Triaxiality and the determination of the cubic shape parameter $K_3$ from five observables

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(Dated: August 9, 2004)

Abstract

The absolute and the relative quadrupole shape invariants $q_3$ and $K_3$ provide a model independent measure of triaxiality for $\beta$-rigid nuclei. We will show that one can obtain $q_3$ and $K_3$ from a small number of observables. The approximations which are made will be shown to hold within a few percent both in the rigid triaxial rotor model and the interacting boson model. The shape parameter $K_3$ is given for an exemplary set of nuclei and is translated into effective values of the geometrical deformation parameters $\beta$ and $\gamma$.

PACS numbers: 21.10.Ky, 21.60.Ev, 21.60.Fw
I. INTRODUCTION

One basic property of the nucleus is its geometric shape. Therefore, the nuclear shape, whether it is spherical, prolate, oblate, axially symmetric, or triaxial, is a key property of the ground state, as well as of excited states of the nucleus. Quantifying the nuclear shape, one usually turns to the well known geometric deformation parameters $\beta$ and $\gamma$. These are deduced from a comparison of data with, e.g., the Davydov-Fillipov model of a rigid triaxial rotor [1]. This approach incorporates a major problem. A rigid rotor model cannot account for vibrations of the nuclear shape, which is a strong limitation. But, even if a model is able to describe also vibrations in the deformation parameters as, e.g., by the Bohr Hamiltonian [2], the geometric interpretation of the interacting boson model [3, 18], or the GCM [4], a second problem arises. In general the shape parameters $\beta$ and $\gamma$ do not have fixed values, because the nuclei have in general not a rigid shape but they are vibrating. Thus it is useful to consider alternative parameters related to the shape of a nucleus, namely quadrupole shape invariants [5, 6, 7], which are model independent, and which are direct observables. In this paper we will discuss mainly the quadratic and cubic shape parameters $q_2$ and $q_3$. We will focus on the relative cubic shape parameter $K_3 = q_3/q_2^{3/2}$, which is independent of the nuclear radius $R_0$ and the charge $e$. We will show that it is possible to obtain $q_2$, $q_3$ and $K_3$ with good accuracy from only few data. The cubic shape parameter $K_3$ is related to triaxiality and will be given for a variety of nuclei. Its connection to the geometrical deformation parameters will be discussed.

Quadrupole shape invariants were introduced by Kumar [5] and widely used by Cline and co-workers, e.g. [6]. They are expectation values in a given nuclear eigenstate of higher order moments of the $E2$ transition operator, which is usually taken to be the quadrupole operator. Considering the ground state they are defined as

\[ q_2 = e^2 \langle 0^+_1 | (Q \cdot Q) | 0^+_1 \rangle, \]  
\[ q_3 = \sqrt{\frac{35}{2}} e^3 \langle 0^+_1 | [QQQ]^{(0)} | 0^+_1 \rangle, \]  
\[ q_4 = e^4 \langle 0^+_1 | (Q \cdot Q) (Q \cdot Q) | 0^+_1 \rangle, \]

where the dot denotes a scalar product and brackets denote tensorial coupling, $Q$ is the quadrupole operator and $e$ the elementary electric charge. Higher order moments can also be defined and are related to fluctuations in $q_3$, but will not be discussed here. The moments
$q_2$ and $q_3$ can be written in terms of averages of geometrical deformation parameters as

$$q_2 = e^2 Q_0^2 \langle \beta^2 \rangle = e^2 Q_0^2 \beta_{eff}^2$$

and

$$q_3 = e^3 Q_0^3 \langle \beta^3 \cos(3\gamma) \rangle .$$

with

$$Q_0 = 3Z R_0^2 / (4\pi) .$$

These quadrupole shape invariants can be renormalized to the second order invariant $q_2$ by

$$K_n = \frac{q_n}{q_2^{n/2}} ,$$

omitting the nuclear radius or the electric charge in this form. These quantities can in principle be obtained directly from data, but this is difficult in praxis because a large number of E2 matrix elements including signs is involved in expressions (1-3). This can be seen expanding the invariants $q_n$ into sums over E2 matrix elements, which is shown here for $q_2$ and $q_3$:

$$q_2 = e^2 \sum_i \langle 0^+_1 \parallel Q \parallel 2^+_i \rangle \langle 2^+_i \parallel Q \parallel 0^+_1 \rangle ,$$

$$q_3 = \sqrt{\frac{7}{10}} e^3 \sum_{i,j} \langle 0^+_1 \parallel Q \parallel 2^+_i \rangle \langle 2^+_i \parallel Q \parallel 2^+_j \rangle \langle 2^+_j \parallel Q \parallel 0^+_1 \rangle .$$

An evaluation of $q_2$ and $q_3$ using extensive sets of experimental quadrupole matrix elements from multiple Coulomb excitation has been done for some nuclei by D. Cline and co-workers, e.g. in [6, 9, 10]. Of course, the existence of such extensive data sets is the favorable, but it is not the general case. Thus, there is great interest to obtain the shape invariants from more restricted sets of data.

