)/k$, $\sin(Kr)/K$ for the Jost solutions $f_{.1}, f_{.2}$ and regular solutions $\varphi_{.1}, \varphi_{.2}$, respectively. Proofs of convergence, unicity, derivability, etc, are slightly cumbersome and cannot be published here. But they are straightforward, using conditions and majorations similar to those used in the one channel case. It will be noticed that, whenever $|k| \rightarrow \infty$ and simultaneously $|K| \rightarrow \infty$, trivial asymptotic solutions read,

$$f_{11}^\infty(r) = \exp(ikr), f_{22}^\infty(r) = \exp(iKr), \varphi_{11}^\infty(r) = \frac{\sin(kr)}{k}, \varphi_{22}^\infty(r) = \frac{\sin(Kr)}{K}, f_{21}^\infty(r) = f_{12}^\infty(r) = \varphi_{21}^\infty(r) = \varphi_{12}^\infty(r) = 0. \quad (45)$$

Obvious boundary conditions resulting from Eqs.(44a,44b) when $r \rightarrow \infty$ read,

$$f_{11}(k, r) \rightarrow \exp(ikr), f_{21}(k, r) \rightarrow 0, f_{12}(k, r) \rightarrow 0, f_{22}(k, r) \rightarrow \exp(iKr), \quad (46a)$$

$$f'_{11}(k, r) \rightarrow ik \exp(ikr), f'_{21}(k, r) \rightarrow 0, f'_{12}(k, r) \rightarrow 0, f'_{22}(k, r) \rightarrow iK \exp(iKr), \quad (46b)$$

where the superscript ' again means d/dr . In turn, obvious boundary conditions resulting from Eqs.(44c, 44d) read,

$$\varphi_{11}(k, 0) = \varphi_{21}(k, 0) = \varphi_{12}(k, 0) = \varphi_{22}(k, 0) = \varphi'_{21}(k, 0) = \varphi'_{12}(k, 0) = 0, \varphi'_{11}(k, 0) = \varphi'_{22}(k, 0) = 1. \quad (47)$$

It will be convenient in the following to arrange the Jost solutions into a matrix $\mathbf{f}^+ = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ whose first column $f_{.1}$, resp. second column $f_{.2}$ represents that solution of Eqs.(44a,44b) with a source term in the first, resp. second channel. A similar matrix will be used for the regular solutions, $\Psi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$. It will be noticed that Ψ is even under a simultaneous reversal of k and K , while the same reversal converts $\mathbf{f}^+$ into a distinct matrix $\mathbf{f}^-$. (It is understood for such definitions of $\mathbf{f}^\pm$ that the reversal of k into $-k$ means that K also becomes $-K$.) Wronskians can now be arranged into matrices, $\mathcal{W}[\mathbf{f}, \mathbf{g}] = \mathbf{f} \mathbf{g}' - \mathbf{f}' \mathbf{g}$, and calculated either at $r = 0$ or for $r \rightarrow \infty$. It is convenient to define, $\mathbf{W} \equiv \mathcal{W}[\mathbf{f}^+, \mathbf{f}^-]$, then $\mathbf{W}_+ \equiv \mathcal{W}[\mathbf{f}^+, \Psi]$ and also $\mathbf{W}_- \equiv \mathcal{W}[\mathbf{f}^-, \Psi]$, and obtain, together with $\mathcal{W}[\mathbf{f}^+, \mathbf{f}^+] = \mathcal{W}[\mathbf{f}^-, \mathbf{f}^-] = 0$,

$$\mathbf{W} = \begin{pmatrix} -2ik & 0 \\ 0 & -2iK \end{pmatrix}, \mathbf{W}_+ = \begin{pmatrix} f_{11}(k, 0) & f_{21}(k, 0) \\ f_{12}(K, 0) & f_{22}(K, 0) \end{pmatrix}, \mathbf{W}_- = \begin{pmatrix} f_{11}(-k, 0) & f_{21}(-k, 0) \\ f_{12}(-K, 0) & f_{22}(-K, 0) \end{pmatrix}. \quad (48)$$

Two slight differences with the one channel case occur for Wronskians, namely i) now, *inside* the Ψ subspace, *inside* the $\mathbf{f}^+$ subspace and *inside* the $\mathbf{f}^-$ subspace, but not from one to another of two such subspaces, two linearly independent solutions may have a vanishing Wronskian, see for instance $\mathcal{W}[\varphi_{.1}, \varphi_{.2}]$, and ii) $\mathcal{W}[\mathbf{f}, \mathbf{g}]$ and $\mathcal{W}[\mathbf{g}, \mathbf{f}]$ differ by not just a $-$ sign, but also by a transposition. Notice also our slight notational change in $f_{12}(\pm K, 0)$ and $f_{22}(\pm K, 0)$, which emphasizes the primary role played by K for the Jost solution $f_{.2}$. Indeed, $f_{.2}$ is insensitive to the reversal of k into $-k$. Naturally, in the same way, $f_{.1}$ is an even function of K .

An expansion of the regular solution matrix Ψ in terms of the Jost solution matrices $\mathbf{f}^\pm$ derives easily from Eqs.(48),

$$\Psi = \mathbf{f}^- \mathbf{W}^{-1} \mathbf{W}_+ - \mathbf{f}^+ \mathbf{W}^{-1} \mathbf{W}_-. \quad (49)$$

From this formula, Eq.(49), the regularity condition, $\Psi(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, gives a constraint on the values of the Jost functions at the origin,

$$K [f_{11}(k, 0) f_{21}(-k, 0) - f_{21}(k, 0) f_{11}(-k, 0)] = k [f_{22}(K, 0) f_{12}(-K, 0) - f_{12}(K, 0) f_{22}(-K, 0)]. \quad (50)$$

It is also easy to verify from Eq.(43) and Eqs.(47) that $\mathcal{W}[\Psi, \Psi] = 0$, hence the symmetry property,

$$\tilde{\mathbf{W}}_+ \mathbf{W}^{-1} \mathbf{W}_- = \tilde{\mathbf{W}}_- \mathbf{W}^{-1} \mathbf{W}_+. \quad (51)$$

Another symmetry property results from expanding in terms of $\mathbf{f}^+(r)$ and $\mathbf{f}^-(r)$ the (matrix) solution $\Psi_s(r)$ defined, like Ψ , by zeroes for the functions and diagonal units for the derivatives, but at some arbitrary position s rather than the origin. Such boundary conditions read, $\Psi_s(s) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and, $d/dr \Psi_s|_s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. With Wronskians calculated at s , naturally, the analog of Eq.(49) reads,

$$\Psi_s(r) = \mathbf{f}^-(r) \mathbf{W}^{-1} \tilde{\mathbf{f}}^+(s) - \mathbf{f}^+(r) \mathbf{W}^{-1} \tilde{\mathbf{f}}^-(s). \quad (52)$$

Since $\Psi_s(s)$ vanishes, the matrix,

$$\mathbf{f}^-(s) \mathbf{W}^{-1} \tilde{\mathbf{f}}^+(s) = \mathbf{f}^+(s) \mathbf{W}^{-1} \tilde{\mathbf{f}}^-(s), \quad (53)$$

is necessarily symmetric. This holds $\forall s$ since s was chosen arbitrarily. For $s = 0$, Eq.(51) is a special case of Eq.(53).

