Time asymmetric spacetimes near null and spatial infinity. I. Expansions of developments of conformally flat data.

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Abstract

The conformal Einstein equations and the representation of spatial infinity as a cylinder introduced by Friedrich are used to analyse the behaviour of the gravitational field near null and spatial infinity for the development of data which are asymptotically Euclidean, conformally flat and time asymmetric. Our analysis allows for initial data whose second fundamental form is more general than the one given by the standard Bowen-York Ansatz. The Conformal Einstein equations imply, upon evaluation on the cylinder at spatial infinity, a hierarchy of transport equations which can be used to calculate asymptotic expansions for the gravitational field in a recursive way. It is found that the solutions to these transport equations develop logarithmic divergences at certain critical sets where null infinity meets spatial infinity. Associated to these, there is a series of quantities expressible in terms of the initial data (obstructions), which if zero, preclude the appearance of some of the logarithmic divergences. The obstructions are, in general, time asymmetric. That is, the obstructions at the intersection of future null infinity with spatial infinity are in general different from those obtained at the intersection of past null infinity with spatial infinity. The latter allows for the possibility of having spacetimes where future and past null infinity have different degrees of smoothness. Finally, it is shown that if both sets of obstructions vanish up to a certain order, then the initial data has to be asymptotically Schwarzschildian to some degree.

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1 Introduction

The regular initial value problem near spatial infinity introduced by Friedrich in [15] has proved an invaluable tool for the understanding of the behaviour of the gravitational field in the region of spacetime near spatial and null infinities. This setting, relying entirely on the properties of conformal structure, is such that the equations and data are regular with spatial and null infinities having a finite representation with their structure and location known a priori. In order to keep the amount of computations at bay, Friedrich’s original analysis was restricted to time symmetric initial data sets which admit a smooth conformal compactification at infinity. Central to his discussion, laid a representation of spatial infinity as a cylinder on which the Conformal Einstein equations fully reduce to a set of transport equations. The crucial feature of these equations is that through its solutions it is possible to relate properties of the spacetime at null infinity with properties of the initial data sets, effectively allowing us to identify the parts of the initial data responsible for the non-smoothness of null infinity. In particular, in [15] a regularity condition

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was formulated, that if satisfied, would preclude the appearance of a certain type of logarithmic divergences in the solutions of the transport equations at the sets where null infinity intersects spatial infinity. Because of the underlying hyperbolic nature of the Conformal Einstein equations, it is to be expected that the logarithmic divergences would propagate along the generators of null infinity and leave an imprint on various radiative properties of the spacetime, the decay of the components of the Weyl tensor among them.

The solutions of the transport equations at spatial infinity can be calculated by means of what is essentially a decomposition of the unknowns in terms of spherical harmonics. The feasibility of this approach was explored in [19]. Leaving aside the calculational complexities, this approach is completely algorithmic, and hence, entirely amenable to an implementation in a computer algebra system. In [24] this idea was put in practice for conformally flat initial data sets which are conformally flat. This family of initial data sets would satisfy trivially the regularity condition found in [15]. The calculations performed by means of scripts written in the computer algebra system Maple revealed that, generically, further logarithmic divergences would arise. Associated with these divergences there is a hierarchy of quantities written entirely in terms of the initial data (obstructions), that, if set to zero, would eliminate some of the logarithmic divergences from the solutions of the transport equations. Furthermore, an initial data set for which the obstructions vanish up to a certain order, \( p_* \), would be found to be asymptotic Schwarzschildian to a certain order \( s_* \), which would depend on \( p_* \)—i.e. \( s_*=s_*(p_*) \). This leads to conjecture that the only time symmetric, conformally flat initial data sets yielding a development admitting a conformal compactification are the Schwarzschildian ones.

