Unified Approach to Dense Matter

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We apply the Skyrme model to dense hadronic matter, which provides a unified approach to high density, valid in the large $N_c$ limit. In our picture, dense hadronic matter is described by the classical soliton configuration with minimum energy for the given baryon number density. By incorporating the meson fluctuations on such ground state we obtain an effective Lagrangian for meson dynamics in a dense medium. Our starting point has been the Skyrme model defined in terms of pions, thereafter we have extended and improved the model by incorporating other degrees of freedom such as dilaton, kaons and vector mesons.

1. Introduction

At high temperature and/or density, hadrons are expected to possess properties that are very different from those at normal conditions. Understanding the properties of hadrons in such extreme conditions is currently an important issue not only in nuclear and particle physics but also in many other related fields such as astrophysics. Data from the high energy heavy ion colliders, astronomical observations on compact stars and some theoretical developments have shown that the phase diagram of hadronic matter is far richer and more interesting than initially expected. Lattice QCD calculations have been carried out successfully at high temperature, however similar calculations at high density have not yet been possible. Theoretical developments have unveiled such interesting QCD phases as color superconductivity. Moreover effective theories can be derived for these extreme conditions, using macroscopic degrees of freedom, by matching them to QCD at a scale close to the chiral scale $\Lambda_{\chi} \sim 4\pi f_\pi \sim 1$ GeV.

We have followed a different path to dense matter studies\textsuperscript{1} by using as our starting point a model Lagrangian, in the spirit of Skyrme, which describes hadronic matter and meson dynamics respecting the symmetries of QCD. The parameters of the model are fixed by meson dynamics at zero baryon number density. À la Skyrme, baryons arise from a soliton solution, the skyrmion, with the topological winding number describing the baryon number. In our scheme dense matter is approximated by a system of skyrmions with a given baryon number density whose ground state arises as a crystal configuration\textsuperscript{11}. Starting from this ground state our approach provides insight on the intrinsic in-medium dependence of meson dynamics. We have studied (i) the in-medium properties of the mesons and (ii) the role of the other degrees of freedom besides pions in the description of matter as it becomes denser.

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\textsuperscript{1}Due to the limitation in length, we only refer to our work, on which the talk is based, and to a few others whose results are quoted explicitly.
2. Model Lagrangian

The original Skyrme model Lagrangian \[ L \] reads

\[
L \chi = -\frac{f^2}{4} \text{Tr}(L_{\mu}L^\mu) + \frac{1}{32\pi^2} \text{Tr}[L_{\mu}, L_{\nu}]^2 \\
+ \frac{f^2 m_\pi^2}{4} \text{Tr}(U + U^\dagger - 2), 
\]

where \( L_{\mu} = U^\dagger \partial_\mu U \) and \( U = \exp(i \vec{\tau} \cdot \vec{\pi}) \in SU(2) \) is a nonlinear realization of the pion fields, \( f_\pi \) the decay constant and \( m_\pi \) the pion mass. The second term with \( e \), the Skyrme parameter, was introduced to stabilize the soliton solution.

The dilaton field \( \chi \) can be incorporated in the model to make it consistent with the scale symmetry with respect to \( U \rightarrow U \exp(i \vec{\tau} \cdot \vec{\mu}) \). The Lagrangian \( L \chi \) then becomes modified as

\[
L_{\chi} = -\frac{f^2}{4} \left( \frac{\chi}{f_\chi} \right)^2 \text{Tr}(L_{\mu}L^\mu) \\
+ \frac{1}{32\pi^2} \text{Tr}[L_{\mu}, L_{\nu}]^2 \\
+ \frac{f^2 m_\pi^2}{4} \left( \frac{\chi}{f_\chi} \right)^3 \text{Tr}(U + U^\dagger - 2) \\
+ \frac{f_\chi}{2} \partial_\mu \chi \partial^\mu \chi - V(\chi), 
\]

where \( \chi = f_\chi + \vec{\pi} \cdot \vec{\pi} \) and \( f_\chi \) is the dilaton mass and \( f_\chi \) its decay constant.

