Stable Pentaquarks from Strange Chiral Multiplets

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Abstract

The assumption of strong diquark correlations in the QCD spectrum suggests flavor multiplets of hadrons that are degenerate in the chiral limit. Generally it would be unnatural for there to be degeneracy in the hadron spectrum that is not protected by a QCD symmetry. Here we show—for pentaquarks constructed from diquarks—that these degeneracies can be naturally protected by the full chiral symmetry of QCD. The resulting chiral multiplet structure recovers the ideally-mixed pentaquark mass spectrum of the diquark model, and interestingly, requires that the axial couplings of the pentaquarks to states outside the degenerate multiplets vanish in the chiral limit. This result suggests that if these hadrons exist, they are stable in the chiral limit and therefore have widths that scale as the fourth power of the kaon mass over the chiral symmetry breaking scale. Natural-size widths are of order a few MeV.
I. INTRODUCTION

Partly motivated by color-superconducting phases of QCD at asymptotic baryon densities \[1, 2, 3\], Jaffe and Wilczek (J-W) have recently argued \[4\] that there are strong diquark correlations in low-energy QCD and that these correlations may show up as novel hadronic states that do not readily fit into the standard constituent-quark picture. The recent highly-controversial observation \[5, 6, 7, 8, 9, 10, 11, 12, 13, 14\] of exotic “pentaquark” states has provided renewed interest in the idea that diquark correlations are important for hadron spectroscopy \[4, 15, 16, 17, 18, 19\]. While the pentaquark is now recognized by the particle-data group \[20\], there are several null results from high-energy experiments \[21, 22, 23, 24\]. Experiments with an expected ten-fold increase in statistics using deuterium and hydrogen targets are currently being analyzed from CLAS at JLab. It is hoped that high statistics will determine whether or not pentaquarks exist. \[1\] Lattice QCD simulations have been performed \[26, 27, 28, 29, 30\], however, to date, the results are inconclusive.

A puzzling feature that any theoretical description of pentaquarks must face is their extremely narrow widths; measurements of the widths are limited by the current experimental resolution \[5, 6, 7, 8, 9, 10, 11, 12, 13, 14\]. An important point regarding the expected size of the pentaquark widths has been made in Ref. \[31\]. We will repeat the argument here. In estimating the \(\Theta^+ NK\) axial coupling, one should not compare to, for instance, the \(\Delta N\pi\) axial coupling which is a number of order one, but rather to the \(N^* N\pi\) axial coupling, where \(N^*\) is an excited baryon, which is generically a number much less than one \[2\]. One useful way to think about this empirical fact is offered by the large-\(N_c\) approximation \[33\], where \(\Theta\) and \(N\) appear in different irreducible representations of the spin-flavor symmetry group, as do \(N\) and \(N^*\), while \(N\) and \(\Delta\) are in the same irreducible representation. This symmetry structure implies that the \(\Theta^+ NK\) and \(N^* N\pi\) axial couplings are suppressed in the large-\(N_c\) counting as compared to the \(\Delta N\pi\) axial coupling \[31\]. This paper will address the issue of the expected size of the pentaquark widths using symmetry arguments that, while unrelated to the large-\(N_c\) approximation, are similar in spirit.

The diquark picture leads one to expect degeneracies in the chiral limit between \(SU(3)\) multiplets that are not obviously protected by any QCD symmetry. (Here we will consider light quarks only, \(q \sim u, d, s\).) For instance, for tetraquarks \((qq\bar{q}\bar{q})\) one expects a degenerate nonet, \(1 \oplus 8\), and for pentaquarks \((qqqq\bar{q})\) one expects a degenerate \(8 \oplus 10\) of positive parity as well as a degenerate \(1 \oplus 8\) of negative parity. These multiplets are expected to appear with a variety of spin content. The expectations of degeneracy are based on the assumption that flavor-dependent interactions between the quarks and the antiquarks are absent. Of course, it would be unnatural for there to be degeneracy in the hadron spectrum that is not protected by a symmetry of QCD. For instance, in the large-\(N_c\) limit \[33\] one naturally expects singlet-adjoint degeneracy in the meson spectrum; that is because the anomaly is suppressed in this limit and the flavor symmetries enhance from \(SU(3)\) to \(U(3)\). Here we will see that the \(SU(3)_R \otimes SU(3)_L\) chiral symmetry of QCD can require singlet-adjoint and octet-antidecuplet degeneracy in the low-energy theory.