A situation of special interest occurs when the determinant, $\det \mathbf{W}_+$, as a function of k , vanishes while simultaneously $\Im k > 0$ and $\Im K > 0$. For then, there exists, according to Eq.(49), a particular superposition of $\varphi_{.1}$ and $\varphi_{.2}$ that has no component upon the subspace described by $\mathbf{f}^-$. Hence it is both regular for $r = 0$ and exponentially decreasing when $r \rightarrow \infty$. Such roots κ_ν locate the bound states and those resonances of interest, made square integrable.

It is now useful to calculate the Green's function (now a matrix of functions, naturally). Elementary manipulations, explained in Appendix I, show that $\mathbf{G}(k, r, r')$, defined by,

$$\mathbf{G}(k, r, r') = \Psi(r) \mathbf{W}_+^{-1} \tilde{\mathbf{f}}^+(r') \text{ if } r < r', \quad \mathbf{G}(k, r, r') = \mathbf{f}^+(r) \tilde{\mathbf{W}}_+^{-1} \tilde{\Psi}(r') \text{ if } r > r', \quad (54)$$

satisfies the condition,

$$(E - \mathbf{H}) \mathbf{G} = - \begin{pmatrix} \Delta(r - r') & 0 \\ 0 & \Delta(r - r') \end{pmatrix}. \quad (55)$$

When acting upon a test function (with two components!), $\mathbf{G}$ returns an image whose both components are regular when $r \rightarrow 0$ and show “outgoing fluxes” when $r \rightarrow \infty$. This is easily seen from Eqs.(54). However, $\Re k$ and/or $\Re K$ can become negative under analytic continuation. This converts outgoing fluxes into ingoing ones, obviously.

Recall that $E \equiv k^2$. Besides the usual cut in the complex E -plane, an additional cut must now be considered, extending “horizontally” from the rotated threshold $\tilde{E}$ to infinity “on the right side”.

FIG. 1. E -plane. Physical cuts (blue and red) for the 2-channel case when $\theta = \pi/6$ and $E^* = 4$. Additional cut (yellow) for a 3-channel case with threshold $E_3^* = 6$.

Define $\rho = \Re k$, so that $k = \rho + i\eta$. In the complex k -plane, the conditions which define the second cut become,

$$\Re(K^2) = \rho^2 - \eta^2 - E^* \cos(2\theta) \geq 0 \quad \text{and} \quad \Im(K^2) = 2\rho\eta - E^* \sin(2\theta) = 0. \quad (56)$$

This makes a hyperbolic arc in the first quadrant, and its symmetric arc in the third quadrant. The first arc starts from $\exp(i\theta)\sqrt{E^*}$ and extends to infinity, on the right side again, with the positive real semi-axis as an asymptote.

Actually, it is more convenient to use a representation in terms of a momentum P that is symmetrical with respect to k and K , namely $P = (k + K)/2$. Define the real, positive number $q = \sqrt{E^*}/2$ and then $Q = q \exp(i\theta)$ so that $\tilde{E} = 4Q^2$. The definition, $K^2 \equiv k^2 - \tilde{E}$, results into, $(k - K)/2 = Q^2/P$, and finally, $k = P + Q^2/P$ and $K = P - Q^2/P$. This representation, incidentally, is valid even if $\ell \neq 0$. In any case, it has several interesting properties, obviously, under symmetry operations such as transforming P into $-P$ or into $\pm Q^2/P$. But its main property is that there is no need of cuts in the P -plane. Indeed, let p and φ be the modulus and argument of P . The conditions, $\Im k = 0$, then $\Im K = 0$ for physical values of k and K read,

$$p \sin \varphi \pm q^2 \sin(2\theta - \varphi)/p = 0, \quad (57)$$

and accordingly the cuts of the E -plane are both unfolded when represented in the P -plane. The $+$ sign in Eq.(57) corresponds to the former k -cut and one gets an easy plot for the blue curve in Figure 2. The $-$ sign, in turn, defines an easy plot for the former K -cut, now unfolded into the red curve.

FIG. 2. P -plane, 2-channel case. Integration contour of the Green's function when $\theta = \pi/6$ and $E^* = 4$.

The following properties are easily proven, i) the blue and red curves contain the points iQ and Q , where k and K vanish respectively, ii) the blue and red curves are asymptotic to the negative and positive real semi-axes, respectively, iii) both curves contain the origin of the P -plane, with a common tangent with slope 2θ , and have no other common point, iv) along the blue curve, k smoothly runs from $-\infty$ (asymptotic branch) to $+\infty$ (origin), and $\Im K$ remains positive, v) similarly, along the red curve, K increases smoothly from $-\infty$ (origin) to $+\infty$ (asymptotic branch) while $\Im k$ remains positive, vi) both $\Im k$ and $\Im K$ are positive in the region of the P -plane lying “above” these curves, vii) furthermore, the blue curve can be completed by its parity reversed image about the origin, and $\Im k$ is positive above this completed curve, and negative below; a similar property holds for the red curve and $\Im K$; this completion is of academic interest only, for this paper shall just take advantage of the region described by property vi).

Further properties of this P -representation are, viii) under the transforms, P into $-P$ and P into $\pm Q^2/P$, the integration measure, $kdk = KdK = (P - Q^4/P^3)dP$, is invariant, ix) the transform, P into $-Q^2/P$, exchanges the

blue branch, running from $-\infty$ to iQ , and the blue segment, running from the origin to iQ ; it maps the pair of values $\{-k, K\}$ into the pair $\{k, K\}$; (while k is real on the red branch, K is complex with $\Im K > 0$;) x) the transform, P into Q^2/P , exchanges the red segment, running from Q to the origin, and the the red branch, running from Q to ∞ ; it maps the pair of values $\{k, -K\}$ into $\{k, K\}$; (while K is real on the red branch, k is complex with $\Im k > 0$.)

Consider now the integral,

$$\mathcal{J}(r, r') \equiv \int_{\mathcal{C}} \left(P - \frac{Q^4}{P^3} \right) dP \mathbf{G}(P, r, r'), \quad (58)$$

where now $\mathcal{C}$ is a contour, oriented anti-clockwise, containing the blue curve, then the red curve and a half-circle at infinity in the upper half P -plane, see again Figure 2. Notice that $\mathbf{G}$ is now labeled by P , which makes a more symmetric label than either k or K . Notice also the integration weight, which equals both kdk and KdK .