II. APPROXIMATIONS

The basic idea is to invoke the $Q$-phonon scheme as has been discussed in [11]. This scheme was suggested by T. Otsuka [12], and was developed by a Köln-Tokyo collaboration, e.g. [13, 14, 15, 16]. The $Q$-phonon scheme implies that the wave functions of low-lying states are exhausted by only a few multiple $Q$-phonon configurations, where a $Q$-phonon itself is an excitation by the quadrupole operator. The $2^+_1$ state in an even-even nucleus is dominantly a one-$Q$-phonon state. It was shown [14, 15] that the $Q$-phonon scheme holds with good accuracy for the lowest levels of collective nuclei. Here, as we will consider only
the very lowest states, we keep within the non-orthogonalized $Q$-phonon scheme \[16, 17\], which will be shown to be sufficient for our purpose. The $Q$-phonon scheme gives a simple selection rule, namely, that an E2 transition between two states may change the number of $Q$-phonons in first order only by one, \textit{i.e.} \(\Delta Q = 1\). Neglecting all $Q$-forbidden transition matrix elements with \(\Delta Q \geq 2\) gives the first order approximation. We will denote quantities given in this first order approximation by a superscript (1). This leads to a drastic truncation in the matrix elements needed in the expansions in Eqs. (7,8), \textit{e.g.}, \(q_2\) as given in Eq. (7) is approximated by

\[
q_2 \approx q_2^{(1)} = e^2 \langle 2^+_1 || Q || 0^+_1 \rangle^2 = B(E2; 0^+_1 \rightarrow 2^+_1) ,
\]

(9)
because transitions from the two-phonon \(2^+_2\) state or even higher-lying \(2^+\) states to the ground state are $Q$-forbidden in first order. Eq. (9) reflects the well-known fact that in most even-even nuclei the largest part of the E2 excitation strength is concentrated in the first excited \(2^+\) state. In the rigid rotor this B(E2) value is known to be directly proportional to the squared $\beta$-deformation by Eq. (4). In the case of non-rigid $\beta$-deformation Eq. (4) defines an effective deformation parameter $\beta_{\text{eff}}$ or, making use of the approximation (9), an approximate $\beta_{\text{eff}}^{(1)}$.

Using the $Q$-phonon scheme in first order for \(q_3\) one obtains

\[
q_3^{(1)} = \sqrt{\frac{7}{10}} e^3 \langle 2^+_1 || Q || 0^+_1 \rangle^2 \cdot \langle 2^+_1 || Q || 2^+_1 \rangle .
\]

(10)

Then, approximating the $K_3$-parameter following its definition in Eq. (6) \((n = 3)\) results in

\[
K_3^{(1)} = \frac{q_3^{(1)}}{(q_2^{(1)})^{3/2}} = \sqrt{\frac{7}{10}} \frac{\langle 2^+_1 || Q || 2^+_1 \rangle}{\langle 2^+_1 || Q || 0^+_1 \rangle} ,
\]

(11)

which is calculated from the ratio of the quadrupole moment of the \(2^+_1\) state and its E2 matrix element to the ground state. It turns out, \textit{e.g.}, checking this approximation within the rigid triaxial rotor model or the IBM-1, that such a rude truncation of the sum given in Eq. (8) is not sufficient for a good approximation to $K_3$, as we will show in sections III and IV. Therefore, we used a second order approximation, allowing in each term of the sum only one $Q$-forbidden matrix element with \(\Delta Q = 2\). Doing so, we derive a second order approximation for \(q_3\) as

\[
q_3^{(2)} = q_3^{\text{appr.}} = \sqrt{\frac{7}{10}} e^3 [(\langle 2^+_1 || Q || 0^+_1 \rangle^2 \cdot \langle 2^+_1 || Q || 2^+_1 \rangle + 2 \cdot \langle 0^+_1 || Q || 2^+_1 \rangle \cdot \langle 2^+_2 || Q || 2^+_1 \rangle \cdot \langle 2^+_1 || Q || 0^+_1 \rangle] ,
\]

(12)
which still includes only few E2 matrix elements. In the following we will always denote the second order approximation with the superscript \textit{appr.} instead of (2), as it is the only one we use. Note that the approximation to $q_2$ in second order approximation is the same as in first order, as a $Q$-forbidden matrix element would always appear squared and such terms are not included in this approximation, and we get

$$q_2^{\text{appr.}} = q_2^{(1)} \text{ and } \beta_{\text{eff}}^{\text{appr.}} = \beta_{\text{eff}}^{(1)}. \quad (13)$$

Dividing $q_3^{\text{appr.}}$ from Eq. (12) by $q_2^{\text{appr.}}$, we get a second order approximation for $K_3$ that includes only four different E2 matrix elements, involving the lowest two excited $2^+$ states and the ground state.

