ANALYTICAL APPROACH TO SU(2) YANG-MILLS THERMODYNAMICS

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We propose an analytical approach to SU(2) Yang-Mills thermodynamics. The existence of a macroscopic and rigid adjoint Higgs field, generated by dilute trivial-holonomy calorons at large temperature $T$ (electric phase), implies a twofold degeneracy of the ground state which signals a broken electric $Z_2$ symmetry. A finite energy density $\propto T$ of the ground state arises due to caloron interaction. An evolution equation for the effective gauge coupling, derived from thermodynamical self-consistency, predicts a second-order like transition (seen in lattice simulations) at $T_c$ to a phase where monopoles are condensed and off-Cartan excitations decoupled. In this magnetic phase the ground state is unique and dominates the pressure (negative total pressure). While the magnetic phase has a massive, propagating 'photon' it confines fundamental matter (pre-confinement). The temperature dependence of the magnetic gauge coupling predicts the transition to the confining phase at $T_C \sim T_c^{1/9}$ where center-vortex loops condense and the 'photon' decouples. We believe that this transition is 'swallowed' by finite-size artefacts in lattice simulations. No thermodynamical connection exists between the confining and the magnetic phase.

The objective is to propose a macroscopic, effective theory for SU(2) Yang-Mills thermodynamics which can be generalized to SU(N) [1]. The approach is similar in spirit to the idea that superconductivity is macroscopically described by a U(1) Higgs theory [2,3]. We predict the phase structure of SU(2) Yang-Mills theory, the (quasiparticle) spectrum of its excitations, and the $T$ dependence of thermodynamical quantities. As a result, the theory is shown to come in three rather than two phases. We predict a vanishing entropy density and an equation of state $\rho = -P$ at $T_C$ and negative pressure $P$ throughout the magnetic phase ($T_C \leq T \leq T_c$). An over-exponentially growing density of states (intersecting and single center-vortex loops) implies that the limiting temperature $T_C \sim \Lambda_{YM}$ of the confining phase can only then be exceeded if the spatial homogeneity
of the system is sacrificed. The situation of two disconnected thermodynamical regimes in a pure SU(2) gauge theory has an analogue in the $N = 1$ SUSY YM theory where a separating pole in the exactly known beta function exists.

The basic assumption is that a dilute-gas ensemble of calorons with trivial-holonomy (THC) macroscopically forms a composite and adjoint Higgs field $\phi$ at high $T$ (electric phase). By dilute gas we mean that in a minimal, local definition (lowest possible mass dimension) of $\phi$,

$$\langle \text{tr} N F_{\mu\nu}(x) t^a F_{\nu\lambda}(x) F_{\lambda\mu}(x) \rangle_{\text{THC}},$$

the average is performed over a single THC and its zero-mode deformations only. Fluctuations, that would lift the action of a given THC configuration $A_{\text{THC}}^\beta$ above the BPS bound, are discarded in Eq. (1) because they mediate interactions between calorons to be considered later at the macroscopic level. The above assumption would be superfluous if it could be shown that at a given temperature $T$ nontrivial solutions to the gap-equation (1) with $|\phi| = |\phi|(T)$ exist for a certain range of values of the fundamental gauge coupling $\bar{g} \equiv \sqrt{\sum_i g_i^2}$. This is the objective of future research. While the (nonfluctuating, see below) field $\phi$ generates a $T$-dependent mass for topologically trivial modes $W^\pm$ on tree-level (thermal quasiparticles, solution to the infrared problem of thermal perturbation theory) the 'photon' $Z_0$ remains tree-level massless in the electric phase. The 'condensate' $|\phi|$ represents a compositeness scale governing the maximal off-shellness of gauge-modes and the maximal center-of-mass energy flowing into a vertex. This yields a converging loop expansion of thermodynamical quantities despite a large value of the effective coupling constant $e$.