On the one hand, summing upon all the residues obtained at roots κ_ν of $\det \mathbf{W}_+$ above the blue and red curves in this upper half-plane, one trivially finds the result,

$$\mathcal{J}(r, r') = -i\pi \sum_{\nu} \frac{\Psi(\kappa_\nu, r) \Lambda_\nu \tilde{\Lambda}_\nu \tilde{\Psi}(\kappa_\nu, r')}{\int_0^\infty dr'' \tilde{\Lambda}_\nu \tilde{\Psi}(\kappa_\nu, r'') \Psi(\kappa_\nu, r'') \Lambda_\nu}. \quad (59)$$

This result, Eq.(59), must be read as follows, i) the column vector Λ_ν is the null, right-hand side eigenvector of $\mathbf{W}_+$, namely $\mathbf{W}_+ \Lambda_\nu = 0$, then ii) the column vector $\Psi \Lambda_\nu$ of wave functions is the wave function of the bound state or resonance, and iii) the denominator is the corresponding “Euclidean-like square norm”, which we will assume non vanishing (this corresponds to the hypothesis of single, isolated poles). All these are labeled by ν , naturally, a discrete index, or as well by κ_ν , an isolated root of $\mathbf{W}_+$ as a function of P . We give in Appendix II the proof relating the denominator in Eq.(59) to the residue of the integrand at its poles, see Eq.(58).

On the other hand, a direct calculation of the contribution $\mathcal{J}_{hc}$ of the half-circle is trivial. Indeed, for $|P| \rightarrow \infty$, the integration weight $kdk = KdK$ reduces to PdP while both k and K reduce to P . Furthermore one may use the asymptotic forms of $\mathbf{f}^+$ and Ψ ,

$$\mathbf{f}^\infty = \begin{pmatrix} \exp(ikr) & 0 \\ 0 & \exp(iKr) \end{pmatrix}, \quad \Psi^\infty = \begin{pmatrix} \sin(kr)/k & 0 \\ 0 & \sin(Kr)/K \end{pmatrix}, \quad (60)$$

see Eqs.(45), while k and K boil down to P . Simultaneously $\mathbf{W}_+$ boils down to the unit matrix. An argument identical to that used for the one channel contribution $\mathcal{I}_{hc}$ now gives,

$$\mathcal{J}_{hc}(r, r') = -i\pi \begin{pmatrix} \Delta(r - r') & 0 \\ 0 & \Delta(r - r') \end{pmatrix}. \quad (61)$$

There remains to obtain the “blue” and “red” contributions. The blue, resp. red curve integrates the discontinuity of the Green’s function across the first, resp. second channel cut, see Figure 1. We first investigate the red integral for the case, $r < r'$. Recall that K is real along this curve. For $P \rightarrow 0$ along the curve, $K \rightarrow -\infty$, and simultaneously $\Re k \rightarrow \infty$, while $\Im k \rightarrow 0^+$. The integral under study reads, in a condensed notation,

$$\mathcal{J}_{red}(r, r') = \int_Q^\infty \left(P - \frac{Q^4}{P^3} \right) dP \Psi(r) \left[\mathbf{W}_+^{-1} \tilde{\mathbf{f}}^+(r') - \mathbf{W}_m^{-1} \tilde{\mathbf{f}}^m(r') \right], \quad (62)$$

where we have taken advantage of the transform, P into Q^2/P , which converts K into $-K$ and leaves k invariant. Under the same transform, Ψ and $f_{,1}$ are invariant, and the red curve segment between Q and the origin becomes the red branch running from Q to $+\infty$. The “mixed” matrix, $\mathbf{f}^m \equiv \begin{pmatrix} f_{11}(k, r) & f_{12}(-K, r) \\ f_{21}(k, r) & f_{22}(-K, r) \end{pmatrix}$ and the corresponding

Wronskian, $\mathbf{W}_m \equiv \begin{pmatrix} f_{11}(k, 0) & f_{21}(k, 0) \\ f_{12}(-K, 0) & f_{22}(-K, 0) \end{pmatrix}$, account for the integrand along the red segment. The bracket $\tilde{\mathbf{D}} \equiv [\mathbf{W}_+^{-1} \tilde{\mathbf{f}}^+ - \mathbf{W}_m^{-1} \tilde{\mathbf{f}}^m]$ in the right-hand side of Eq.(62) makes explicit the discontinuity of $\mathbf{G}$ across the red cut. It is clear that, $\mathbf{D} = \mathbf{f}^+ \tilde{\mathbf{W}}_+^{-1} - \mathbf{f}^m \tilde{\mathbf{W}}_m^{-1}$, relates with a dimension 3 subspace of solutions at most. It is now tempting to further reduce the product, $\Psi(r) \tilde{\mathbf{D}}(r')$, to a rank 1, symmetric, tensor product, $\varphi_{red}(r) \tilde{\varphi}_{red}(r')$, of two component

wave-functions, and obtain a formula as elegant as Eq.(36), with obviously, for a “level density” in its denominator, the product of determinants, $\det \mathbf{W}_+ \det \mathbf{W}_m$.

With 2×2 matrices, it is trivial to find the inverse matrix,

$$\mathbf{W}_+^{-1} = \begin{pmatrix} f_{22}(K, 0)/d_+ & -f_{21}(k, 0)/d_+ \\ -f_{12}(K, 0)/d_+ & f_{11}(k, 0)/d_+ \end{pmatrix}, \quad d_+ \equiv \det \mathbf{W}_+ = f_{11}(k, 0) f_{22}(K, 0) - f_{21}(k, 0) f_{12}(K, 0), \quad (63)$$

and, when K becomes $-K$, similar formulae,

$$\mathbf{W}_m^{-1} = \begin{pmatrix} f_{22}(-K, 0)/d_m & -f_{21}(k, 0)/d_m \\ -f_{12}(-K, 0)/d_m & f_{11}(k, 0)/d_m \end{pmatrix}, \quad d_m \equiv \det \mathbf{W}_m = f_{11}(k, 0) f_{22}(-K, 0) - f_{21}(k, 0) f_{12}(-K, 0), \quad (64)$$

Thus a slightly cumbersome, but explicit formula for $\tilde{\mathbf{D}}$ is found. An analysis of this result for $\tilde{\mathbf{D}}$ by means of its eigenvectors then shows that $\tilde{\mathbf{D}}$ is a rank 1, fully separable matrix. We just give the result, which can be easily verified directly,

$$d_+ d_m \tilde{\mathbf{D}} = \begin{pmatrix} f_{21}(k, 0) \\ -f_{11}(k, 0) \end{pmatrix} \tilde{\varphi}_{red}(r'), \quad \varphi_{red}(r') = \begin{pmatrix} d_t f_{11}(k, r') - d_m f_{12}(K, r') + d_+ f_{12}(-K, r') \\ d_t f_{21}(k, r') - d_m f_{22}(K, r') + d_+ f_{22}(-K, r') \end{pmatrix}, \quad (65)$$

where we make use of a third determinant, $d_t = f_{12}(K, 0) f_{22}(-K, 0) - f_{12}(-K, 0) f_{22}(K, 0)$. Notice that φ_{red} is an odd function of K . Moreover, upon calculating $\varphi_{red}(r')$ at the origin, $r' = 0$, via the values of d_+ , d_m and d_t , one finds easily that φ_{red} is a regular solution, $\varphi_{red}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. More details are found in Appendix III.