The vector mesons, \( \rho \) and \( \omega \), can be included into the model as dynamical gauge bosons of a hidden local gauge symmetry which requires the doubling of the degrees of freedom as \( U = \xi_L \xi_R \). One of such Lagrangian is, for example \( L_{\pi, \rho, \omega} \)

\[
L_{\pi, \rho, \omega} = \frac{f_\pi^2}{4} \left( \frac{\chi}{f_\chi} \right)^2 \text{Tr}(\xi_L D_\mu \xi_L^\dagger - \xi_R D_\mu \xi_R^\dagger)^2 \\
+a \frac{f_\rho^2}{4} \left( \frac{\chi}{f_\chi} \right)^2 \text{Tr}(\xi_L D_\mu \xi_L^\dagger + \xi_R D_\mu \xi_R^\dagger)^2 \\
+ \frac{f_\omega^2 m_\omega^2}{4} \left( \frac{\chi}{f_\chi} \right)^3 \text{Tr}(\xi_L^\dagger \xi_R + \xi_R^\dagger \xi_L - 2) \\
+ \frac{N_c g}{2} \omega_\mu B^\mu - \frac{1}{4} \tilde{\rho}_{\mu \nu} \tilde{\rho}^{\mu \nu} - \frac{1}{4} \omega_{\mu \nu} \omega^{\mu \nu} \\
+ \frac{f_\chi}{2} \partial_\mu \chi \partial^\mu \chi - V(\chi), 
\]

with \( D_\mu = \partial_\mu - \frac{i}{2} \vec{\tau} \cdot \vec{\rho} - \frac{1}{2} \omega_\mu \), \( \tilde{\rho}_{\mu \nu} = \partial_\mu \tilde{\rho}_\nu - \partial_\nu \tilde{\rho}_\mu + g \tilde{\rho}_\mu \times \tilde{\rho}_\nu \), \( \omega_{\mu \nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \), where \( B_\mu \) is the topological baryon number current. The quartic Skyrme term of \( L \) is not present, because its stabilizing role is played here by the vector mesons.

One may decouple the vector mesons and/or dilaton field from the other by making the corresponding particles infinitely heavy. In this limit, the dilaton field is frozen to its vacuum value \( \chi = f_\chi \) and rho mesons are constrained to

\[
i \vec{\tau} \cdot \vec{\rho}_\mu = \frac{1}{g} (\partial_\mu \xi_L \xi_L^\dagger + \partial_\mu \xi_R \xi_R^\dagger) 
\]

as kind of composite particles made of two pions. Then, the model Lagrangian \( L \) reduces to \( L_{\chi} \) or \( L_{\pi, \rho, \omega} \), where the kinetic energy term of the rho vectors becomes the Skyrme term.

3. Dense Skyrmion Matter

These nonlinear meson Lagrangian supports soliton solutions, skyrmions, carrying nontrivial topological winding numbers. Once we accept Skyrmie’s conjecture of interpreting the winding number as the baryon number, we may describe a dense baryonic matter as a system made of many skyrmions.

In the classical picture, the lowest energy state of the multi-skyrmion system is a crystal and there has been intensive work in late 80’s with the model containing pions only \( L \). Well-separated two skyrmions have lowest energy when they are relatively rotated in the isospin space about an axis perpendicular to the line joining their centers. We can imagine that the lowest energy state of skyrmion matter in a relatively low density is in an FCC(face centered cubic) crystal phase where well localized single skyrmions are arranged on each lattice site in a way that 12 nearest skyrmions have such lowest energy relative orientations. At higher density, on the other hand, skyrmion tails start overlapping and the system undergoes a phase transition to a more symmetric configuration the so called the “half-skyrmion” cubic crystal. There, one half of the baryon number carried by the single skyrmion is concentrated at original FCC site where \( U = -1 \) while the other is concentrated on the links where \( U = +1 \). Now, the system has an additional symmetry with respect to \( U \rightarrow -U \), which results in
At the density $\rho$, the average value of $m$ in the pseudogap phase for a while but the system changes to the pseudogap phase. As the density increases, the average value of $\chi/f_{\pi}$ drops quickly and reaches zero at the density $\rho_c$, where the system changes to the pseudogap phase. We call this phase as “pseudogap” phase to distinguish it from the genuine chiral symmetry restored phase, where the chiral circle shrinks to a point at $\sigma = 0, \phi = 0$.

In the model with dilaton field which plays a role of the “radial” field for $U$, the restriction on chiral radius becomes released. The pseudogap phase still remains as a transient process unless the dilaton field vanishes, while the chiral circle still has a fixed radius $f_{\pi}$.

The vector mesons (especially the omega meson) also play important roles in dense skyrmion matter. Numerical results obtained in various models are presented in Figure 2. In the $\pi\rho$ model, as the density of the system increases (and the lattice constant decreases), the system energy per baryon, $E/B$, changes slightly. Its value is close to the energy of a single skyrmion up to somewhat high densities. The main reason for this is that the size of the skyrmion is very small because of the absence of strong repulsive terms in the model. Thus, the skyrmions in the lattice interact only at very high densities where their tails overlap. In the $\pi\rho\chi$ model without $\omega$, the dilaton field plays an important role. Skyrmion matter undergoes an abrupt phase transition at the density where the expectation value of the dilaton field vanishes $\langle \chi \rangle = 0$.

