Eccentric inflation and WMAP

Roman V. Buniy

Department of Physics and Astronomy, Vanderbilt University, Nashville, TN 37235, USA

For uniform arrangements of magnetic fields, strings, or domain walls (together with the cosmological constant and non-relativistic matter), exact solutions to the Einstein equations are shown to lead to a universe with ellipsoidal expansion. We argue the results can be used to explain some features in the WMAP data. The magnetic field case is the easiest to motivate and has the highest possibility of yielding reliable constraints on observational cosmology.

I. INTRODUCTION

The first year WMAP results [3–5] contain interesting large-scale features which warrant further attention [6, 7]. One glaring feature is the suppression of power at large angular scales (\(\theta \gtrsim 60^\circ\)), which is reflected most distinctly in the reduction of the quadrupole \(C_2\) and octopole \(C_3\). This effect was also seen in the COBE results [8, 9]. While one can argue that such an effect could just be statistical, it does not seem unreasonable to try to model such behavior by altering the cosmological model from the standard big bang plus inflation scenario. More intriguingly, the quadrupole \(C_2\) and octopole \(C_3\) are found to be aligned. In particular, the \(l = 2\) and \(3\) powers are found to be concentrated in a plane \(P\) inclined about 30° to the Galactic plane. In a coordinate system in which the equator is in the plane \(P\), the \(l = 2\) and \(3\) powers are primarily in the \(m = \pm l\) modes. The axis of this system defines a unique ray and supports an idea of power on the axis is suppressed relative to the power in the orthogonal plane. These effects seem to suggest one (longitudinal) direction may have expanded differently from the other two (transverse) directions in \(P\).

We approach the issue of global anisotropy of the Universe by a simple modification of the conventional Friedman-Robertson-Walker (FRW) model. To achieve this, one has to consider an energy-momentum tensor which either is spatially non-spherical or spontaneously becomes non-spherical. Such a situation could occur when magnetic fields [10] or cosmic defects [11] are present. Moreover, it is known that large scale magnetic fields exist in the universe, perhaps up to cosmological scales [10, 12].

As a modest step toward understanding the form, significance and implications of an asymmetric universe, we will modify the standard spherically symmetric FRW cosmology to a form with only planar symmetry [13]. Our choice of energy-momentum will result in non-spherical expansion from an initially spherical symmetric configuration: an initial co-moving sphere will evolve into an ellipsoid that can be either prolate or oblate depending on the choice of matter content.

For the sake of clarity, we first give general equations for cosmologies with planar symmetry, (The universe looks the same from all points but they all have a preferred direction.) and then investigate one case in detail—a universe filled with dust, uniform magnetic fields and cosmological constant. This is perhaps the most easily motivated, exactly solvable case to consider and it will give us a context in which to couch the discussion of other examples with planar symmetry and cases where planar symmetry is broken.

To set the stage, consider an early epoch in the universe at the onset of cosmic inflation, where strong magnetic fields have been produced in a phase transition [14–18]. Assuming the magnitude of the magnetic field and vacuum energy (\(\Lambda\)) densities initially are about the same, we will find that eventually \(\Lambda\) dominates. It was estimated [16] that the initial magnetic field energy produced in the electroweak phase transition was within an order of magnitude of the critical density. Other phase transitions may have even higher initial field values [17, 18], or high densities of cosmic defects. Hence, it is not unphysical to consider a universe with B-fields and \(\Lambda\) of comparable magnitudes. If the B-fields are aligned in domains, then some degree of inflation is sufficient to push all but one domain outside the horizon.

*The calculations presented in this talk (at Coral Gables 2003, Fort Lauderdale, FL) appeared in short form in Ref. [1] and a full exposition is to appear in Ref. [2].

†Electronic address: roman.buniy@vanderbilt.edu
II. PLANAR GEOMETRY

To make the simplest directionally anisotropic universe, we modify FRW spherical symmetry of space-time into planar symmetry; the most general form for the metric then is [13] \( (g_{\mu\nu}) = \text{diag}(1, -e^{2a}, -e^{2b}, -e^{2c}) \), where \( a \) and \( b \) are functions of \( t \) and \( z \); the \( xy \)-plane is the plane of symmetry. We also impose translational symmetry along the \( z \)-axis; the functions \( a \) and \( b \) now depend only on \( t \).