It may seem perplexing that chiral symmetry can protect a degeneracy in the low-energy theory as the hadronic Hamiltonian clearly has a contribution that breaks chiral symmetry, even in the chiral limit. However, by working in collinear Lorentz frames, one can

\[1\] A recent critical review of the current state-of-affairs is given in Ref. \[25\].

\[2\] For a tabulation of these axial couplings, see Ref. \[32\].
see that there are no non-vanishing matrix elements of the symmetry-breaking part of the Hamiltonian between hadrons that are in irreducible representations of the chiral group. Therefore the states within irreducible representations can remain degenerate as a consequence of chiral symmetry even though the Hamiltonian has a large symmetry-breaking piece. Now the remarkable thing is that states in different chiral multiplets cannot communicate by Goldstone-boson emission and absorption. Therefore, in what we will refer to as “the natural J-W scenario”, with chiral symmetry protecting the degeneracy between flavor multiplets, the chiral-limit axial couplings of the tetraquarks and the pentaquarks to the ordinary mesons and baryons must vanish. In particular, this implies that the tetraquarks and pentaquarks built from diquarks are stable in the chiral limit. Their decay widths to the ordinary mesons and baryons generally scale as $M_q^2$, where $M_q$ is the quark mass matrix, and are therefore suppressed.

This paper is organized as follows. In Sec. II we construct the pentaquark interpolating fields from diquarks and write down the leading-order (LO) operators in chiral perturbation theory ($\chi$PT) that govern the pentaquark axial transitions. In Sec. III we introduce technology which allows one to extract consequences of $SU(3)_R \otimes SU(3)_L$ for hadrons and we consider possible representations filled out by the pentaquarks. We then focus on the diquark scenario. In Sec. IV we recover the pentaquark axial couplings obtained from the chiral algebra in the diquark scenario using generalized Adler-Weisberger sum rules. We conclude in Sec. V.

II. FIELDS AND OPERATORS

A. Diquark Field

The basic assumption of the J-W picture is that quarks correlate strongly in the channel which is antisymmetric in color, spin, and flavor. For light quarks the resulting bosonic diquark, $Q$, is a color and flavor-$SU(3)$ antitriplet with $J^P = 0^+$. That is,

$$Q^{a\alpha} = \epsilon^{abc} \epsilon^{\alpha\beta\gamma} q_b^{\beta} q_c^{\gamma}, \quad (1)$$

where roman indices are fundamental flavor and greek indices are fundamental color. This diquark object is argued by J-W to be an important degree of freedom in low-energy QCD. We will now construct pentaquark interpolating fields out of diquarks.

B. Pentaquark Fields

As diquarks are flavor-$SU(3)$ antitriplets, the only way to make an exotic pentaquark out of two diquarks and an antiquark is to combine the diquarks symmetrically in flavor, $(\mathbf{3} \otimes \mathbf{3})_S = \mathbf{6}$, and then couple the antiquark. The flavor content of the resulting lowest-lying $qqqqq$ states is then a degenerate $\mathbf{6} \otimes \mathbf{3} = \mathbf{8} \oplus \mathbf{10}$ with spin-parity $\frac{1^+}{2}$. Neglecting the color indices, the $\mathbf{6}$ may be represented by the symmetric tensor $[34],

$$S^{ab} \equiv \frac{1}{2\sqrt{2}} \left( Q^a Q^b + Q^b Q^a \right). \quad (2)$$
We can then contract the $\bar{6}$ with the $\bar{3}$ antiquark to give the interpolator, $T^{abc}$, for the $8_{\uparrow}10_{\uparrow}$ pentaquarks:

$$T^{abc} = \frac{1}{\sqrt{2}} S^{ab} \bar{q} = P^{abc} + \frac{1}{\sqrt{6}} \left( \epsilon^{abc} O^{a}_{a} + \epsilon^{dac} O^{b}_{d} \right), \quad (3)$$

where

$$P^{abc} = \frac{1}{3 \sqrt{2}} \left( S^{ab} \bar{q} + S^{ac} \bar{q}^{b} + S^{bc} \bar{q}^{a} \right);$$

$$O^{a}_{b} = \frac{1}{\sqrt{3}} \epsilon_{bed} S^{ac} \bar{q}^{d}. \quad (4)$$

In terms of the hadronic description we have the antidecuple

$$P^{333} = \Theta^{+}, \quad P^{133} = \frac{1}{\sqrt{3}} N_{10}^{0}, \quad P^{233} = \frac{1}{\sqrt{3}} N_{10}^{+}$$

$$p^{113} = \frac{1}{\sqrt{3}} \Sigma_{10}^{-}, \quad p^{123} = \frac{1}{\sqrt{6}} \Sigma_{10}^{0}, \quad p^{223} = \frac{1}{\sqrt{3}} \Sigma_{10}^{+}$$

$$p^{111} = \Xi_{10}^{-}, \quad p^{112} = \frac{1}{\sqrt{3}} \Xi_{10}^{-}, \quad p^{122} = \frac{1}{\sqrt{3}} \Xi_{10}^{0}, \quad p^{222} = \Xi_{10}^{+}, \quad (5)$$

and the octet

$$\hat{O} = \begin{pmatrix} \frac{1}{\sqrt{6}} \Lambda_{\varnothing} + \frac{1}{\sqrt{2}} \Sigma_{\varnothing}^{0} & \Sigma_{\varnothing}^{+} & p_{\varnothing} \\ \Sigma_{\varnothing}^{-} & \frac{1}{\sqrt{6}} \Lambda_{\varnothing} - \frac{1}{\sqrt{2}} \Sigma_{\varnothing}^{0} & n_{\varnothing} \\ \Xi_{\varnothing}^{-} & \Xi_{\varnothing}^{0} & -\frac{2}{\sqrt{6}} \Lambda_{\varnothing} \end{pmatrix}. \quad (6)$$

A priori, in QCD, we would expect that the $P-\mathcal{O}$ mass splitting in the chiral limit is of order $\Lambda_{QCD}$. In the diquark picture of J-W the octet and antidecuple are degenerate in the chiral limit. This implies that the low-energy effective field theory should be formulated using the field $T^{abc}$ rather the $P$ and $\mathcal{O}$ fields separately; i.e. there should exist a symmetry which places $P$ and $\mathcal{O}$ in a single 18-dimensional multiplet. We will see below that chiral symmetry can fill this role.