In the $\pi\rho\omega\chi$ model, the situation changes dramatically. Above all, $\omega$ provides a strong repulsion which inflates each single skyrmion. The tails of the bigger skyrmions overlap providing attraction to the system in the intermediate range.

In both the $\pi\rho\omega$ and the $\pi\rho\omega\chi$ models, at high density, the interaction reduces $E/B$ to 85% of the $B = 1$ skyrmion mass. This value should be compared with 94% in the $\pi\rho$ model. In the $\pi\rho\chi$-model, $E/B$ goes down to 74% of the $B = 1$ skyrmion mass, but in this case it is due to the dramatic behavior of the dilaton field. On the other hand, the omega suppresses the role of the dilaton field. It could provide only a small attraction at intermediate densities. Moreover, the phase transition towards its vanishing expectation value, $\langle \chi \rangle = 0$ does not take place. Instead, its value grows at high density!

The reason for this can be found in the role played by omega in $\chi/f_{\pi}$. In the static configuration, omega produces a potential, whose source is the baryon number density, which mediates the self-interaction energy of the baryon number distribution. Thus, unless it is screened properly by the omega mass, the periodic source filling infinite space will lead to an infinite self-energy. To reduce the energy of the system, the effective $\omega$ mass must grow at high density, for which $\chi$ must grow too. Note the factor $(\chi/f_{\pi})^2$ in the omega mass term in Lagrangian (3).
On the other hand, these classical crystalline structures are quite far from normal nuclear matter which is known to be a Fermi liquid at low temperature. In order for skyrmion matter to be identified with nuclear matter we have to quantize and thermalize the classical system. Since it is a system of solitons in a meson field theory, it is not sufficient to quantize the meson fluctuations. We need to introduce and quantize proper collective variables not only to complement the broken symmetries of the whole skyrmion system but also to describe the dynamics of the single skyrmions in order to obtain a realistic picture of nuclear matter. For extended objects, we may need an infinitely large number of dynamical variables such as the positions of their center of mass, their relative orientations, their sizes, their deformations, etc. Among them, those degrees of freedom that describe translations and rotations of the single skyrmion play the dominant role at low energy. Thus, we need at least 6 variables for each skyrmion. For a multi-skyrmion system, the simplest way of introducing collective variables for the position and orientation of each single skyrmion is through the use of the product Ansatz, the old idea of Skyrme[7]. There, a multi-skyrmion solution can be approximated by products of single skyrmion solutions centered at the corresponding positions and rotated to have the corresponding orientation. However, the product Ansatz works well only when the skyrmions are sufficiently separated. Furthermore, due to the non-commutativity of the matrix products, it is difficult to use it in multi-skyrmion systems.

Another scheme which can be used to study multi-skyrmion systems is the Antiyah-Manton Ansatz [12]. In this scheme, skyrmions of baryon number $N$ are obtained by calculating the holonomy of Yang-Mills instantons of charge $N$, which has been used in describing successfully few-nucleon systems and also nuclear matter. One advantage of the Atiyah-Manton ansatz is that it provides a natural framework to introduce the proper dynamical variables for the skyrmions through the parameters describing the multi-instanton configuration. Contrary to the non-commutative product ansatz, some multi-instanton solutions are given in a commutative manner. Furthermore, multi-instanton solutions have been investigated widely and many useful solutions have been found.

We have tried this idea in Ref.[1]. 'tHooft ansatz on multi-instanton solution is adopted to produce the FCC instanton crystal. To avoid the divergence coming from the infinite number of instantons and to incorporate the relative orientations to each instantons, the ansatz is slightly modified. Then, the Antiyah-Manton procedure is applied to transform the instanton crystal into a skyrmie crystal. As expected, the resulting skyrmion crystal is FCC for low baryon number density. At high density, however, it becomes a half-skyrmion CC (approximately).

One great advantage of this procedure to generate the skyrmion crystal is that it is really made up of single objects located at specified positions and with specified rotations. Now, we can, for example, move a single skyrmion and investigate how the system changes. Shown in Fig.3 is the potential energy $V(d)$ when a single skyrmion is moved away by a distance $d$ from its stable position. Two extreme cases are shown. In the case of a dense system ($L_{FCC} = 5.0$), the energy changes abruptly. For small $d$, it is almost quadratic in $d$. It implies that the dense system is in the crystal phase. On the other hand, in the case of dilute system ($L_{FCC} = 10.0$), the system energy...
remains almost constant up to some distance $d$, which implies that the system is in a gas or liquid phase. If we allow all the variables to vary freely to seek the minimum energy configuration, the system will end up in a disordered phase, in which a few skyrmions will form a finite cluster. Furthermore, we will be able to investigate the thermal properties of the skyrmion system and quantize those collective variables.