Stationary (vacuum) spacetimes, and in particular the static ones, are known to admit a smooth null infinity—see [11]. In order to be able to consider static solutions one has to move away from the class of conformally flat initial data sets as the Schwarzschild solution is the only static solution having slices which are conformally flat near infinity. In [24] the calculations carried out in [21] were extended to the case of non-conformally flat spacetimes which satisfy Friedrich’s regularity condition. As expected, the calculations rendered a generalised hierarchy of obstructions, which, if satisfied, imply asymptotic stationarity to a certain order. Thus, it seems that time symmetric initial data sets would render developments admitting a null infinity if and only if they are asymptotic static. Providing a proof of this conjecture would constitute a remarkable feat.

Why so much interest in spacetimes admitting a smooth null infinity? The driving force of the programme initiated in [15] is to verify the so-called Penrose proposal. Loosely speaking, Penrose’s proposal states that the gravitational field of spacetimes describing isolated systems—e.g. two coalescing black holes—should satisfy in its asymptotic region the Peeling behaviour. That is, the components \( \bar{\Psi}_n \) of the Weyl tensor (where here we are using the standard Newman-Penrose notation) should decay along the generators of future oriented light cones as

\[
\bar{\Psi}_n = O\left( \frac{1}{r^{n+1}} \right), \quad n = 0, \ldots, 4, \tag{1}
\]

where \( r \) is an affine parameter along the generators of the light cones. As it was already pointed out in [25], spacetimes with obstructions like the ones found in [24] cannot be expected to peel. Thus, the question is now how restrictive is Penrose’s proposal when discussing systems of physical interest? In any case, it should be noted that non-static spacetimes satisfying the peeling behaviour do exist. These have been constructed by Chruściel & Delay from initial data sets which are asymptotic Schwarzschildian outside a compact set. The initial data sets have been obtained by using a refinement of the deformation techniques firstly introduced by Corvino in [6]. Perhaps one day numerical simulations will allow us to contrast the physical differences (if any) of a spacetime evolved, say, from Bowen-York initial data—which shall not peel—and those of

1This condition is expressed in terms of the Cotton-York (Bach) tensor and its derivatives to all orders. More precisely,

\[ S(D_{i_1} \cdots D_{i_q} B_{jk})(i) = 0 \quad \text{for} \quad q = 0, 1, \ldots, \]

where \( D_i \) denotes the covariant derivative of the metric on the initial hypersurface, \( B_{jk} \) is its Cotton tensor, and \( S \) denotes the operation of taking the symmetric, tracefree part of a tensor.

2Note, however, that the original proof had a gap which was filled in [7].
the time development of a data and initial data set which exactly is Bowen-York inside a compact set, and exactly Kerrian in the asymptotic region—which, arguably, should peel.

The present article and a companion one—which we shall refer as to paper II—extend the analysis initiated in [15, 24, 23] to the case of non-time symmetric (or time asymmetric) initial data sets. In particular, the present article is concerned with those time asymmetric initial data sets which are conformally flat. Thus, it encompasses, but it is not limited to, some of the initial data sets which are routinely used in the numeric simulations of spacetimes with black holes like the Bowen-York and Brand-Brügmann [3, 4]. We shall consider second fundamental forms with higher order multipole moments like the one used in [10]. On the other hand, paper II shall deal with time asymmetric initial data sets which are non-conformally flat, like the one that can be derived from the Kerr metric written in Boyer-Lindquist coordinates. Most of the ideas and methods to be employed in paper I will be developed and discussed in the present article.

A computer algebra based approach, similar to the one used in [24, 23] will reveal the existence of a hierarchy of obstructions to the smoothness of null infinity. In contrast to the time symmetric case where the vanishing of the obstructions implied a certain degree of regularity at both future and past null infinity, the obstructions arising when contemplating non-time symmetric initial data can be time asymmetric. That is, there is a hierarchy of obstructions ensuring the smoothness of $\mathcal{I}^+$ and another one for $\mathcal{I}^-$, and these do not imply each other in general. This novel feature opens the possibility of the existence of spacetimes where the two disconnected parts of null infinity have different degrees of regularity. The existence of this kind of situations had been speculated long time ago by Walker & Will [26, 27].