(Examples of planar-symmetric spaces include space uniformly filled with either uniform magnetic fields, static aligned strings, or static stacked walls; i.e., the defects are at rest with respect to the cosmic background frame. This situation with defects is artificial or at best contrived, but could perhaps arise in brane world physics where static walls could be static or walls beyond the horizon could be connected by static strings. We will not pursue these details here. Of course, to these any spherically-symmetric contributions can be added: vacuum energy, matter, radiation.)

To support the symmetry of space-time, the energy-momentum tensor for the matter has to have the same symmetry, \( (T^a_{\mu\nu}) = (8\pi G)^{-1}\text{diag}(\xi, \eta, \eta, \zeta) \). Here the energy density \( \xi \), transversal \( \eta \) and longitudinal \( \zeta \) tension densities are functions of time only. The corresponding Einstein equations are

\[
\dot{a}^2 + 2\dot{a}b = \xi, \quad \dot{b} = \eta, \quad 2\dot{a} + 3a^2 = \zeta. \tag{1-3}
\]

We also need the equation expressing covariant conservation of the energy-momentum [a direct consequence of Eqs. (1)–(3)]:

\[
\dot{\xi} + 2\dot{a}(\xi - \eta) + \dot{b}(\xi - \zeta) = 0. \tag{4}
\]

For an adiabatic process, the entropy in a comoving volume is conserved. For the matter with the equation of state \( p = w\rho \), this leads to [20] \( \rho = \rho_i e^{-(1+w)(2a+b)} \) and, as a result, the anisotropic part of energy-momentum is conserved as well:

\[
\dot{\xi} + 2\dot{a}(\xi - \eta) + \dot{b}(\xi - \zeta) = 0. \tag{5}
\]

For the cases of magnetic field, strings and walls, the set of quantities \((\xi, \eta, \zeta)\) take, respectively, the values \((\epsilon, -\epsilon, \epsilon)\), \((\epsilon, 0, \epsilon)\) and \((\epsilon, \epsilon, 0)\).

Various relations and inequalities based on energy conditions, many of which involve the eccentricity of the expansion can be derived for general \( \xi, \eta \) and \( \zeta \); for details, see Ref. [2].

III. MAGNETIC FIELD

Extending the analysis of Ref. [1], we now present the exact solution for the case of cosmological constant, dust \((w = 0)\) and uniform B-field [2].

Conservation of the anisotropic part of the energy-momentum, Eq. (5), gives \( a = \frac{1}{3}\ln(\epsilon_i/\epsilon) \). Using this result in Eq. (3), we find that \( \epsilon(t) \) is given implicitly by

\[
t - t_i = \frac{1}{4} \int_{\epsilon}^{\epsilon_i} \frac{\lambda \epsilon^2 + \frac{2}{3} \epsilon_i^{-\frac{2}{3}} (4\epsilon_i + \rho_i) \epsilon_i^{\frac{1}{3}} - \epsilon^3}{\epsilon} \frac{\lambda \epsilon^2 - \epsilon_i^{-\frac{2}{3}} (4\epsilon_i + \rho_i) \epsilon_i^{\frac{1}{3}} + 3\epsilon_i^{\frac{2}{3}}}{\lambda \epsilon^2 + \epsilon_i^{\frac{2}{3}}} + \ln[1 + F_m(\epsilon)], \tag{6}
\]

Eq. (1) and \( \rho = \rho_i e^{-(2a+b)} \) give

\[
b = \frac{1}{2} \ln \left( \frac{\lambda e^{-\frac{2}{3}} + (4\epsilon_i + \rho_i)\epsilon_i^{-\frac{2}{3}} e^{\frac{2}{3}} - 3\epsilon_i^{\frac{2}{3}}}{(\lambda + \rho_i)\epsilon_i^{-\frac{2}{3}} + \epsilon_i^{\frac{2}{3}}} + \ln[1 + F_m(\epsilon)] \right), \tag{7}
\]

where

\[
F_m(\epsilon) = \frac{3\rho_i}{8\epsilon_i} \left( 1 + \frac{\lambda + \rho_i}{\epsilon_i} \right)^{\frac{2}{3}} \int_{\epsilon/\epsilon_i}^{1} dx x^{-\frac{2}{3}} \left[ \frac{\lambda}{\epsilon_i} + \left( 4 + \frac{\rho_i}{\epsilon_i} \right) x^{\frac{2}{3}} - 3x \right]^{-\frac{2}{3}}. \tag{8}
\]