4. Mesons in Dense Matter

By using the same model Lagrangian defined at zero baryon number density, we may investigate the properties of the mesons in dense matter. The Lagrangians (1)-(3) are originally constructed to describe the meson dynamics in vacuum, which is defined as

$$U = 1, \quad \chi = f \chi, \quad \tilde{\rho}_\mu = \omega_\mu = 0.$$  

The fluctuations on top of this vacuum can describe the corresponding particles in zero baryon number space, for which we determine the physical parameters of the model Lagrangian, such as the particle masses, decay constants and coupling constants.

Now, as we have described in the previous section, we have another classical background configuration for the matter with non vanishing baryon number density. Let’s denote the solution as $U_{(0)}$, $\chi_{(0)}$, $\tilde{\rho}_{\mu(0)}$ and $\omega_{\mu(0)}$, respectively. We can incorporate the meson fluctuation on top of this background as

$$U = \sqrt{U_\pi U_{(0)}} \sqrt{U_\pi^*}, \quad \chi = \chi_{(0)} + \chi^*, \quad \tilde{\rho}_\mu = \tilde{\rho}_{\mu(0)} + \tilde{\rho}_{\mu}^*, \quad \omega_\mu = \omega_{\mu(0)} + \omega^\mu,$$

where the ‘starred’ fields are describing the corresponding particles in medium.

We will illustrate the basic strategy of the approach by using the simplest model with only pions. Substituting eqs. (6) into the Lagrangian (1) and keeping the terms up to the second order in the fluctuating fields, we obtain

$$L^* = \frac{1}{2} G_{ab}(\vec{r}) \partial_\mu \pi^*_a \partial^\mu \pi^*_b + \frac{1}{2} m_{\pi}^* \sigma(\vec{r}) \pi^*_a \pi^*_a + \cdots,$$  

where $G_{ab}(\vec{r})$ and $\sigma(\vec{r})$ are the background potentials provided by the dense skyrmion matter. For the fluctuating pions in free space, both potentials are just 1. This Lagrangian tells us how the properties of the pions change in the dense matter. For example, $G_{ab}(\vec{r})$ in front of the pion kinetic term can be absorbed into the wavefunction renormalization, which can be interpreted as a ratio between the local effective pion decay constant in dense medium to that in free space. The pion mass term gets a similar correction from dense matter. To the first order in the background potentials, we can translate (7) into an effective Lagrangian for the pions in dense medium with in-medium physical parameters as

$$\frac{f_\pi^*}{f_\pi} = \sqrt{\langle G_{aa} \rangle}, \quad \frac{m_\pi^*}{m_\pi} = \sqrt{\frac{\langle \sigma \rangle}{\langle G_{aa} \rangle}}.$$  

In Figure 4 we show the ratios of the in-medium parameters of pions relative to its free-space values. The pion mass is almost constant at low density but decreases as the density increases to much higher values. The pion decay constant drops quite fast but after some density it stays with value $\sim 2/3$ indicating that the system is in the pseudogap phase mentioned in Section 3. A similar process can be applied after incorporating the dilaton [3], where the vanishing of $\chi$ can restore the chiral symmetry completely.

One may treat the interaction of the fluctuating fields with the background potentials in a more
systematic way. As an example, in Ref.[4] the in-medium pion velocity is studied, where the background interactions are taken into account up to the second order. The breakdown of the Lorentz symmetry due to the presence of medium makes the pion velocity deviate from that in free space.

5. Summary

We have developed a unified approach to dense matter within the Skyrme philosophy, where systems of baryons and mesons can be described by a single Lagrangian. In our approach dense baryonic matter is approximated by skyrmion matter in the lowest energy configuration for a given baryon number density. By incorporating in it fluctuating mesons we can get some insight on meson dynamics in a dense medium. Our approach enables us to study this dynamics beyond the first order in the baryon number density. One can continue to work in this direction by incorporating more degrees of freedom, by improving the way of treating matter beyond the crystal solution, and so on.

However, before closing the presentation, we must clearly lay down the scope of our work. We do not claim that the results obtained at present describe reality. The most fundamental problem we phase is that our “ground state” for matter is a crystal not a Fermi liquid. Our aim has been to assume a state for matter, given by a classical solution of a theory considered to be valid at large $N_c$, and have studied the implications for its excitations. Our work should be taken as representing the first step towards a more realistic treatment of a dense matter theory.

REFERENCES