C. Pentaquark Effective Lagrangian

The pentaquark octet is described by a two-index tensor, $\mathcal{O}_{a}^{b}$, and transforms as

$$\mathcal{O}_{a}^{b} \rightarrow (U^{a'})_{a}^{b} (U^{\dagger})_{b'}^{b} \mathcal{O}_{a'}^{b'} \quad (7)$$

with respect to the diagonal flavor-$SU(3)$ subgroup of $SU(3)_{R} \otimes SU(3)_{L}$. The pentaquark octet matrix is given in Eq. (6). The baryon octet is described by a two-index tensor, $B_{a}^{b}$, which transforms in the same way as $\mathcal{O}_{a}^{b}$ with respect to flavor and is given by

$$\hat{B} = \begin{pmatrix} \frac{1}{\sqrt{6}} \Lambda + \frac{1}{\sqrt{2}} \Sigma^{0} & \Sigma^{+} & p \\ \Sigma^{-} & \frac{1}{\sqrt{6}} \Lambda - \frac{1}{\sqrt{2}} \Sigma^{0} & n \\ \Xi^{-} & \Xi^{0} & -\frac{2}{\sqrt{6}} \Lambda \end{pmatrix}. \quad (8)$$


The antidecuplet pentaquarks are described by a three-index completely-symmetric tensor, $P^{abc}$, and transform as

$$P^{abc} \rightarrow (U^\dagger)^a_a (U^\dagger)^b_b (U^\dagger)^c_c P^{a'b'c'}$$

with respect to flavor-$SU(3)$. The components of the antidecuplet tensor are given in Eq. (5).

At LO in the three-flavor chiral expansion, the relevant axial matrix elements are parametrized by the pentaquark self-couplings, $D^O$, $F^O$, $H^P$, $C^PO$ and the couplings of the pentaquarks to the baryons, $C^PB$, $D^OB$ and $F^OB$. Assuming $J^p = \frac{1}{2}^+$ pentaquarks, the LO $\chi$PT lagrangian \[35, 36, 37, 38\] is

$$L = 2D^O \bar{O} S^\mu \{A_\mu, O\} + 2F^O \bar{O} S^\mu [A_\mu, O] + 2H^P \bar{P} (S \cdot A) P + 2C^PO [\bar{P} (S \cdot A) O + h.c.] + 2C^PB [\bar{P} (S \cdot A) B + h.c.] + 2D^OB [\bar{B} S^\mu \{A_\mu, O\} + h.c.] + 2F^OB [\bar{B} S^\mu [A_\mu, O] + h.c.] ,$$

where $A_\mu$ is the axial-vector field that contains the Goldstone bosons, and $S_\mu$ is the usual spin operator. This lagrangian is defined in the chiral limit; when the quark masses are turned on mixing occurs and the diagonalization of the mass matrix shifts the axial couplings.

### III. $SU(3) \otimes SU(3)$ REPRESENTATIONS

#### A. The Charge Algebra

In this section we will develop minimal technology that will allow us to extract consequences of chiral symmetry in low-energy QCD. Consider three-flavor QCD in the chiral limit. Weinberg and others have shown that by working in Lorentz frames in which all momenta are collinear, one may use the full $SU(3)_R \otimes SU(3)_L$ symmetry of QCD to classify hadrons \[39, 40, 41, 42, 43, 44, 45, 46, 47\]. As helicity is conserved in collinear frames, the chiral classification is for each helicity, as we will see. The presence of the full chiral-symmetry group in the low-energy theory is related to the special asymptotic behavior of certain Goldstone-boson-hadron scattering amplitudes. Hence, not surprisingly, consequences of chiral symmetry may also be extracted from the study of generalized Adler-Weisberger sum rules \[48, 49\], whose validity follows from the special asymptotic constraints. We express the algebra of $SU(3)_R \otimes SU(3)_L$ via \[4\]

$$[X^\lambda_\alpha, X^\lambda_\beta] = if_{\alpha\beta\gamma} T_\gamma \quad , \quad [T_\alpha, X^\lambda_\beta] = if_{\alpha\beta\gamma} X^\lambda_\gamma \quad , \quad [T_\alpha, T_\beta] = if_{\alpha\beta\gamma} T_\gamma$$

where $T_\alpha = \lambda_\alpha/2$. The object $X^\lambda_\alpha$ acts as an axial-vector current or charge in the low-energy theory; its matrix elements between hadron states mediate Goldstone boson emission and absorption. The Feynman amplitude for a Goldstone-boson transition between hadrons $A$ and $B$ of helicity $\lambda$ is

$$M(A \rightarrow B \pi_\alpha; \lambda) = \frac{1}{F_\pi} (M_A^2 - M_B^2) \langle A, \lambda | X^\lambda_\alpha | B, \lambda \rangle ,$$

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3 See Refs. \[35, 36, 37, 38\] for index contractions and other details.