A problem that appears is that the signs of the E2 matrix elements are needed, which are not known in most cases. Usually we know only the B(E2) values which are

$$B(E2; J_i \rightarrow J_f) = \frac{1}{2J_i + 1} e^2(\langle J_f || Q || J_i \rangle)^2. \quad (14)$$

This ambiguity in the signs can be avoided by using a relation between the signs of four matrix elements, suggested by Jolos and von Brentano [19]:

$$\text{sign}(\langle 2^+_1 || Q || 2^+_1 \rangle) = -\text{sign}(\langle 0^+_1 || Q || 2^+_2 \rangle \langle 2^+_2 || Q || 2^+_1 \rangle \langle 2^+_1 || Q || 0^+_1 \rangle). \quad (15)$$

This relation gives the relative phase of the two terms in Eq. (12). There is still an overall sign of $K_3$, which is the sign of the quadrupole moment of the $2^+_1$ state, deciding between prolate and oblate deformation. Then, the second order approximation for $K_3^{\text{appr.}}$ is

$$K_3^{\text{appr.}} = \sqrt{\frac{7}{10}} \text{sign}(Q(2^+_1)) \left[ \frac{B(E2; 2^+_1 \rightarrow 2^+_2)}{B(E2; 2^+_1 \rightarrow 0^+_1)} - \frac{\sqrt{B(E2; 2^+_2 \rightarrow 0^+_1) \cdot B(E2; 2^+_2 \rightarrow 2^+_1)}}{B(E2; 2^+_1 \rightarrow 0^+_1)} \right], \quad (16)$$

where we use an alternative but elegant definition of the squared quadrupole moment following Eq. (14),

$$B(E2; 2^+_1 \rightarrow 2^+_2) = \frac{1}{5} e^2(\langle 2^+_1 || Q || 2^+_2 \rangle)^2 = \frac{35}{32\pi} Q(2^+_1)^2. \quad (17)$$

The approximation formula for $K_3$ given in Eq. (16) is the key result of this work. It allows to measure this observable directly and in a model independent way from only few data. These are four absolute B(E2) values, namely $B(E2; 2^+_1 \rightarrow 0^+_1)$, $B(E2; 2^+_2 \rightarrow 0^+_1)$, $B(E2; 2^+_1 \rightarrow 2^+_2)$, and $B(E2; 2^+_1 \rightarrow 2^+_1)$, and the sign of the quadrupole moment of the
$2_1^+$ state, which we consider as a fifth observable. This new method to determine $K_3$ is of particular importance because $K_3$ is closely connected to the triaxiality of nuclei, i.e. to $\gamma$-deformation. For axial symmetry $K_3 = -1$ for prolate ($\gamma = 0^\circ$) and $K_3 = +1$ for oblate ($\gamma = 60^\circ$) nuclei, while $K_3$ drops to zero at a maximum triaxiality of $\gamma = 30^\circ$. This holds for geometrical models like the Davydov-Fillipov triaxial rotor model, as well as for the dynamical symmetries of the IBM. One major difference between these two models is that the IBM describes non-rigid $\beta$- and $\gamma$-deformation, e.g., in the U(5) vibrational limit and the O(6) limit of $\gamma$-soft nuclei, in both of which $K_3$ vanishes. In the SU(3) and SU(3) dynamical symmetries of the IBM, which correspond to the prolate and oblate axially symmetric rigid rotors, respectively, the same values for $K_3$ are derived as in the geometrical model. In the following we will check to which extent

$$K_3 \approx K_3^{appr.}$$

holds, using as a test the rigid triaxial rotor model of Davydov and Fillipov and the IBM-1.