This separability of $\tilde{\mathbf{D}}$ generates at once the separability of the matrix product, $\Psi \tilde{\mathbf{D}}$. Indeed, the matrix $\Psi(r)$ acts upon a column vector, $\begin{pmatrix} f_{21}(k, 0) \\ -f_{11}(k, 0) \end{pmatrix}$, which does not depend on r' . All factors $K dK/(d_+ d_m)$ being understood, our integrand thus factorizes into the tensor product of a vector function of r ,

$$\varphi_{reg}(r) = \varphi_{.1}(r) f_{21}(k, 0) - \varphi_{.2}(r) f_{11}(k, 0), \quad (66)$$

and a vector function of r' , namely $\varphi_{red}(r')$. Furthermore it turns out that this makes a symmetric product of the form, $\varphi_{red}(r) \tilde{\varphi}_{red}(r')$, namely actually φ_{reg} is proportional to φ_{red} . To prove this proportionality, we use Eq.(49) and obtain, after a slightly tedious calculation, φ_{reg} in terms of our four Jost solutions (three, actually). Then we compare with Eq.(65), after taking advantage of Eq.(50). The brute force result is,

$$\varphi_{reg} = \varphi_{red}/(2iK). \quad (67)$$

Because of this symmetry under an exchange of r and r' , there is no need to study the case, $r > r'$. Hence the red contribution reads, with an integral along that branch of the red curve extending from Q to $+\infty$,

$$\mathcal{J}_{red}(r, r') = \int_Q^\infty dK \frac{\varphi_{red}(r) \varphi_{red}(r')}{2i d_+ d_m}. \quad (68)$$

Obviously again, a similar factorization occurs for the “blue” contribution, with now auxiliary determinants, $d_n = f_{11}(-k, 0) f_{22}(K, 0) - f_{21}(-k, 0) f_{12}(K, 0)$ and $d_f = f_{11}(k, 0) f_{21}(-k, 0) - f_{11}(-k, 0) f_{21}(k, 0)$, then $\varphi_{reg} = \varphi_{.1} f_{22}(K, 0) - \varphi_{.2} f_{12}(K, 0) = \varphi_{blu}/(2ik)$, and, with an integral along the blue “segment” from iQ to 0,

$$\mathcal{J}_{blu}(r, r') = \int_{iQ}^0 dk \frac{\varphi_{blu}(r) \varphi_{blu}(r')}{2i d_+ d_n}, \quad \varphi_{blu}(r') = \begin{pmatrix} d_n f_{11}(k, r') - d_+ f_{11}(-k, r') + d_f f_{12}(K, r') \\ d_n f_{21}(k, r') - d_+ f_{21}(-k, r') + d_f f_{22}(K, r') \end{pmatrix}. \quad (69)$$

We thus obtain, for the continuum, integrals whose integrand is a product of i) a linear superposition of regular solutions and ii) a linear superposition of Jost solutions which turns out to be also regular. These superpositions are proportional to each other, hence self-dual.

To obtain the resolution of the identity, add $\mathcal{J}_{blu}$, $\mathcal{J}_{red}$ and $\mathcal{J}_{hc}$, see Eqs.(69,68,61), and compare with $\mathcal{J}$, see Eq.(59). To summarize this long Section, the resolution reads, $i(\mathcal{J} - \mathcal{J}_{blu} - \mathcal{J}_{red})/\pi$, in terms of mixtures of those regular CSM solutions defined by Eqs.(44c,44d). The contributions of the bound states and resonances have the usual pseudo projector forms, compare Eq.(34) and Eq.(59). For those background contributions which remain after resonance subtractions, the two channel calculations are not much more complicated, see for instance Eq.(65) and Eq.(67), than those needed for the one channel case, see Eq.(36).

IV. DISCUSSION AND CONCLUSION

There are two steps in this work. The first one is a somewhat cumbersome, but elementary reminder of the properties of regular and irregular solutions of Schrödinger equations driven by some simple local potentials used in nuclear physics, such as finite sums of Gaussians and their CSM analytical continuations. Indeed, only few of the other nuclear interactions used in practical calculations can be continued analytically, and we found that a “refamiliarization” with some jargon was in order; it was impossible to explain the P -representation for two channels without already a long, cautious reminder of the one channel tools. Simultaneously the corresponding Green’s function is visited again, for the one channel derivation, still with elementary methods, of the CSM resolution of the identity by means of eigenfunctions of such CSM equations. The first step was also necessary because those potentials, specific to the CSM, are complex, non Hermitian, and because the topology of “CSM cuts” differs from that of those Hermitian problems already treated in the available literature. It turns out, fortunately, that very little adjustments are needed to turn “Hermitian proofs” into “CSM proofs” for the existence and analyticity properties of CSM solutions. This makes the second step, that of the two channel completeness, reasonably easy.

For less “friendly” potentials than the FSG ones, our one channel arguments survive, since the integration contours $\mathcal{C}$ are located in the upper half-plane, where the Jost function exists under weak conditions, such as the convergence of $\int_0^\infty dr' |U(r')|$, see Eq.(11). Actually, for more practical generalizations [8], the convergence of $\int_0^\infty dr' r' |U(r')|$, or even that of $\int_0^\infty dr' r'^2 |U(r')|$, are requested, but these are still weak conditions. They extend considerably beyond FSG the class of potentials admissible for CSM. We sketch in Appendix IV the argument, related to the case $k = 0$, where the convergence of $\int_0^\infty dr' r' |U(r')|$ is a useful constraint.

For pedagogical reasons, we restricted our discussion to the case of s -waves. Higher orbital momenta $\ell \neq 0$ only bring small differences in the algebra, such as Hankel functions rather than simple exponentials, sines and cosines. The radial Schrödinger equation now reads,

$$[k^2 + d^2/dr^2 - \ell(\ell+1)/r^2] F_\ell(k, r) = U(r) F_\ell(k, r), \quad (70)$$

and a well-known “centrifugal phase shift”, $-\ell\pi/2$, appears in asymptotic forms such as, $\exp(ikr - \ell\pi/2)$, $\sin(kr - \ell\pi/2)$, etc. It will be noticed that the centrifugal potential is invariant under CSM, since we use Eq.(1). Indeed, trivially, $\exp(2i\theta) \ell(\ell+1)/[r \exp(i\theta)]^2 = \ell(\ell+1)/r^2$. The Sturm analysis of second order linear differential equations as regards the behavior of solutions at $r \rightarrow 0$ and $r \rightarrow \infty$ is thus valid for any value of θ . Majoration schemes necessary for proofs of existence, convergence, etc. remain the same as those discussed for $\ell = 0$. It must be stressed that all integral equations to be handled will be of the form,

$$F_\ell(k, r) = F_{0\ell}(k, r) + k^{-1} \int_0^\infty dr' g_\ell(r, r') U(r') F_\ell(k, r'), \quad (71)$$

where g_ℓ is made of products of homogeneous solutions (left-hand side) of Eq.(70). Hence the value of ℓ appears only through majorations of g_ℓ , which turn out to be quite similar to Eqs.(7). Criteria for majorations are again satisfied by the existence of a converging integral $\int_0^\infty dr' |U(r')| \exp(\Omega r')$, where Ω is any real, positive number. For FSG potentials, the result is trivial: all our considerations for s -waves extend to $\ell \neq 0$.