4 Here we use greek indices for adjoint flavor.
where \( F_\pi = 93 \) MeV. The LO pentaquark axial-vector current may be found from the \( \chi \)PT
lagrangian of Eq. (10) and is given by

\[
(X_\alpha^\dagger)_{LO} = D_\alpha \mathcal{O}_\alpha^\dagger \{ T_\alpha, \mathcal{O}_\alpha \} + \mathcal{F}_\alpha \mathcal{O}_\alpha^\dagger \{ T_\alpha, \mathcal{O}_\alpha \} + \mathcal{H}_P \, P_\alpha^\dagger \, T_\alpha \, P_\alpha \\
+ \mathcal{C}_\text{PO} \left[ \mathcal{O}_\alpha \, T_\alpha \, P_\alpha + \text{h.c.} \right] + \mathcal{C}_\text{PB} \left[ B_\alpha^\dagger \, T_\alpha \, P_\alpha + \text{h.c.} \right] \\
+ D_\alpha \mathcal{O}_\alpha \left[ B_\alpha^\dagger \, \{ T_\alpha, \mathcal{O}_\alpha \} + \text{h.c.} \right] + \mathcal{F}_\alpha \mathcal{O}_\alpha \left[ B_\alpha^\dagger \, \{ T_\alpha, \mathcal{O}_\alpha \} + \text{h.c.} \right] ,
\]

(13)

where we have projected out the \( \lambda = 1/2 \) (\( \uparrow \)) current. The \( \lambda = -1/2 \) (\( \downarrow \)) current is given by \( X_\alpha^\dagger = -X_\alpha^\dagger \). As an example, the helicity-\( \uparrow \) transition \( \Theta^+ \rightarrow K^+ n \) is mediated by the axial-vector matrix element

\[
\langle \Theta^+ \uparrow | X_4^\dagger - i X_5^\dagger | n \uparrow \rangle = -\mathcal{C}_\text{PB} .
\]

(14)

The constraints imposed by the algebra of Eq. (11) on the current of Eq. (13) are determined by how the pentaquark states are placed in representations of the chiral group. We should therefore consider the allowed chiral representations. Now the quarks transform as \( (1, 3) \) and \( (3, 1) \), with respect to \( (SU(3)_R, SU(3)_L) \). In a helicity-conserving frame, helicity and chirality are the same. Therefore, if the helicity-\( \lambda \) component of a hadron is in the \( (R_1, R_2) \) representation of \( SU(3)_R \otimes SU(3)_L \) where \( R_{1,2} \) are flavor-\( SU(3) \) representations, then the \( -\lambda \) component of the hadron is in the \( (R_2, R_1) \) irreducible representation. We will now review the allowed chiral representation for the baryons. The flavor-\( SU(3) \) decomposition of a baryon interpolating operator is

\[
qqq \sim 1 \oplus 8_2 \oplus 10 ,
\]

(15)

where the subscript denotes multiplicity. Therefore, generally, we expect the baryons to transform as linear combinations of \( (3,3), (3,\bar{3}), (3,6), (6,3), (8,1), (1,8), (10,1), (1,10) \) and \( (1,1) \) irreducible representations as these are the only representations of \( SU(3)_R \otimes SU(3)_L \) that contain the flavor multiplets of the baryon interpolator and no others.

The \( SU(3) \) decomposition of a pentaquark interpolating operator is

\[
qqqqq \bar{q} \sim 1_3 \oplus 8_8 \oplus 10_4 \oplus \overline{10}_2 \oplus 27_3 \oplus 35 .
\]

(16)

In addition to the chiral representations allowed for the baryons, the pentaquarks may also be in linear combinations of \( (10, 1), (1, \overline{10}), (\overline{3}, \bar{3}), (6, \bar{3}), (21, 1), (1, 21), (35, 1), (1, 35), (8, 8), (8, 10), (10, 8), (8, \overline{10}), (10, 8), (6, \bar{6}) \) and \( (6, 6) \) irreducible representations of \( SU(3)_R \otimes SU(3)_L \).