### III. THE RIGID TRIAXIAL ROTOR MODEL

The Hamiltonian of the Davydov-Fillipov rotor model is

$$H_{geo} = A_1 J_1^2 + A_2 J_2^2 + A_3 J_3^2,$$

(19)

where $J_n$ are the projections of the spin $J$ on the three symmetry axes, and where the parameters $A_k$ are connected to the moments of inertia $\Theta_k$ by

$$A_k = \frac{\hbar}{2\Theta_k}.\$$

(20)

The moments of inertia can further be written in terms of the geometrical deformation parameters $\beta$ and $\gamma$,

$$\Theta_k = 4B\beta^2 \sin^2\left(\gamma - \frac{2k}{3}\pi\right).$$

(21)

The $E2$ transition operator is given by

$$T(E2)_{geo} = eQ_{2\mu} = eQ_0 \beta \left[D_{\mu0}^{2*} \cos(\gamma) + \sqrt{2} (D_{\mu2}^{2*} + D_{\mu-2}^{2*}) \sin(\gamma)\right],$$

(22)

where the $D_{\mu\nu}^2$ are the Wigner-$D$-matrices and $Q_0$ is given by Eq. (5). We stress that this model with rigid $\beta$- and $\gamma$-deformations is applicable to only a limited number of nuclei.
Nevertheless, apart from our discussion of the ground state deformation, the model is often applied to highly excited and strongly and also super-deformed bands as well, for which, in principle, our approach of \( K \)-parameters may also apply.

In our calculations we vary the parameter \( \gamma \) over the range \( \gamma \in [0^\circ, 30^\circ] \), covering the range of prolate axially symmetric and triaxial structures inherent to the model. The results for \( \gamma \in [30^\circ, 60^\circ] \) are fully symmetric to those given and thus omitted. The choice of \( \beta \) is arbitrary, as in the rigid case \( K_3 \) is independent of \( \beta \) and is given by

\[
K_3 = -\frac{\beta^3 \cos(3\gamma)}{(\beta^2)^{3/2}} = -\cos(3\gamma) .
\] (23)

In a similar way one defines an approximate deformation \( \gamma^{\text{appr.}} \) from \( K_3^{\text{appr.}} \) by

\[
K_3^{\text{appr.}} = -\cos(3\gamma^{\text{appr.}}) .
\] (24)

In order to avoid the division by zero, we use the ratio

\[
R_{geo}^{K_3} = \frac{1 + |K_3^{\text{appr.}}|}{1 + |K_3|}
\] (25)

as a measure of the quality of the approximation (18). The solid curve in the left panel of Figure 1 shows the quantity \( R_{geo}^{K_3} \) versus the deformation parameter \( \gamma \), calculated numerically using the code DAVIDOV [20]. In the axially symmetric limit at \( \gamma = 0^\circ \) the approximation is exact. This also holds for the case of maximum triaxiality at \( \gamma = 30^\circ \), while \( R_{geo}^{K_3} \) is small for all intermediate cases with a deviation from one of 8% in maximum. The dashed curve represents the same calculation, but using the first order approximation \( K_3^{(1)} \) from Eq. (11). The deviation of the first order approximation is clearly much larger with a maximum of about 30%, showing that the use of a second order approximation is unavoidable for transitional nuclei. On the right hand side of Figure 1 the corresponding absolute deviation of \( \gamma^{\text{appr.}} \) derived from Eq. (24) from the real \( \gamma \)-values in the model is shown as a solid curve. The maximum deviation is below 3.5° at \( \gamma \approx 15^\circ \). Again, the deviation is much larger using only the first approximation, given as a dashed curve.

IV. THE INTERACTING BOSON MODEL

Now, we check the quality of \( K_3^{\text{appr.}} \) in the IBM-1, within the Extended Consistent Q Formalism (ECQF) [21, 22] and the Hamiltonian [8]

\[
H_{IBM} = (1 - \zeta) n_d - \frac{\zeta}{4N} Q^x \cdot Q^x ,
\] (26)
depending on only two structural parameters, $\zeta$ and $\chi$, and omitting an overall energy scale. The E2 transition operator in the ECQF is chosen to be proportional to the quadrupole operator in the Hamiltonian,

$$T(E2)_{IBM} = e_B Q^\chi = e_B [(s^+ \tilde{d} + d^+ s) + \chi (d^+ \tilde{d})] ,$$

with an effective boson charge $e_B$, and $n_d = (d^+ \tilde{d})$ is the boson number operator. Varying the values of $\zeta$ and $\chi$ over the full range of symmetries ($\zeta \in [0,1], \chi \in [-\sqrt{7}/2, \sqrt{7}/2]$), one covers the dynamical symmetry limits of the IBM, namely $U(5)$ ($\zeta = 0, \chi$), the prolate (oblate) $SU(3)$ ($SU(3)$) ($\zeta = 1, \chi = \mp \sqrt{7}/2$), and $O(6)$ ($\zeta = 1, \chi = 0$), as well as the transitional structures in between. In analogy to Eq. (25), we define the ratio