At this stage, the reader should be convinced that, besides a few pedagogic modifications, the only differences between the proofs offered by [8] and those described here consist of two elements, namely

- i) our potential U is complex and a symmetry such as, $j(-k^*) = j(k)^*$, is obviously lost; but most analyticity properties of f, φ , etc. depend on $|V|$ only for [8] and on $|U|$ in our case, hence almost identical derivations;
- ii) we used the ABC theorems [1] to locate the discrete spectrum at trivially rotated positions deduced from the discrete spectrum described by [8].

This scheme of proofs was made possible by the consideration of a slightly modified CSM Hamiltonian, as shown by Eq.(1). Locations of zeroes of the Jost function in the case of optical potentials were already studied by [9], for instance, in cases where potentials are more general than CSM ones. In [9] the ABC theorems could not be used and multiple poles were not excluded. Our present CSM case, however, through the argument of concentric circular arcs, clearly gives a transparent and safe solution for the simple nature and the locations of Jost function zeroes. Furthermore, while the poles now rotate by 2θ in the energy plane, the orientation of the cuts representing the continuum remains fixed. This makes it slightly easier than in [2] to find integration contours suitable for a proof of the resolution of the identity. Moreover, once the resonances have become square integrable and can be treated on the

same footing as bound states, the regularization advocated by [2] becomes unnecessary, hence a further simplification of practical calculations.

For coupled channels with thresholds, our main subject, the situation is more intricate. As discussed by [10], the mixture of channels with different values of ℓ may demand additional restrictions on the potentials, in order to ensure convergences at $r = 0$. Alternately, one may modify the coupled equations by means of counter-terms, see [10]. But these are technicalities and our P -representation is still available for two channels if one or both of them correspond(s) to $\ell > 0$; we can safely consider that the orbital momentum ℓ will bring no essential complication in a search for resolutions of the identity. Rather, two more serious difficulties arise for coupled channels with thresholds, because of topological structures of the problem. The first difficulty occurs already at the stage of two channels, namely, across each cut, the sign change of one momentum only. For instance, see Figure 1, the upper rim of the red cut symbolically reads $\{k, K\}$, while the lower rim reads $\{k, -K\}$. This complicates the description of the discontinuity across the cut, but not in an unacceptable manner. We solved this problem. Indeed, as shown in Appendix III, the discontinuity turns out to be only a (rank one) dyadic of regular solutions. The second difficulty, much more fundamental, arises as soon as there are three channels, see Figure 1. For the cuts merge at infinity. This makes a triple point for three channels. It seems difficult to find a generalization of our P -representation and create a contour where, in analogy with the two channel case, a “blue” line would start from $-\infty$ on the real axis, would be followed by a “red” line, followed in turn by a “yellow” line (for the third cut), finishing at $+\infty$ on the real axis. A triple point is obviously incompatible with such a naive scheme, and the cusp of the contour in the two channel case gives already a warning. Our pedestrian approach must stop after the two channel level. While completeness for more than two channels may occur [7], a generalization of a representation as simple, if possible, as the P -one is needed. Fortunately, two channel couplings make a large fraction of the cases to be studied in nuclear theory for multichannel resonances. Our P -representation facilitates numerical calculations.

This pedestrian CSM spectral resolution of the identity, while limited for two channels, still brings interesting further possibilities. It allows a study of effective forces in one of the channels if the second one is projected onto the first one by means of the usual algebra of Feshbach and Bloch-Horowitz projectors and propagators. Namely, in a standard notation with local potentials $U_{11}, U_{12}, \dots, U_{22}$, the first potential U_{11} becomes an effective, non local and energy dependent potential, $\mathcal{U} = U_{11} + U_{12}(K^2 - T - U_{22})^{-1}U_{21}$. Besides the non locality of $\mathcal{U}$, its energy dependence makes it complicated to build an identity resolution in channel 1 by means of eigenstates of $T + \mathcal{U}$. But that two channel identity operator, obtained from our two channel proof of completeness, is energy independent. It can be projected onto the first channel to study resonances of $\mathcal{U}$. This resolution of the identity may be of interest for detailed studies of phenomena resulting from delicate couplings between two channels, a case of frequent practical interest.

It can be concluded that, at least for a large class of problems, the CSM, while providing a safe representation of resonances, shows the advantage of spectrum completeness.

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- [1] Aguilar, J. and Combes, J.M., *Commun. Math. Phys.* **22** (1971) 269; Aguilar, J. and Balslev, E., *Commun. Math. Phys.* **22** (1971) 280
 - [2] Berggren, T., *Phys. Lett.* **B44** (1973) 23; Romo, W. J., *Nucl. Phys.* **A237** (1975) 275
 - [3] Myo, T., Ohnishi, A. and Katō, K., *Progr. Th. Phys.* **99** (1998) 801
 - [4] Myo, T., Katō, K., Aoyama, S. and Ikeda, K., *Phys. Rev. C* **63** (2001) 054313-1-10
 - [5] Weidenmüller, H.A., *Ann. of Phys.* **28** (1964) 60
 - [6] Ho, Y.K., *Phys. Reports* **99** (1983) 1; see also the reference list of the paper
 - [7] Moiseyev, N., *Phys. Reports* **302** (1998) 211; see also the reference lists of the paper
 - [8] Newton, R.G. *J. Math. Phys.* **1** (1960) 319
 - [9] S. Joffily *Nucl. Phys.* **A215** (1973) 301
 - [10] Newton, R.G. *Scattering Theory of Waves and Particles* (1982) Springer Verlag (New-York, Heidelberg, Berlin) ISBN 0-387-10950-1, 3-540-10950-1; see in particular chapters 12, 15 and 17.