In Ref. [47] it was conjectured that the chiral multiplet filled out by a given hadron is the minimal representation that contains only the flavor multiplets of the interpolator for that hadron. Evidence was provided in two-flavor QCD for light and heavy-light systems that this is indeed the case. Given the vast multiplicity of the pentaquark interpolator of Eq. (16), it is not clear that the conjecture of Ref. [47] has anything useful to say about it. By constrast, in the diquark picture, the \( SU(3) \) decomposition of a pentaquark interpolating operator is

\[
QQ\bar{q} \sim (1 \oplus 8)_- \oplus (8 \oplus \overline{10})_+ ,
\]

(17)

where the subscripts denote parity. Therefore, here we expect the pentaquarks to transform as linear combinations of \( (3,3), (3,\bar{3}), (3,6), (6,3), (8,1), (1,8), (10,1), (1,10) \) and \( (1,1) \)
irreducible representations. A degenerate octet and antidecuplet naturally fall into $(\bar{3}, 6)$ and $(6, \bar{3})$ irreducible representations and a degenerate singlet and octet naturally fall into $(\bar{3}, 3)$ and $(3, \bar{3})$ irreducible representations. The diquark interpretation of pentaquarks is therefore consistent with the conjecture of Ref. [47]. Below we will place the $8 \oplus 10$ pentaquarks in the $(3, \bar{6})$ representation of $SU(3)_R \otimes SU(3)_L$.

B. The Mass-Squared Matrix

In helicity-conserving frames, spontaneous chiral symmetry breaking appears in the hadronic Hamiltonian as an operator that transforms non-trivially with respect to $SU(3)_R \otimes SU(3)_L$. We will assume that the hadronic mass-squared matrix \(^5\) can be decomposed as:

\[
\hat{M}^2 = \hat{M}_1^2 + \hat{M}_{33}^2,
\]

where $\hat{M}_1^2$ transforms as a singlet, $(1, 1)$, and $\hat{M}_{33}^2$ transforms as $(\bar{3}, 3) \oplus (3, \bar{3})$. This last assumption is not crucial; it is sufficient for our purposes that $\hat{M}^2$ contain a piece that transforms non-trivially with respect to the chiral group in the sense that $[X^\lambda_\alpha, \hat{M}^2] \neq 0$ since in the absence of this piece there would be no spontaneous chiral-symmetry breaking in the low-energy theory. This then allows us to prove several useful lemmas [39, 40, 41, 42, 43, 44, 45, 46, 47]. Define $[X^\lambda_\alpha, \hat{M}^2] \equiv \hat{M}^2_\alpha$. Taking the matrix element of this relation between hadron states $A$ and $B$ with helicity-$\lambda$ gives

\[
(\hat{M}_B^2 - \hat{M}_A^2)\langle A, \lambda | X^\lambda_\alpha | B, \lambda \rangle = \langle A, \lambda | \hat{M}_\alpha^2 | B, \lambda \rangle.
\]

**LEMMA 1:** Say $A$ and $B$ are in an irreducible representation. Then the right-hand side of Eq. (19) vanishes as $A$ and $B$ overlap only through the singlet part of the mass-squared matrix, $\hat{M}_1^2$. As $X_\alpha$ acts as a symmetry generator, $\langle A | X_\alpha | B \rangle$ is nonvanishing for $A$ and $B$ in the same irreducible representation. It follows that if hadrons $A$ and $B$ are in an irreducible representation, they must be degenerate.

**LEMMA 2:** Say $A$ and $B$ are in a reducible representation. A representation is reducible if any two irreducible representations that make up the reducible multiplet have a nonvanishing overlap with the symmetry breaking part of the mass-squared matrix, $\hat{M}_{33}^2$. Hence the right hand side of Eq. (19) will be nonvanishing, as will the matrix element for Goldstone boson transitions via Eq. (12). If $A$ and $B$ are in different chiral representations, the right hand side of Eq. (19) vanishes. Therefore, *hadrons in different chiral representations do not communicate by Goldstone boson emission and absorption.*

These lemmas will be crucial in what follows.

C. Explicit Breaking Effects

We will now consider quark-mass corrections. In QCD the quark mass matrix, $\hat{M}_q$, transforms as $\hat{M}_q \rightarrow L \hat{M}_q R^\dagger$ with respect to $SU(3)_R \otimes SU(3)_L$. The quark mass matrix is

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\(^5\) We work with the mass-squared matrix as that is the relevant quantity in helicity conserving frames. An easy way to see this is to take the infinite-momentum limit of the relativistic energy dispersion relation.
\[ \hat{M}_q = \text{diag}(m, m, m_s) \text{ where } m \text{ is the common light-quark mass, and } m_s \text{ is the strange quark mass.} \]

In order to properly introduce explicit breaking effects in the helicity conserving theory, it is convenient to introduce in addition a spurion, \( \langle c \rangle \), with dimensions of mass which transforms like the quark mass matrix and acts like an effective condensate parameter in the low-energy theory. Inclusion of explicit breaking effects is then achieved by allowing invariant operators with insertions of \( \langle c \rangle \hat{M}_q \), which transforms as

\[
((\langle c \rangle M_q)_L \rightarrow L ((\langle c \rangle M_q)_L L^\dagger, (\langle c \rangle M_q)_R \rightarrow R ((\langle c \rangle M_q)_R R^\dagger).
\]

This transformation rule amounts to assuming that explicit breaking effects are purely octet with respect to flavor-\( SU(3) \). We should therefore recover the Gell-Mann-Okubo formula for the baryon octet and the equal-spacing relations for the baryon decuplet. In what follows we define \( X \equiv \langle s \rangle \hat{M}_q \). Our final form for the hadronic mass-squared matrix is

\[
\hat{M}^2 = \hat{M}^2_1 + \hat{M}^2_{33} + \hat{M}^2_{q8},
\]

where \( \hat{M}^2_{q8} \) is from the quark masses and transforms as \((1, 8) \oplus (8, 1)\). The lemmas proved above continue to hold away from the chiral limit but now \( \hat{M}^2_\alpha \) has a component that transforms as \((1, 8) \oplus (8, 1)\) and which gives rise to new mixing between chiral multiplets. We will see an example of this mixing below.