A fixed color orientation of $\phi$ forms a finite domain induced by a 'seed' caloron. At the points where at least four domains meet isolated zeros of $\phi$ exist with the associated magnetic monopoles being mappings from $S_2$ onto the coset spaces $\{\text{SU}(2)/U(1)\}$. Microscopically, isolated monopoles are generated by nontrivial-holonomy calorons ($A_0(|\vec{x}| \rightarrow \infty) \neq 0$ and regular gauge copies) which form and dissociate during domain collisions.

The integration over the instanton scale $\tilde{\rho}$ in Eq. (1) is cut off in the UV at the compositeness scale $|\phi|^{-1}$. 

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with excitations: $A_\beta = A_\beta^{THC} + a_\beta$. The macroscopic action is

$$S_E = \int_0^{1/T} d\tau \int d^3x \left( \frac{1}{2} \text{tr}_N G_{\mu\nu} G_{\mu\nu} + \text{tr}_N D_\mu \phi D_\mu \phi + V_E(\phi) \right), \quad (2)$$

where $G_{\mu\nu}^{a} = \partial_\mu a_\nu^a - \partial_\nu a_\mu^a - e f^{abc} a_\mu^b a_\nu^c$ denotes the field strength of a top.\trivialmode $a_\beta$, $e$ is the effective coupling constant, $D_\beta \phi = \partial_\beta + ie[\phi, a_\beta]$, and $\text{tr}_N t^a t^b = 1/2 \delta^{ab}$. The potential $V_E(\phi) \equiv \text{tr}_N v_E^l v_E$ is uniquely determined by the requirement of spatially homogeneous, periodic-in-euclidean-time $\tau$ solutions ($\tau$ independent modulus, $0 \leq \tau \leq 1/T$) to the BPS equation $\partial_\phi \phi = 0$. Macroscopic BPS saturation derives from the zero-energy property of THCs (microscopically BPS) of which $\phi$ is composed (in absence of other gauge-field fluctuations), the other properties follow from thermodynamical equilibrium. The potential reads

$$V_E = \text{tr} v_E^l v_E \equiv A_E^3 \text{tr} (\phi^2)^{-1}, \quad (3)$$

where $A_E$ denotes a dynamically generated mass scale determined by a boundary condition to the thermodynamical evolution. Up to gauge transformations the ‘square-root’ $v_E$ is given as $v_E \equiv i \Lambda_E^3 \lambda_1 \phi/|\phi|^2$ where $\lambda_i (i = 1, 2, 3)$ denote the Pauli matrices and $|\phi| \equiv 1/2 \text{tr} \phi^2$. Solutions to the BPS equation $\partial_\tau \phi = 0$ are labelled by nonzero winding numbers $l \in \mathbb{Z}$:

$$\phi_l(\tau) = \sqrt{\frac{A_E^3}{2\pi T|l|}} \lambda_1 \exp(-2\pi i T l \lambda_1 \tau). \quad (4)$$

On the solutions $\phi_l$ we have $\partial^2_\phi V_E/T^2 = 12\pi^2 l^2$ and $\partial^2_\phi V_E/|\phi|^2 = 3l^3 \lambda_1^2$. We thus conclude that $\phi_l$ does not fluctuate thermodynamically and quantum mechanically ($\lambda_E \equiv 2\pi T/A_E$ is much larger than unity, see $\lambda_l^+$). We restrict to lowest winding $|l| = 1$. Interactions between THC are accounted for macroscopically by taking $\phi_1$ as a background in the equation of motion $D_\mu G_{\mu\beta} = 2ie[\phi, D_\beta \phi]$. On the macroscopic level there must not be a net field strength in the thermal ground state, and thus the only admissible solution $a^{bg,1}_\beta$ is pure gauge, $G_{\mu\beta}[a^{bg,1}_\beta] = 0 = D_\beta \phi$. We have $a^{bg,1}_\beta = \frac{2}{7} T \delta_{\beta4} \lambda_1$. On the configuration $a^{bg,1}_\beta$ the Polyakov loop is $P = -1$. To assign ($T$ dependent and $\tau$ independent) masses $m^2_{3/2} = -2 e^2 \text{tr} [\phi, t^{1,2}] [\phi, t^{1,2}]$ to off-Cartan fluctuations a singular gauge transformation to unitary gauge needs to be performed $\lambda^+$. The Polyakov loop transforms as $P = -1 \rightarrow P = +1$. The singular gauge transformation does not change the periodicity of the fluctuations $a_\beta$. Thus it is irrelevant whether one integrates out $a_\beta$ in winding or unitary gauge in a
loop expansion of thermodynamical quantities. The two distinct ground states $P = \pm 1 \neq 0$ together with the associated gauge-field fluctuations are physically equivalent. In unitary gauge there is a physical interpretation of $\alpha_{\beta}$ and integrating them out is simple. The existence of two ground states signals a broken electric $Z_2$ symmetry and thus deconfinement. By virtue of $D_{\beta}\phi = 0$ the vanishing energy density (pressure) of the hypothetical ground state, that is composed of noninteracting THCs only, is shifted to the finite energy density (pressure) $(-)V_E = 4\pi\Lambda_E^3 T$ of the physical ground state by THC interaction. As a consequence, the covariant BPS equation $D_\tau\phi = v_E$ is not satisfied by the above configurations mirroring that ground-state physics on its own is thermodynamically incomplete (gauge-field excitations are emitted and absorbed in addition).