Appendix I

Let $\tau(r) = \begin{pmatrix} \tau_1(r) \\ \tau_2(r) \end{pmatrix}$ be a test function. To define the Green's function we solve the inversion equation,

$$(E - \mathbf{H})\chi = -\tau, \quad (72)$$

where the components $\chi_1(r), \chi_2(r)$ of χ must be regular at the origin and “outgoing” at infinity. For this we let $\tilde{E}$ be absorbed by U_{22} and set χ to be a mixture of the regular solutions, $\varphi_{.1}(r), \varphi_{.2}(r)$, and outgoing Jost solutions, $f_{.1}(r), f_{.2}(r)$. The mixture coefficients are column vectors $A(r)$ and $B(r)$,

$$\chi = \Psi A + \mathbf{f}^+ B, \quad \Psi A' + \mathbf{f}^+ B' = 0. \quad (73)$$

We added here the usual constraint, $\Psi A' + \mathbf{f}^+ B' = 0$, where the superscript $'$ means d/dr . The constraint allows the usual, most simple form of Eq.(72),

$$\Psi' A' + \mathbf{f}^{+'} B' = -\tau, \quad \Psi A' + \mathbf{f}^+ B' = 0, \quad (74)$$

hence a possibility of solving for A' and B' . Indeed, left multiply the first of these Eqs.(74) by $\tilde{\mathbf{f}}^+$ and the second one by $-\tilde{\mathbf{f}}^{+'}$. Add the results. This gives, $\mathcal{W}[\mathbf{f}^+, \Psi] A' = -\tilde{\mathbf{f}}^+ \tau$, hence $A(r) = \int_r^\infty dr' \mathbf{W}_+^{-1} \tilde{\mathbf{f}}^+(r') \tau(r')$. In turn, left multiply the same first, resp. second of Eqs.(74) by $\tilde{\Psi}$, resp. $-\tilde{\Psi}'$, and obtain, $\mathcal{W}[\Psi, \mathbf{f}^+] B' = -\tilde{\Psi} \tau$, hence $B(r) = \int_0^r dr' \tilde{\mathbf{W}}_+^{-1} \tilde{\Psi}(r') \tau(r')$. This proves Eqs.(54).

While it is well known that the derivative of the Green's function must have discontinuities, it may be convenient to verify the continuity of $\mathbf{G}$ for $r = r'$, since two distinct formulae result from Eqs.(54), namely

$$\mathbf{G}(E, r, r) = \Psi(r) \mathbf{W}_+^{-1} \tilde{\mathbf{f}}^+(r), \quad \mathbf{G}(E, r, r) = \mathbf{f}^+(r) \tilde{\mathbf{W}}_+^{-1} \tilde{\Psi}(r). \quad (75)$$

These are transpose of each other. It is then enough to show that the first of these formulae, for instance, makes a symmetric matrix. Take advantage of Eq.(49) and find,

$$\mathbf{G}(E, r, r) = \mathbf{f}^- \mathbf{W}^{-1} \tilde{\mathbf{f}}^+ - \mathbf{f}^+ \tilde{\mathbf{W}}_+^{-1} \tilde{\mathbf{W}}_+ \mathbf{W}^{-1} \mathbf{W}_- \mathbf{W}_+^{-1} \tilde{\mathbf{f}}^+. \quad (76)$$

The first term in the right-hand side is symmetric, because of Eq.(53). The second term in the same right-hand side is also obviously symmetric, because of Eq.(51).

Appendix II

To make easier the calculation of residues at poles of the integrand in Eq.(58), we first notice that, in terms of the energy, $E = k^2 = K^2 + \tilde{E}$, we are dealing with a slightly simpler integral,

$$\mathcal{J}(r, r') = \frac{1}{2} \int dE \mathbf{G}(E, r, r'), \quad (77)$$

provided, naturally, that its contour in the E -representation and the definition of $\mathbf{G}$ along this contour match the definition, Eq.(58). Hence, residues will be obtained from derivatives d/dE in this Appendix.

In the same spirit, we shall use short notations in which the dependence of $\Psi, \mathbf{f}^+, \mathbf{f}^-, \mathbf{W}, \mathbf{W}_+, \mathbf{W}_-$ upon E along the E -contour will be understood. However, at those energies $E_\nu = \kappa_\nu^2$ where a pole occurs, we shall use an explicit subscript ν to specify that such quantities $\Psi, \dots, \mathbf{W}_-$ are evaluated at E_ν .

Poles occur because of $\mathbf{W}_+^{-1}$, hence, see Eq.(54), we must only find the residue,

$$\mathcal{R}_\nu = \lim_{E \rightarrow E_\nu} (E - E_\nu) \mathbf{W}_+^{-1}, \quad (78)$$

and form the matrix product, $\Psi_\nu \mathcal{R}_\nu \tilde{\mathbf{f}}_\nu^+$ and its transpose $\mathbf{f}_\nu^+ \tilde{\mathcal{R}}_\nu \tilde{\Psi}_\nu$.

Assume now that $\mathbf{W}_+$ is diagonalizable, with right, resp. left, eigenvectors Λ_1, Λ_2 , resp. Λ'_1, Λ'_2 , written as columns. Denote the corresponding eigenvalues by w_1, w_2 , hence,

$$\mathbf{W}_+ \Lambda_i = w_i \Lambda_i, \quad \tilde{\Lambda}'_i \mathbf{W}_+ = w_i \tilde{\Lambda}'_i, \quad i = 1, 2. \quad (79)$$

Biorthonormalize these eigenvectors with a symmetric (Euclidean-like) scalar product so that, again with δ as a Kronecker symbol,

$$\tilde{\Lambda}'_i \Lambda_j = \delta_{ij}, \quad \mathbf{W}_+^{-1} = \frac{\Lambda_1 \tilde{\Lambda}'_1}{w_1} + \frac{\Lambda_2 \tilde{\Lambda}'_2}{w_2}. \quad (80)$$

The case of non diagonalizability where such manipulations are not available is so special that we can discard it for our generic study.

Now choose w_1 to be that eigenvalue vanishing when $E \rightarrow E_\nu$. Then only the term, $\Lambda_1 \tilde{\Lambda}'_1/w_1$, is divergent, hence

$$\mathcal{R}_\nu = \lim (E - E_\nu) \Lambda_1 \tilde{\Lambda}'_1/w_1. \quad (81)$$

Since only $1/w_1$ diverges, the same residue reads,

$$\mathcal{R}_\nu = \Lambda_{1\nu} \tilde{\Lambda}'_{1\nu} r_\nu, \quad r_\nu \equiv \lim (E - E_\nu)/w_1 = 1/(dw_1/dE)|_\nu. \quad (82)$$

Because of the biorthonormalization of the eigenvectors, see Eqs.(80), and because of the stationarity of eigenvalues with respect to infinitesimal variations of such eigenvectors, the derivative, $(dw_1/dE)|_\nu$, reads,

$$(dw_1/dE)|_\nu = \tilde{\Lambda}'_{1\nu} (d\mathbf{W}_+/dE)|_\nu \Lambda_{1\nu}, \quad (83)$$

where it may be stressed that $\mathbf{W}_+$ is still considered as a function of E , but that the mixture of its matrix elements is defined by fixed vectors $\Lambda_{1\nu}, \tilde{\Lambda}'_{1\nu}$. This brings us back to the derivative of a Wronskian, with manipulations most similar to those of the one dimensional case.