### D. Pentaquark Octet and Antidecuplet in a \((3, 6)\)

#### 1. Field Content and the Mass-Squared Matrix

In this section we will put the helicity-\( \uparrow \) \( O \) and \( P \) in the \((3, 6)\) irreducible representation of \( SU(3)_R \otimes SU(3)_L \). We introduce the three-index tensor, \( T^{a,bc} \), which transforms as

\[
T^{a,bc} \rightarrow (R^\dagger)^a_{\alpha'} (L^\dagger)^{\beta}_{\beta'} (L_{\gamma})^c_{\gamma'} T^{a',b',c'}. \]

The helicity-\( \downarrow \) \( O \) and \( P \) may be placed in an analogous tensor transforming in the \((6, 3)\) representation. The tensor \( T \) must be symmetric under the interchange of the two left handed indices, while there is no symmetry condition for the interchange of right and left handed indices. In terms of tensors transforming as an \( SU(3) \) octet, \( O \), and an antidecuplet, \( P, T \) can be written as

\[
T^{a,bc} = P^{abc} + \frac{1}{\sqrt{6}} \left( O_d^c \epsilon^{abd} + O_d^b \epsilon^{acd} \right). \]

This is precisely the interpolator for a degenerate \( 8 \oplus \overline{10} \) which we constructed in the diquark picture and is given in Eq. (3). Neglecting the kinetic term, the free Lagrangian is

\[
\mathcal{L}_{\uparrow} = -M^2_{1T} T^\dagger_{a,bc} T^{a,bc}. \]

Clearly \( O \) and \( P \) are degenerate. As the octet and the antidecuplet are assumed to be \( \frac{T}{T} \), explicit breaking will induce mixing. If we turn on the quark mass matrix we must account for the operators

\[
\mathcal{L}^{M_q}_{\uparrow} = -\alpha_T T^\dagger_{a,bc} X^a_d T^{d,bc} - \beta_T T^\dagger_{a,bc} X^b_d T^{a,dc} - \gamma_T T^\dagger_{a,bc} X^a_d T^{a,bc} X^d. \]

\[ \mathcal{L}^{M_q}_{\uparrow} \]

8
Notice that the operator with coefficient \( \gamma_T \), while \( O(M_q) \), is not \( SU(3) \) violating and can therefore be absorbed into a redefinition of \( M_T^2 \). We will nevertheless keep it for reasons of comparison. The symmetry breaking operators induce mixing between \( N_{10} \) and \( N_O \) and between \( \Sigma_{10} \) and \( \Sigma_O \). Defining the mixed mass eigenstates \( N_{1,2} \) and \( \Sigma_{1,2} \) as

\[
N_O = \sin \theta_N N_1 + \cos \theta_N N_2, \quad \Sigma_O = \sin \theta_\Sigma \Sigma_1 + \cos \theta_\Sigma \Sigma_2, \\
N_{10} = -\sin \theta_N N_2 + \cos \theta_N N_1, \quad \Sigma_{10} = -\sin \theta_\Sigma \Sigma_2 + \cos \theta_\Sigma \Sigma_1, 
\]

we then find off-diagonal operators of the form

\[
\frac{1}{12} \left( 2 \alpha_T - \beta_T \right) (m - m_s) \left( 2 \sqrt{2} \cos 2\theta_T - \sin 2\theta_T \right) \Gamma_1 \Gamma_2 + h.c. 
\]

where \( \Gamma \equiv N, \Sigma \). The mass-squared matrix is diagonalized by choosing \( \sin \theta_T = -\sqrt{\frac{2}{3}} \) and \( \cos \theta_T = \sqrt{\frac{1}{3}} \), which corresponds to ideal mixing. Setting \( \langle c \rangle = 1 \), we then find the masses to be

\[
M_{N_1}^2 = M_{1T}^2 + \alpha_T m + \beta_T m_s + \gamma_T (2m + m_s), \\
M_{N_2}^2 = M_{1T}^2 + \alpha_T m + \frac{1}{2} \beta_T (m + m_s) + \gamma_T (2m + m_s), \\
M_{\Sigma_1}^2 = M_{1T}^2 + \alpha_T m_s + \beta_T m + \gamma_T (2m + m_s), \\
M_{\Sigma_2}^2 = M_{\Delta_0}^2 = M_{1T}^2 + \alpha_T m + \frac{1}{2} \beta_T (m + m_s) + \gamma_T (2m + m_s), \\
M_{\Sigma^*}^2 = M_{1T}^2 + \alpha_T m_s + \beta_T m_s + \gamma_T (2m + m_s). 
\]