$$R_{IBM}^{K_3} = \frac{1 + |K_3^{\text{appr.}}|}{1 + |K_3|} ,$$

which has been calculated over the full parameter space using the code PhINT [23]. Again, $K_3^{\text{appr.}}$ is defined by Eq. (16). The results are shown in the top part of Figure 2 for $N = 10$ bosons. They are given for $\chi < 0$, because the results for $\pm \chi$ are fully symmetric, as the change in sign is equivalent to the symmetry transformation $d \rightarrow -d$ (keeping $s \rightarrow s$). The use of positive $\chi$ values corresponds to the choice of $\gamma > 30^\circ$ in the geometrical model. Deviations of $K_3^{\text{appr.}}$ from the exact $K_3$ values are small in all cases, the deviation of $R_{IBM}^{K_3}$ from 1 is below 7%. For comparison, Figure 3 shows $R_{IBM}^{K_3(1)}$, which is defined analog to Eq. (28), but where the first order approximation, i.e. Eq. (11), is used. Like in the geometrical model it is seen, that the deviations from the exact value of $K_3$ are much larger in the first order approximation. Thus, in general it is necessary to use $K_3^{\text{appr.}}$ from Eq. (16).

The deviation $R_{IBM}^{K_3}$ peaks in a region around $SU(3)$. A reason for this behavior is found by a close look at this region. The middle part of Figure 2 shows the values of $K_3$ and $K_3^{\text{appr.}}$ on the $U(5)$–$SU(3)$ (left) and $O(6)$–$SU(3)$ (right) transition legs. It is obvious that the maximum deviation of $R_{IBM}^{K_3}$ from unity appears in those regions, in which $K_3$ changes most rapidly. These are exactly those regions that are connected to the shape/phase transition between spherical and axially symmetric nuclei, or between prolate and oblate deformations, as discussed, e.g., in [11, 24, 25, 26, 27, 28]. This means that in the IBM the approximation $K_3^{\text{appr.}}$ misses the exact value of $K_3$ somewhat when leaving the rotational limit. However, overall deviations of $R_{IBM}^{K_3}$ from unity are small and the approximation (18) is well fulfilled.

From comparison with the geometrical model an effective $\gamma$-deformation can be defined
[8] from $K_3$ by

$$K_3^\text{appr} = \frac{\langle \beta^3 \cos(3\gamma) \rangle}{\langle \beta^2 \rangle^{3/2}} = -\cos(3\gamma_{\text{eff}}),$$

and an approximate value $\gamma_{\text{eff}}^\text{appr}$ can be defined analog from $K_3^\text{appr}$. The differences between the exact and the approximate $\gamma$-values, $\gamma_{\text{eff}} - \gamma_{\text{eff}}^\text{appr}$, are included in the bottom part of Figure 2 and show good agreement. The deviation of $\gamma_{\text{eff}}^\text{appr}$ from $\gamma_{\text{eff}}$ is always smaller than 2.5°.

These effective $\gamma$-values are not and cannot be equivalent to those given by Eqs. (23,24), because $K_3$ is not generally independent of $\beta$-deformation and fluctuations in $\beta$ occur, especially for vibrational nuclei. Moreover, in case of rigid $\beta$ (on the SU(3)-O(6) transitional line) $K_3$ is a measure of $\langle \cos(3\gamma) \rangle$, while in case of rigid $\gamma$ it is a measure of $\langle \beta^3 \rangle / (\langle \beta^2 \rangle^{3/2}$. The effect of a $\beta$-vibration is only effectively taken out in the translation to the geometric model by Eq. (29). However, if fluctuations in $\beta$ are small, which is the case past the phase transition towards deformed nuclei (typically for $\zeta > 0.6$), a factorization of the averages over $\beta$ and $\cos(3\gamma)$ should work, and we can assume

$$\langle \beta^3 \cos(3\gamma) \rangle = \langle \beta^3 \rangle \langle \cos(3\gamma) \rangle$$

and

$$\frac{\langle \beta^3 \rangle}{\langle \beta^2 \rangle^{3/2}} = 1,$$

making $\gamma_{\text{eff}}$ comparable to the geometrical $\gamma$-deformation.

V. $K_3$ FOR VARIOUS NUCLEI

A. Direct measure of $K_3$

For the two considered models we have shown that $K_3^\text{appr}$ is a good approximation to the value of the cubic shape parameter $K_3$. Thus we assume this to hold also in other collective models such as the GCM or the Bohr Hamiltonian. Only few observables have to be obtained in order to derive $K_3^\text{appr}$, namely the lifetime of the $2_1^+$ state, the lifetime and the branching ratio of the $2_2^+$ state, and the quadrupole moment of the $2_1^+$ state. Besides the modulus of $K_3$ also its sign is interesting, which is obtained from the sign of the quadrupole moment of the $2_1^+$ state. This quadrupole moment itself is not easy to obtain, therefore it is a challenge to measure triaxiality. Especially for vibrational or $\gamma$-soft nuclei, where the quadrupole moment is small, high quality data is needed. Thus, an approximate value of $K_3$ is so far known for a number of nuclei in or near the valley of stability only. For a set
of nuclei that belong to various symmetry regions the \( K_3^{\text{appr.}} \)-parameter has been calculated from tabulated data. The results are given in Table I, together with effective \( \gamma \)-deformation parameters derived from Eq. (29), and effective \( \beta \)-deformations from Eqs. (4,9).