Consider indeed the states, $\xi \equiv \Psi \Lambda_{1\nu}$ and $F \equiv \mathbf{f}^+ \Lambda'_{1\nu}$. These depend upon E because of Ψ and $\mathbf{f}$, respectively, while $\Lambda_{1\nu}$ and $\Lambda'_{1\nu}$, eigenvectors at E_ν , are fixed. At E_ν , the condition, $\mathbf{W}_+ \Lambda_{1\nu} = \mathcal{W}[\mathbf{f}^+, \xi] = 0$, means that ξ , a regular solution, is also “outgoing”. Simultaneously, the condition, $\Lambda'_{1\nu} \mathbf{W}_+ = \mathcal{W}[F, \Psi]$ means that F , an outgoing solution, becomes regular. Furthermore, still for $E = E_\nu$, the condition, $\Lambda'_{1\nu} \mathbf{W}_+ \Lambda_{1\nu} = 0$ also reads, $\mathcal{W}[F, \xi] = 0$. The latter result indicates that F and ξ are proportional to each other, despite the possibility of vanishing Wronskians for independent solutions inside the $\mathbf{f}^+$ subspace or inside the Ψ subspace. Indeed, F and ξ cannot be independent; for, if they were independent, the two subspaces would coincide completely, and the second eigenvalue, w_2 , would also vanish. Such an abnormal case being excluded, there exists therefore a number c such that $F = c \xi$.

We will now use “prime” superscripts $'$ for both derivatives with respect to E and r , with suitable subscripts. The right-hand side of Eq.(83) is a derivative of a Wronskian and reads,

$$d\mathcal{W}[F, \xi]/dE|_{E_\nu} = \tilde{F}'_E(E_\nu, r) \xi'_r(E_\nu, r) - \tilde{F}''_{Er}(E_\nu, r) \xi(E_\nu, r) + \tilde{F}(E_\nu, r) \xi''_{Er}(E_\nu, r) - \tilde{F}'_r(E_\nu, r) \xi'_E(E_\nu, r). \quad (84)$$

Since at E_ν there exists a number c such that $F = c \xi$, and, naturally, $F'_r = c \xi'_r$, this becomes,

$$d\mathcal{W}[F, \xi]/dE|_{E_\nu} = c^{-1} \left[\tilde{F}'_E(E_\nu, r) F'_r(E_\nu, r) - \tilde{F}''_{Er}(E_\nu, r) F(E_\nu, r) \right] + c \left[\tilde{\xi}(E_\nu, r) \xi''_{Er}(E_\nu, r) - \tilde{\xi}'_r(E_\nu, r) \xi'_E(E_\nu, r) \right]. \quad (85)$$

The two brackets $[]$ are Wronskians of F'_E, F and ξ, ξ'_E , respectively. But, since F'_E , resp. ξ'_E , is not driven by the Schrödinger equation of F , resp. ξ , such Wronskians are functions of r . As a preliminary step for their calculation, take the derivative with respect to E of the corresponding Schrödinger equations, transposed if necessary,

$$\tilde{F}'_E(E_\nu, r) [E_\nu + d^2/dr^2 - \mathbf{U}(r)] + \tilde{F}(E_\nu, r) = 0, \quad [E_\nu + d^2/dr^2 - \mathbf{U}(r)] \xi'_E(E_\nu, r) + \xi(E_\nu, r) = 0, \quad (86)$$

keeping in mind that $\tilde{E}$ is absorbed by U_{22} and that $\mathbf{H}$ is symmetric. Then right, resp. left multiply the first, resp. second of these Eqs.(86) by $F(E_\nu, r)$, resp. $\xi(E_\nu, r)$. One can replace $[E_\nu - \mathbf{U}(r)] F(E_\nu, r)$, resp. $\xi(E_\nu, r) [E_\nu - \mathbf{U}(r)]$, by $-d^2 F(E_\nu, r)/dr^2$, resp. $-d^2 \xi(E_\nu, r)/dr^2$. This gives, in a transparent, condensed notation,

$$\tilde{F}'_E F''_{r^2} - \tilde{F}'''_{Er^2} F = \tilde{F} F, \quad \tilde{\xi} \xi'''_{Er^2} - \tilde{\xi}''_{r^2} \xi'_E = -\tilde{\xi} \xi. \quad (87)$$

Now, obviously, the left-hand sides of Eqs.(87) are the derivatives, with respect to r , of the brackets of Eq.(85),

$$\tilde{F}'_E F''_{r^2} - \tilde{F}'''_{Er^2} F = d/dr \left[\tilde{F}'_E F'_r - \tilde{F}''_{Er} F \right], \quad \tilde{\xi} \xi'''_{Er^2} - \tilde{\xi}''_{r^2} \xi'_E = d/dr \left[\tilde{\xi} \xi''_{Er} - \tilde{\xi}'_r \xi'_E \right]. \quad (88)$$

Since both $\Im k$ and $\Im K$ are positive, all asymptotic forms, $\exp(ikr)$, ..., $ir \exp(iKr) dK/dE$ contributing to F and $\tilde{F}'_E$ vanish exponentially when $r \rightarrow \infty$. Hence, upon integrating between r and ∞ the first of Eqs.(88), one finds,

$$\tilde{F}'_E(E_\nu, r) F'_r(E_\nu, r) - \tilde{F}''_{Er}(E_\nu, r) F(E_\nu, r) = - \int_r^\infty dr' \tilde{F}(E_\nu, r') F(E_\nu, r'). \quad (89)$$

Furthermore, both $\xi(E_\nu, r)$ and $\xi'_{Er}(E_\nu, r)$ vanish when $r = 0$. For the former, this is because of the very definition of the φ 's by such a boundary condition. For the latter, this is because, again by the definition of ξ as a fixed superposition of the φ 's, its derivative with respect to r at the origin, $\xi'_r(E_\nu, 0)$, does not depend on E . Accordingly,

$$\tilde{\xi}(E_\nu, r) \xi''_{Er}(E_\nu, r) - \tilde{\xi}'_r(E_\nu, r) \xi'_E(E_\nu, r) = - \int_0^r dr' \tilde{\xi}(E_\nu, r') \xi(E_\nu, r'). \quad (90)$$

The sum of the two bracket contributions to the right-hand side of Eq.(85) finally gives,

$$d\mathcal{W}[F, \xi]/dE|_{E_\nu} = -c \int_0^\infty dr' \tilde{\xi}(E_\nu, r') \xi(E_\nu, r'). \quad (91)$$

It is unlikely that $\mathbf{U}$ would let the integral, $\int_0^\infty dr' \tilde{\xi}(E_\nu, r') \xi(E_\nu, r')$, a complex number, vanish for some of the roots E_ν . It can be safely concluded that, for most practical cases, the derivative, $d\mathcal{W}[F, \xi]/dE|_{E_\nu}$, does not vanish when the determinant, $\det \mathbf{W}_+$, itself vanishes. In case of a serious doubt, a numerical, and sufficiently precise calculation of the right-hand side of Eq.(91) should remove some ambiguity. Generically, roots of $\det \mathbf{W}_+$ will be isolated and a discrete spectrum will be found with square integrable states, unifying both bound states and narrow resonances.