One finds, for instance, the mass-squared relations

\[
M_{N_1}^2 + M_{\Sigma_1}^2 = M_{N_2}^2 + M_{\Sigma_2}^2 = M_{10}^2 + M_{\Delta_0}^2. 
\]

It is easy to check that this mass-squared spectrum is equivalent to that found in Ref. [5] by Jaffe and Wilczek [6]. For discussion of phenomenology we refer the reader to Ref. [5] and to Ref. [4]. It is worth recalling our input: we have assumed an octet and an antidecuplet in the irreducible \((3, 6)\) representation of \( SU(3) \otimes SU(3) \) with purely-octet symmetry breaking.

2. Axial Couplings in the Chiral Limit

The main point of this paper is to observe that the chiral multiplet structure which reproduces the ideally-mixed J-W model mass spectrum constrains the axial couplings as well. We will now consider the axial couplings in the chiral limit. Now the left-handed current transforms as an octet under \( SU(3)_L \), a \((1, 8)\), and the right-handed current transforms as an octet under \( SU(3)_R \), a \((8, 1)\). We introduce \( T^L_\alpha \) and \( T^R_\alpha \), that transform as

\[
T^L_\alpha \rightarrow L T^L_\alpha L^T, \quad T^R_\alpha \rightarrow R T^R_\alpha R^T. 
\]

\[\text{We can choose our arbitrary constant } \gamma_T = -\beta_T \text{ so that } M_{N_1}^2 \text{ does not depend on the strange quark mass. Setting } m = 0 \text{ and making the identification } M_0 = M_{1T} = \alpha_T \text{ and } \alpha = -\beta_T/2\alpha_T + 1 \text{ with } M_0 \text{ and } \alpha \text{ the J-W parameters, one immediately recovers their results, to } O(M_q). \]
under $SU(3)_R \otimes SU(3)_L$. The vector and axial-vector current matrix elements of the $(3, \bar{6})$ are reproduced by the effective currents

\begin{align*}
T^\dagger_\alpha &= T^\dagger_{a,bc} (T^a_{\alpha})_d \ T^{d, bc} + 2 \ T^\dagger_{a, bc} (T^b_{\alpha})_d \ T^{a, dc}, \\
X^\dagger_\alpha &= T^\dagger_{a, bc} (T^a_{\alpha})_d \ T^{d, bc} - 2 \ T^\dagger_{a, bc} (T^b_{\alpha})_d \ T^{a, dc},
\end{align*}

respectively \(^7\). Matching to the $\chi$PT axial-vector current of Eq. (13) one directly finds

\begin{align*}
\mathcal{D}_O &= 1, \quad \mathcal{F}_O = -\frac{2}{3}, \quad \mathcal{H}_P = 1, \quad \mathcal{C}_{PO} = -2\sqrt{\frac{2}{3}}, \\
\mathcal{C}_{PB} &= \mathcal{D}_{OB} = \mathcal{F}_{OB} = 0
\end{align*}

in the chiral limit. The vanishing axial transitions from the pentaquarks to the baryons is a simple consequence of LEMMA 2 as, by construction, the pentaquarks and the baryons are in different chiral representations. The only way to get $\mathcal{C}_{PB}, \mathcal{D}_{OB}, \mathcal{F}_{OB} \neq 0$ in the chiral limit without requiring pentaquark-baryon degeneracy is to place the pentaquarks and baryons in a reducible representation with pentaquark octet-antidecuplet mass-squared splitting of order $M^2_{33} \sim \Lambda^2_{QCD}$. There is no natural way to maintain octet-antidecuplet degeneracy with non-vanishing couplings to the ground-state baryons.

### 3. Quark-Mass Corrections to the Axial Couplings

Away from the chiral limit, mixing will shift the axial couplings within the pentaquark chiral multiplet. Assuming that the ground-state baryons are in a chiral multiplet that mixes with the pentaquarks at $O(M_q)$, then on general grounds one expects

\begin{align*}
\mathcal{C}_{PB} \xrightarrow{M_q \neq 0} \bar{d} \ \frac{M_K^2}{\Lambda^2_\chi}
\end{align*}

away from the chiral limit, where $\bar{d}$ is a dimensionless coupling of order one, $M_K = 494$ MeV is the kaon mass and $\Lambda_\chi \equiv 4\pi F_\pi$. (Notice that $\bar{d}$ contains a chiral logarithm as one-loop effects contribute to the axial currents at $O(M_q)$ in the chiral expansion.) The total width of $\Theta^+$ can be expressed as \(^{36, 37, 38}\)

\begin{align*}
\Gamma(\Theta^+) = (146 \text{ MeV}) \ C^2_{PB} \xrightarrow{M_q \neq 0} (5 \text{ MeV}) \ \bar{d}^2.
\end{align*}

where we have used Eq. \(^{38}\). Therefore, we find that in the natural J-W scenario the $\Theta^+$ width is expected to be of order a few MeV.