A careful inspection of the above solution shows that space is oblate, \( e^{a-b} \geq 1 \), and its oblateness monotonically increases from its initial value (unity) to its asymptotic value. More magnetic field increases the anisotropy; more matter reduces anisotropy, but neither can change an oblate ellipsoid into a prolate one.
The asymptotics for large $t$ are as follows:

$$
\epsilon \sim \epsilon_i e^{-4(\lambda/3)^{1/2}(t-t_i+\tau)},
$$

(9)

$$
a \sim (\lambda/3)^{1/2}(t-t_i+\tau),
$$

(10)

$$
b \sim (\lambda/3)^{1/2}(t-t_i+\tau) - \frac{1}{2}\ln [1 + (\epsilon_i + \rho_i)/\lambda] + \ln [1 + F_m(0)],
$$

(11)

$$
\rho \sim \rho_i [1 + (\epsilon_i + \rho_i)/\lambda]^{\frac{3}{2}} [1 + F_m(0)]^{-1} e^{-3\lambda^{1/2}(t-t_i+\tau)},
$$

(12)

where

$$
\tau = \frac{1}{4} \int_0^{\epsilon_i} \epsilon^\prime \left( \frac{1}{\epsilon_i} \right)^{1/2} - \left[ \frac{1}{2} \lambda \epsilon^2 + \frac{4}{3} \epsilon_i^{-2} (4\epsilon_i + \rho_i) \epsilon^{\frac{3}{2}} - \epsilon^3 \right]^{-1/2}
$$

(13)

IV. STRINGS

From the conservation equation there follows $a = \frac{1}{2} \ln (\epsilon_i/\epsilon)$, and then Eq. (3) gives $\epsilon(t)$ implicitly via

$$
t - t_i = \frac{1}{2} \int_{\epsilon_i}^\epsilon \epsilon^\prime \left( \frac{1}{\epsilon_i} \right)^{1/2} \left[ \frac{1}{3} \lambda \epsilon^2 + \epsilon^3 + \frac{4}{3} \epsilon_i^{-2} (\rho_i - 2\epsilon) \epsilon^{\frac{3}{2}} \right]^{-1/2}.
$$

(14)

Similarly to the case of the magnetic field, we solve Eq. (1) and find

$$
b = \frac{1}{2} \ln \left( \frac{3 + \lambda \epsilon^{-1} + (\rho_i - 2\epsilon_i) \epsilon^{-\frac{1}{2}} \epsilon^{\frac{3}{2}}}{1 + (\lambda + \rho_i) \epsilon_i^{-1}} \right) + \ln [1 + F_s(\epsilon)],
$$

(15)

where

$$
F_s(\epsilon) = \frac{3\rho_i}{4\epsilon_i} \left( 1 + \frac{\lambda + \rho_i}{\epsilon_i} \right)^{\frac{1}{2}} \int_{\epsilon_i}^\epsilon \, dx^\prime \frac{x^\prime}{x_i} \left[ \frac{\lambda}{\epsilon_i} + 3x + \left( \frac{\rho_i}{\epsilon_i} - 2 \right) x^\prime \right]^\frac{1}{2}.
$$

(16)

A careful inspection of the above solution shows that space is oblate, $e^{a-b} \geq 1$, and its oblateness monotonically increases from its initial value (unity) to its asymptotic value. More string density increases the anisotropy; more matter reduces anisotropy, but neither can change an oblate ellipsoid into a prolate one.