Typical rotational nuclei like the heavier Gd or Dy isotopes show \( K_3 \) values around -1 as it is expected for prolate deformed axially symmetric shapes. Also \(^{152}\)Sm and \(^{154}\)Gd, which are attributed \([25, 29]\) to be close to the critical point symmetry X(5) proposed by F. Iachello \([24]\), show this value. The \( K_4 \)-parameter obtained from \( q_4 \) of Eq. (3) and Eq. (6), which can be approximated in a similar way \([11, 19]\), gives a direct measure for \( \beta \)-softness. One finds that \( K_4^{\text{appr.}} = 1 \) for \( \beta \)-rigid nuclei and \( K_4^{\text{appr.}} \sim 1.4 \) for vibrators. For \(^{152}\)Sm and \(^{154}\)Gd one finds \( K_4^{\text{appr.}} \) values of 1.02(3) \([30]\) and 1.088(26) \([29]\), respectively, which, in combination with \( K_3^{\text{appr.}} \), meets the expectations for the vibrator to well-deformed rotor transition, more on the rotational side of the phase transition. Especially \(^{152}\)Sm seems to be on the rotor side (where \( K_3^{\text{appr.}} = -1 \) and \( K_4^{\text{appr.}} = 1 \)) of the phase transition, which seemingly conflicts with the interpretation of this nucleus as being close to the phase transitional point. This may be related to the systematical error made in the approximations for \( K_3 \), which maximizes exactly in the transitional region in the IBM, which may be reflected also in data. However, the systematical error made in the determination of \( K_4 \) should be smaller in that region \([19]\) and a problem remains.

The \( K_3 \) values of the Os isotopes show an evolution from the axially symmetric rotor towards O(6) symmetry with a maximum effective triaxiality of \( \gamma_{\text{eff}} = 30^\circ \). Note, that here one talks of effective \( \gamma \)-deformation, as the nucleus does not have a rigid triaxiality. The more vibrational Pd and Cd isotopes show moderate values of \( K_3 \) with relatively large errors due to the quadrupole moments. Non-zero values are not a contradiction to a more U(5) like structure as they may emanate from finite N effects (see \([8]\))

An surprising conflict appears for \(^{196}\)Pt, which is usually taken as a prime example of O(6) symmetry \([31]\), as well as for the neighboring \(^{194}\)Pt. Both nuclei show rather large, positive quadrupole moments \([9]\) and thus have quite large values of \( K_3^{\text{appr.}} \). This shows that they are on the side of oblate deformation, with a considerable deviation of \( K_3^{\text{appr.}} \) from the expectation value, \( K_3(O(6)) = 0 \). Other observables like the branching ratio of the \( 2^+ \) state or energies agree much better with the O(6) predictions. Therefore, values of \( K_3 \) derived from an IBM fit (see below) agree much better with \( K_3 = 0 \). One cannot argue that this deviation is due to the dependence on \( \beta \)-fluctuations (compare Eq. (29)). On the
SU(3)–O(6) transition line, no $\beta$-fluctuations are allowed, and indeed, the shape invariant $K_4$ approximately equals 1 (see [11]) for both nuclei, which pinpoints $\beta$-rigidity. Again, this may be related to the maximal systematical error close to O(6) seen from Figure 2. But, even if the value of $K_3$ is overpredicted from the approximation, a deviation from O(6) remains. However, we want to stress that these values, e.g. $\gamma_{\text{eff}} = 42^\circ$ instead of $\gamma_{\text{eff}} = 30^\circ$ for $^{196}\text{Pt}$, still indicate a strong triaxiality. It is only the quantitative value of $\gamma_{\text{eff}}$ which is in doubt.

We stress that if one uses only the first order approximation, the value of $K_3$ is missed for transitional nuclei like the Os isotopes, for which the transition $2^+_2 \rightarrow 0^+_1$ is sizeable, e.g., $K_3^{\text{appr.}} = -0.7(1)$ for $^{190}\text{Os}$ in the first order approximation, underestimating triaxiality, while the second order approximation gives $K_3^{\text{appr.}} = -0.35(9)$.