### IV. ADLER-WEISBERGER SUM RULES

To get a better sense of what the predictions for the axial couplings mean we will obtain the same results from a different perspective. Assume that the scattering amplitude for a pion scattering on a hadron target with isospin-one in the t-channel falls off sufficiently rapidly

\(^7\) The coefficients of the operators are obtained by taking matrix elements of the commutator of Eq. (11) between various pentaquark states using Eq. (31).
asymptotically to justify an unsubtracted dispersion relation for the amplitude $^{48, 49}$. The amplitude at threshold simply measures the isospin of the target as guaranteed by chiral symmetry low-energy theorems, while the dispersion integral can be expressed as an integral over the total cross-section. Neglecting the continuum and saturating the Adler-Weisberger sum rules for elastic pion scattering on $p_\mathcal{O}$, $\Sigma_\mathcal{O}^-$, $\Xi_\mathcal{O}^-$ and $N^{1+}_{\mathcal{O}10}$ with only those states within the octet, $\mathcal{O}$, and the antidecuplet, $P$, yields

$$
1 = (D_\mathcal{O} + F_\mathcal{O})^2 + \frac{1}{9} C_{P\mathcal{O}}^2,
2 = \frac{2}{3} D_\mathcal{O}^2 + 2 F_\mathcal{O}^2 + \frac{1}{6} C_{P\mathcal{O}}^2,
-1 = -(D_\mathcal{O} - F_\mathcal{O})^2 + \frac{2}{3} C_{P\mathcal{O}}^2,
1 = \frac{1}{9} \mathcal{H}_P + \frac{1}{3} C_{P\mathcal{O}}^2,
$$

(35)

where the left-hand side measures twice the isospin of the target. One readily checks that these Adler-Weisberger sum rules have two solutions:

$$
|D_\mathcal{O}| = 1, |F_\mathcal{O}| = \frac{2}{3}, |\mathcal{H}_P| = 1, |C_{P\mathcal{O}}| = 2\sqrt{\frac{2}{3}},
$$

(36)

and

$$
|D_\mathcal{O}| = 0, |F_\mathcal{O}| = 1, |\mathcal{H}_P| = 3, |C_{P\mathcal{O}}| = 0.
$$

(37)

The first solution corresponds —up to phases that are further constrained by inelastic Adler-Weisberger sum rules— to placing $\mathcal{O}$ and $P$ in a $(\mathbf{3}, \mathbf{6})$ representation. One may easily verify that the second solution corresponds to the only other possibility: putting $\mathcal{O}$ in a $(\mathbf{1}, \mathbf{8})$ and $P$ in a $(\mathbf{1}, \mathbf{10})$. Of course the second solution does not require octet-antidecuplet degeneracy. Notice that the coupling between the octet and antidecuplet pentaquarks in the second solution vanishes. This is once again due to LEMMA 2.

V. DISCUSSION

The diquark model of pentaquarks of Jaffe and Wilczek $^4$ has a remarkably simple and natural interpretation in terms of the $SU(3)_R \otimes SU(3)_L$ chiral symmetry of QCD. By placing an $\mathbf{8}$ and a $\mathbf{10}$ of flavor-$SU(3)$ in a $(\mathbf{3}, \mathbf{6})$ of $SU(3)_R \otimes SU(3)_L$ one enforces $\mathbf{8} - \mathbf{10}$ degeneracy without resorting to quark-model assumptions. When the quark masses are turned on one finds precisely the ideally-mixed mass spectrum of Ref. $^4$. Moreover, the chiral-multiplet structure completely determines the axial-vector transitions of the pentaquarks. In particular, one finds that the pentaquarks do not communicate with the baryons by Goldstone-boson exchange in the chiral limit. Hence, the pentaquark widths scale as the quark mass matrix squared and are generically small, of order a few MeV. It should be noted that analogous arguments hold for tetraquark and pentaquark nonets built out of diquarks, which naturally fall into $(\mathbf{3}, \mathbf{3})$ and $(\mathbf{3}, \overline{\mathbf{3}})$ irreducible representations of $SU(3)_R \otimes SU(3)_L$ $^8$.

It may well be the case that the true chiral multiplets in which the pentaquarks find themselves are quite different than what is suggested by the diquark picture. However, as

\footnote{This prediction is strongly at odds with the tetraquark interpretation of the controversial broad scalar $f_0(600)$.}
long as the baryons and the pentaquarks are not in the same chiral multiplet, the estimate of the width given in Eq. (34) continues to hold. For instance, the pentaquark octet may be absent, as is the case in the chiral-soliton model \[51, 52, 53, 54\]. The antidecuplet P may then find itself, for instance, in the \((1, 0\bar{10})\) of \(SU(3)_R \otimes SU(3)_L\) which again experiences no axial transitions to the baryons in the chiral limit. However, as no degeneracies among flavor multiplets are expected in the chiral-soliton model, there is no compelling rationale to conclude anything simple about its chiral multiplet structure.

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