Appendix III

The reduction to rank 1 of the discontinuity, $\Psi(r) \tilde{\mathbf{D}}(r')$, (or $\mathbf{D}(r) \tilde{\Psi}(r')$) of $\mathbf{G}(r, r')$ across an arbitrary n -th cut among N cuts, $0 < n \leq N$, can be formalized as follows. First notice that the N components of the n -th Jost solution make the n -th row of the $N \times N$ matrix $\tilde{\mathbf{f}}^{up}(r')$ containing the Jost solutions defined for the upper rim of the cut. As long as all the Jost solutions exist, under suitably fast decrease properties of $\mathbf{U}$, to compensate for negative $\Im k$'s, it makes no difference whether the corresponding momenta are considered as k 's or $-k$'s, since they are anyhow complex except k_n . Then the only momentum which changes sign across the cut is that real momentum k_n driving the source term of this n -th solution. Hence the solution matrix $\tilde{\mathbf{f}}^{lo}(r')$ that describes the solutions on the lower rim is the same as $\tilde{\mathbf{f}}^{up}(r')$, except for its n -th row.

The relevant Wronskian on the upper rim is nothing but, $\mathbf{W}_{up} = \tilde{\mathbf{f}}^{up}(0)$. For the lower rim, we find, $\mathbf{W}_{lo} = \tilde{\mathbf{f}}^{lo}(0)$. They again differ by their n -th row only. The difference, $\tilde{\mathbf{D}}(r') = [\tilde{\mathbf{f}}^{up}(0)]^{-1} \tilde{\mathbf{f}}^{up}(r') - [\tilde{\mathbf{f}}^{lo}(0)]^{-1} \tilde{\mathbf{f}}^{lo}(r')$, obviously vanishes at the origin, $\tilde{\mathbf{D}}(0) = 0$. This is because on each rim the relevant $\tilde{\mathbf{f}}(0)$ gets left multiplied by its respective inverse $[\tilde{\mathbf{f}}(0)]^{-1}$, hence at the origin $\tilde{\mathbf{D}}(0)$ is nothing but the difference of two unit matrices. Because of such a boundary condition, it can already be concluded that $\tilde{\mathbf{D}}(r')$ belongs to the realm of regular solutions, for any number of channels and every channel.

Furthermore, it is not surprising that the difference $\tilde{\mathbf{D}}$ between such strongly similar products reduces to a rank 1 matrix. We verified this by explicit algebraic calculations for $N = 2$ and $N = 3$. But the generalization theorem for any N is easy. Indeed, in the space of $N \times N$ matrices, let $|n\rangle$ be that vector with vanishing components except a 1 at position n . The rank one dyadic, $\mathcal{D} = |n\rangle\langle n|$, is obviously a projector. Furthermore, if $\mathbf{B}$ and $\mathbf{B}'$ are any $N \times N$ matrices, then the products, $\mathcal{D}\mathbf{B}$ and $\mathcal{D}\mathbf{B}'$ are vanishing matrices except for their n -th row. (Actually, this is nothing but a trivial way to extract their n -th row from $\mathbf{B}$ and $\mathbf{B}'$.) Let $\mathbf{A}$ be an invertible matrix. Let $\mathbf{A}'$ be another matrix, invertible or not. Any n -th row modification of $\mathbf{A}$ and $\mathbf{A}'$ can be parameterized by forms such as, for instance, $\mathbf{A} - \mathcal{D}\mathbf{B}$ and $\mathbf{A}' + \mathcal{D}\mathbf{B}'$. We retain only those cases where $\mathbf{A} - \mathcal{D}\mathbf{B}$ is invertible, like $\mathbf{A}$, and investigate the difference,

$$[\mathbf{A} - \mathcal{D}\mathbf{B}]^{-1} [\mathbf{A}' + \mathcal{D}\mathbf{B}'] - \mathbf{A}^{-1} \mathbf{A}'. \quad (92)$$

The cut discontinuity $\tilde{\mathbf{D}}$ is a special case of this difference, Eq.(92). This difference can be rewritten as,

$$[\mathbf{A} \mathbf{1} - \mathbf{A} \mathbf{A}^{-1} \mathcal{D} \mathbf{B}]^{-1} [\mathbf{A}' + \mathcal{D} \mathbf{B}'] - \mathbf{A}^{-1} \mathbf{A}' = [\mathbf{1} - \mathbf{A}^{-1} \mathcal{D} \mathbf{B}]^{-1} \mathbf{A}^{-1} [\mathbf{A}' + \mathcal{D} \mathbf{B}'] - \mathbf{A}^{-1} \mathbf{A}', \quad (93)$$

where $\mathbf{1}$ is the unit $N \times N$ matrix. A trivial expansion, $[\mathbf{1} - \mathbf{A}^{-1}\mathcal{D}\mathbf{B}]^{-1} = 1 + \sum_{i=1}^{\infty} (\mathbf{A}^{-1}\mathcal{D}\mathbf{B})^i$, gives the following result for the difference under study,

$$\left[\sum_{i=1}^{\infty} (\mathbf{A}^{-1}\mathcal{D}\mathbf{B})^i \right] \mathbf{A}^{-1}\mathbf{A}' + \mathbf{A}^{-1}\mathcal{D}\mathbf{B}' + \left[\sum_{i=1}^{\infty} (\mathbf{A}^{-1}\mathcal{D}\mathbf{B})^i \right] \mathbf{A}^{-1}\mathcal{D}\mathbf{B}'. \quad (94)$$

It is trivial to left factorize from this result the common factor, $\mathbf{A}^{-1}\mathcal{D}$, which is a dyadic. This proves the rank 1.

Appendix IV

For $k = 0$ the integral equation (3) is replaced by,

$$f(0, r) = 1 + \int_r^{\infty} dr' (r' - r) U(r') f(0, r'). \quad (95)$$

The iteration method then gives,

$$f(0, r) = \sum_{n=0}^{\infty} f_n(0, r), \quad f_0(0, r) = 1, \quad f_n(0, r) = \int_r^{\infty} dr' (r' - r) U(r') f_{n-1}(0, r). \quad (96)$$

The following majorations hold,

$$|f_n(0, r)| \leq \int_r^{\infty} dr' (r' - r) |U(r')| |f_{n-1}(0, r)|, \quad (97a)$$

$$|f_n(0, r)| \leq \int_r^{\infty} dr_1 (r_1 - r) |U(r_1)| \int_{r_1}^{\infty} dr_2 (r_2 - r_1) |U(r_2)| \dots \int_{r_{n-1}}^{\infty} dr_n (r_n - r_{n-1}) |U(r_n)|, \quad (97b)$$

$$|f_n(0, r)| \leq \left[\int_r^{\infty} dr' (r' - r) |U(r')| \right]^n / n!. \quad (97c)$$

and finally,

$$f(0, r) \leq \sum_{n=0}^{\infty} \frac{1}{n!} [\alpha_1(r)]^n = \exp[\alpha_1(r)], \quad \alpha_1(r) = \int_r^{\infty} dr' (r' - r) |U(r')| \leq \int_0^{\infty} dr' r' |U(r')|. \quad (98)$$

If $\alpha_1(r) < \infty$, the series defining $f(0, r)$ converges absolutely. And because each of its terms is analytic, the sum is also analytic.