Note, that for the electric quadrupole moment of the $2^+_1$ state, $Q(2^+_1)$, is usually not easy to access experimentally. A new relation was found [11], however, which gives a new way to approximately determine $Q(2^+_1)$ or $B(E2; 2^+_1 \rightarrow 2^+_1)$, respectively:

$$B(E2; 2^+_1 \rightarrow 2^+_1)^{\text{appr.}} = B(E2; 4^+_1 \rightarrow 2^+_1) - B(E2; 2^+_2 \rightarrow 2^+_1).$$  \hfill (31)

This relation may be used to obtain the quadrupole moment as an input for $K_3$, but it will give a large uncertainty especially for vibrational or $\gamma$-soft nuclei, which have a small quadrupole moment. So far the relation (31) was only checked in the IBM [11]. Figure 4 shows the deviation

$$R_{E2}^{E2} = 1 - \frac{B(E2; 4^+_1 \rightarrow 2^+_1)}{B(E2; 2^+_1 \rightarrow 2^+_1) + B(E2; 2^+_2 \rightarrow 2^+_1)},$$  \hfill (32)

from the real value calculated within the rigid triaxial rotor model. Also in the geometrical model the agreement is good.

**B. Fit procedure for $K_3$**

In cases where not all of the needed data are present, one may follow another procedure, fitting parameters of a model to the available data for one nucleus, and calculating $K_3$ from the model. Here we used the simple two parameter Hamiltonian of the IBM given in Eq. (26). The two parameters were fitted to the energy ratio

$$R_{4/2} = E(4^+_1)/E(2^+_1)$$  \hfill (33)
and the B(E2) ratio
\[ B(E2; 2^+_2 \rightarrow 0^+_1)/B(E2; 2^+_2 \rightarrow 2^+_1), \] (34)

that are sensitive to changes in structure over wide parameter regions. For the reproduction of the energy ratio an error of 2% was allowed, while for the B(E2) ratio the experimental errors were taken into account, resulting in an allowed parameter range of \( \zeta \) and \( \chi \), in which \( K_3 \) takes various values within a certain range, with an upper a lower limit. In Table I we denote values of \( K_3 \) obtained from the fit as \( K_3^{\text{fit}} \), and give the upper and lower limits allowed from the experimental errors. These values can be compared with the measured \( K_3^{\text{appr.}} \). The values agree reasonably well in most cases, considering the simplicity of the Hamiltonian and the arbitrary choice of the two observables used in the fit. Note, that other observables can be used for the fit. But in some cases the simple Hamiltonian used cannot describe all features of a given nucleus, as, e.g., for the Pt nuclei. Therefore, the Hamiltonian (26) may be extended or another model used.

VI. CONCLUSIONS

To conclude, we discussed measures of triaxiality. In this respect we considered in particular the absolute and relative cubic shape parameters \( q_3 \) and \( K_3 \). The approximative \( K_3^{\text{appr.}} \) was introduced as a direct, model independent observable, which is, if \( \beta \) is rigid, a measure of triaxiality, while \( K_3^{\text{appr.}} \) is more general an observable in all structural limits and the regions between them. \( K_3^{\text{appr.}} \) was shown to be a good approximation to the exact value of \( K_3 \), and can with good accuracy be obtained from only four matrix elements, or from four B(E2) values, one of them equivalent to the modulus of the quadrupole moment \( Q(2^+_2) \), and the sign of \( Q(2^+_1) \). This, manifested in Eq. (16), is the key result of this work. This accuracy of the approximation was checked for the IBM and the rigid triaxial rotor model, and the need of the second order approximation within the \( Q \)-phonon scheme was shown in both models, especially in transitional regions. Effective values of \( \beta \)- and \( \gamma \)-deformation in the ground state, derived from \( q_2^{\text{appr.}} \) and \( K_3^{\text{appr.}} \), respectively, have been deduced from data. For vibrational nuclei geometrical deformation parameters cannot be given, while \( q_2 \) and \( K_3 \) are always well-defined properties of the ground state. Finally, we proposed a way how to derive \( K_3^{\text{appr.}} \) from a model fitting data.

For discussions we thank A. Dewald, A. Gelberg, J. Jolie, R.V. Jolos, B.R. Mottelson,
and I. Stefanescu.

This work was supported by the DFG under contract No. Br 799/12-1.


FIG. 1: $R^{K3}_{QEO}$ calculated for all values of $\gamma$ (solid line on the left hand side), showing that the second order approximation Eq. (18) holds well in the rigid triaxial rotor model. The approximation does not seriously change the value of $\gamma$ (solid line on the right hand side). The dashed lines give the values derived from the use of only the first order approximation.
TABLE I: Approximate $K_3^{\text{appr.}}$-values, the effective approximate $\beta$- and $\gamma$-deformation parameters derived from our approach are listed for a set of nuclei. For $\beta$-deformations, errors are omitted as they are in the order of per mil or smaller, and the systematic error made by assuming $R_0 = 1.2 \text{fm}$ for the nuclear radius is presumably larger. The last two columns give upper and lower limits for the value of $K_3$ fitted to the observables (33,34) as described in section V B.