An expression for the total pressure $P$ in one-loop approximation is derived. Because of the compositeness scale $|\phi_1|$ quantum fluctuations can be neglected and two-loop corrections are tiny ($<2\%$ of the one-loop value). Thermodynamics is consistent (as in the underlying theory) if we impose $\partial_a P = 0$ where the mass parameter $a$ is defined as $a = 2\pi e\lambda^{3/2}_E$. This enforces an evolution of the effective gauge coupling with $T$ which, in implicit form, is governed by

$$\partial_a \lambda_E = -\frac{24 \lambda^2_E a}{(2\pi)^6} D(2a).$$

The function $xD(x)$ denotes a scaled-out derivative w.r.t mass $Ta$ of the thermal component of the one-loop pressure for a bosonic field. The righthand side of Eq. (refeeq) is zero at $a = 0, \infty$ implying the existence of a highest and lowest attainable temperature. Shortly below $\lambda, P$ shows a large bump of height $\sim \lambda, P$ making the assumed importance of THCs in the partition function self-consistent due to their small action. A plateau value $e_{\text{plateau}} = 17.15$ (attractor) exists which is independent of the boundary condition imposed at large temperature $T_P$ (IR ↔ UV decoupling, renormalizability of the underlying theory). The constancy of $e$ after the formation of $\phi_1$ is consistent with the existence of isolated and locally conserved magnetic charge. We note here that the Stefan-Boltzmann limit is reached quickly (with three polarizations for the $W^\pm$ bosons!) due to a power-like approach.

For lack of space we only quote the results for the magnetic phase (see for details) which sets in at $T_c$ where $a \to \infty$: condensation of magnetic monopoles; decoupling of $W^\pm$ bosons; 'photon' $Z_0$ is massless at $T_c$ and massive below; decoupling of $Z_0$, equation of state $\rho = -P$, and vanish-
ing entropy density at $T_C \sim T_c/1.9$ (predicted from an evolution equation similar to Eq. (3) for the magnetic coupling $g$); negative total pressure for $T_C \leq T \leq T_c$. The Polyakov loop is +1 already in winding gauge. Thus only a single ground state exists and the electric $Z_2$ symmetry is restored (confinement of fundamental test charges but existence of a massive, propagating 'photon' $Z_0$). In the confining phase, setting in at $T_C \sim T_c/1.9$ by violent 'reheating' (entropy generation), no propagating gauge mode exists, excitations are made of crossed and uncrossed closed magnetic flux lines (center-vortex loops), and the ground-state pressure is precisely zero once the vortex condensate has settled to the minimum of its potential. Since excitations resemble vacuum diagrams in a $\phi^4$ theory and since the mass spectrum of these states is equidistant it is easy to estimate that the density of states in the confining phase is over-exponentially growing. This implies the existence of the Hagedorn temperature $T_C$ and the thermodynamical disconnectedness of the confining phase from the other two phases.

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References