The asymptotics for large $t$ are as follows:

$$
\epsilon \sim \epsilon_i e^{-2(\lambda/3)^{1/2}(t-t_i+\tau)},
$$

(17)

$$
a \sim (\lambda/3)^{1/2}(t-t_i+\tau),
$$

(18)

$$
b \sim (\lambda/3)^{1/2}(t-t_i+\tau) - \frac{1}{2}\ln [1 + (\epsilon_i + \rho_i)/\lambda] + \ln [1 + F_s(0)],
$$

(19)

$$
\rho \sim \rho_i [1 + (\epsilon_i + \rho_i)/\lambda]^{\frac{3}{2}} [1 + F_s(0)]^{-1} e^{-3\lambda^{1/2}(t-t_i+\tau)},
$$

(20)

where

$$
\tau = \frac{1}{2} \int_0^{\epsilon_i} \epsilon^\prime \left( \frac{1}{\epsilon_i} \right)^{1/2} - \left[ \frac{1}{2} \lambda \epsilon^2 + \frac{4}{3} \epsilon_i^{-2} (\rho_i - 2\epsilon) \epsilon^{\frac{3}{2}} + \epsilon^3 \right]^{-1/2}
$$

(21)

V. WALLS

Energy-momentum conservation gives $b = \ln (\epsilon_i/\epsilon)$. Eq. (3) is easily solved to give

$$
a = \frac{2}{3} \ln \left( \frac{e^{3\lambda^{1/2}(t-t_i)} - \sigma}{1 - \sigma} \right) - (\lambda/3)^{1/2}(t-t_i),
$$

(22)

where

$$
\sigma = \frac{1 + (\rho_i + \epsilon_i)/\lambda}{{[1 + (\rho_i + \epsilon_i)/\lambda]}^{\frac{1}{2}} + 1}.
$$

(23)
Using Eqs. (1) and (22) together with $\rho = \rho_i e^{-(2a+b)}$, we find

$$\epsilon = \epsilon_i \gamma (\gamma^3 - \sigma)^\frac{\lambda}{3} F_w(\gamma),$$

(24)

where

$$F_w(\gamma) = \left[ \frac{(1 - \sigma)^\frac{\lambda}{3}}{1 + \sigma} + \frac{\rho_i (1 - \sigma)^\frac{\lambda}{3} \gamma^3 - 1}{\gamma^3 + 1} + \frac{3\epsilon_i}{2\lambda} \int_1^\gamma dx \left( \frac{x^3 - \sigma}{(x^3 + \sigma)^2} \right)^\frac{\lambda}{3} \right]^{-1}$$

(25)

and $\gamma = e^{(\lambda/3)(t-t_i)}$.

Inspecting the above solution shows that space is prolate, $e^{a-b} \leq 1$, and its prolateness monotonically increases from its initial value (unity) to its asymptotic value. More wall density increases the anisotropy; more matter reduces anisotropy, but neither can change an prolate ellipsoid into an oblate one.

The asymptotics for large $t$ are as follows:

$$\epsilon \sim \epsilon_i e^{-(\lambda/3)(t-t_i)} F_w(\infty),$$

(26)

$$a \sim (\lambda/3)^\frac{\lambda}{3} (t - t_i) - \frac{2}{3} \ln (1 - \sigma),$$

(27)

$$b \sim (\lambda/3)^\frac{\lambda}{3} (t - t_i) - \ln F_w(\infty),$$

(28)

$$\rho \sim \rho_i (1 - \sigma)^\frac{\lambda}{3} e^{\mu(t-t_i)} F_w(\infty).$$

(29)

VI. CONCLUSIONS

To apply the results of this paper, it will be necessary to consider how the spectrum of density perturbations are affected by asymmetric expansion. Since perturbations get laid down by quantum fluctuations and then asymmetrically expanded in our model, any initial spherical perturbation becomes ellipsoidal. After a while the expansion becomes spherically symmetric, but as long as perturbations remain outside the horizon they stay ellipsoidal. Only after they enter our horizon will they be able to adjust (they will probably start to oscillate between prolate and oblate with
frequency that depends on size and overdensity). So if the perturbations are just entering at last scattering they should be ellipsoidal. The smaller they are at last scattering, the more they have oscillated and if damped, the closer to spherical they should be. Hence, the large perturbations (corresponding to small $l$) will have a better memory of our eccentric phase. This would give what seems to be hinted at in the WMAP observations—more distortion of the low $l$ modes.

To summarize, what we need are modes that expanded eccentrically to be entering the horizon at the time of last scattering and then to feed into the Sachs-Wolfe calculation. This is a most interesting and challenging calculation, since it requires a full reanalysis of the density perturbations in eccentric geometry [19].

Acknowledgments

This work is supported in part by U.S. DoE grant number DE-FG05-85ER40226.

[20] The subscript “i” refers to the moment of transition from isotropic to anisotropic dynamics.