<table>
<thead>
<tr>
<th></th>
<th>$K_3^{\text{appr.}}$</th>
<th>$\beta_{\text{eff}}^{\text{appr.}}$</th>
<th>$\gamma_{\text{eff}}^{\text{appr.}}$</th>
<th>$K_3^{\text{fit}}$ upper</th>
<th>$K_3^{\text{fit}}$ lower</th>
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</thead>
<tbody>
<tr>
<td>$^{156}\text{Gd}$</td>
<td>-0.97(5)</td>
<td>0.339</td>
<td>4(4)</td>
<td>-0.86</td>
<td>-0.98</td>
</tr>
<tr>
<td>$^{158}\text{Gd}$</td>
<td>-0.95(6)</td>
<td>0.349</td>
<td>6(6)</td>
<td>-0.83</td>
<td>-1.00</td>
</tr>
<tr>
<td>$^{160}\text{Gd}$</td>
<td>-0.96(3)</td>
<td>0.351</td>
<td>5(3)</td>
<td>-0.85</td>
<td>-1.00</td>
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<tr>
<td>$^{164}\text{Dy}$</td>
<td>-0.93(9)</td>
<td>0.347</td>
<td>7(7)</td>
<td>-0.77</td>
<td>-1.00</td>
</tr>
<tr>
<td>$^{154}\text{Gd}$</td>
<td>-1.00(3)</td>
<td>0.310</td>
<td>1(5)</td>
<td>-0.77</td>
<td>-0.84</td>
</tr>
<tr>
<td>$^{152}\text{Sm}$</td>
<td>-0.98(4)</td>
<td>0.307</td>
<td>4(4)</td>
<td>-0.54</td>
<td>-0.83</td>
</tr>
<tr>
<td>$^{188}\text{Os}$</td>
<td>-0.63(5)</td>
<td>0.185</td>
<td>17(3)</td>
<td>-0.65</td>
<td>-0.76</td>
</tr>
<tr>
<td>$^{190}\text{Os}$</td>
<td>-0.35(9)</td>
<td>0.177</td>
<td>23(3)</td>
<td>-0.53</td>
<td>-0.75</td>
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<tr>
<td>$^{192}\text{Os}$</td>
<td>-0.3(1)</td>
<td>0.167</td>
<td>25(2)</td>
<td>-0.48</td>
<td>-0.71</td>
</tr>
<tr>
<td>$^{194}\text{Pt}$</td>
<td>0.53(4)</td>
<td>0.143</td>
<td>41(1)</td>
<td>0.08</td>
<td>0.14</td>
</tr>
<tr>
<td>$^{196}\text{Pt}$</td>
<td>0.59(7)</td>
<td>0.129</td>
<td>42(2)</td>
<td>0.00</td>
<td>0.02</td>
</tr>
<tr>
<td>$^{106}\text{Pd}$</td>
<td>-0.4(1)</td>
<td>0.230</td>
<td>22(3)</td>
<td>-0.41</td>
<td>-0.55</td>
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<tr>
<td>$^{108}\text{Pd}$</td>
<td>-0.6(4)</td>
<td>0.242</td>
<td>17(16)</td>
<td>-0.23</td>
<td>-0.30</td>
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<td>$^{112}\text{Cd}$</td>
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<td>0.181</td>
<td>22(2)</td>
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<td>-0.83</td>
</tr>
<tr>
<td>$^{114}\text{Cd}$</td>
<td>-0.4(1)</td>
<td>0.184</td>
<td>23(3)</td>
<td>-0.36</td>
<td>-0.59</td>
</tr>
</tbody>
</table>
FIG. 2: Top panel: $R_{IBM}^{K_3}$ calculated over the whole IBM parameter space confirms a good fulfillment of the second order approximation to $K_3$. Middle panel: The two transitional legs for fixed values of $\chi = -\sqrt{7}/2$ (left) and $\zeta = 1$ (right). The approximation misses slightly the phase/shape-transitional parameter region. Bottom panel: Effective $\gamma$-deformations calculated from $K_3$ and $K_3^{\text{appr.}}$ calculated over the whole parameter range. All calculations are for $N = 10$ bosons.

FIG. 3: $R_{IBM}^{K_3(1)}$, the analog to the top panel of Figure 2, but using only the first order approximation. Deviations from unity are much larger than in the second approximation.
FIG. 4: $R_{geo}^{E2}$ calculated for all values of $\gamma$. The E2-relation (32) holds well in the geometrical model.