The analysis undertaken in [20, 25, 23] has shown that the only stationary spacetime that can be expected to admit conformally flat slices is the Schwarzschild spacetime. Thus, based on prior experience, one would expect that the new hierarchy of obstructions that are obtained in this analysis should, if vanishing, restrict the initial data set to be asymptotic Schwarzschildian to a certain order. This is found to be the case only if both the obstructions at $\mathcal{I}^+$ and $\mathcal{I}^-$ vanish. The Schwarzschild data referred to in this context is not the standard time symmetric $t =$ constant slice obtained from writing the Schwarzschild metric in isotropic coordinates, but a non-time symmetric, conformally flat slice having a second fundamental form decaying like

$$\tilde{\chi}_{ab} = O \left( \frac{1}{|y|^{-1}} \right).$$

as $|y| \to \infty$ in some coordinates adapted to the asymptotic region—see below. Whether it is possible to have a spacetime whose future null infinity is fully smooth, without having to restrict the behaviour of the field at past null infinity is a question which would require a substantial refinement of the computer algebra methods used in this article, and which will be left for the future.

This article follows, in as much as it possible, the notation and nomenclature of [15], and in particular that of [18]. It makes use of a number of results and techniques which although they are available in the literature, they may not be, widely known. Therefore, some time is spend in introducing the necessary ideas and concepts. The reader is in any case remitted to the original articles for full details.

The article is structured as follows: in section 2, a discussion of the constraint equations in the light of our intended applications is performed. In particular, some results due to Dain & Friedrich, guaranteeing the existence of the type of initial data under consideration are recalled. In section 3, a discussion of the description of the geometry of the initial hypersurface near spatial infinity in terms of spatial spinors is given. In section 4, a discussion of the solutions to the conformally flat momentum constraint within the space spinor formalism is given. Section 5 does likewise with the Hamiltonian constraint. Section 6 is concerned with the particularities of the propagation equations implied by the Conformal Einstein equations in the region of spacetime

\[^{3}\text{In these two articles, the authors discussed by means of post-Newtonian methods the effects of past null infinity of the incoming gravitational radiation produced by two gravitating bodies coming from infinity, } i^-, \text{ on approximately hyperbolic orbits which then escape to infinity, } i^+.\]
near null and spatial infinity. Section 7 deals with some properties of the asymptotic expansions near null and spatial infinity introduced by Friedrich. Section 8 discusses the results of our computer algebra calculations. Finally, in section 9 some concluding remarks are given.

In order to perform the calculations here described, a number of assumptions have been made. In order to help the reader to keep track of them, they have been clearly marked in the text and numbered from 1 to 5.

1.1 General conventions

In this work we shall be considering spacetimes \((\tilde{M}, \tilde{g}_{\mu\nu})\) arising as the development of some Cauchy initial data \((\tilde{S}, \tilde{h}_{\alpha\beta}, \tilde{\chi}_{\alpha\beta})\). Tilded quantities will refer to quantities in the physical spacetime, whereas untilded ones will denote generically quantities on an unphysical —i.e. conformally rescaled— spacetime. The indices \(\mu, \nu, \lambda, \ldots\) (second half of the Greek alphabet) are spacetime indices taking the values \(0, \ldots, 3\); while \(\alpha, \beta, \gamma, \ldots\) are spatial ones with range \(1, \ldots, 3\). The latin indices \(a, b, c, \ldots\) will be used in spatial expressions which are valid for a particular coordinate system (usually a Cartesian normal one) and take the values \(1, \ldots, 3\). The indices, \(i, j, k, \ldots\) are spatial frame indices ranging \(1, \ldots, 3\), while \(A, B, C, \ldots\) will be spinorial indices taking the values \(0, 1\). Because of the use of spinors, the signature of \(\tilde{g}_{\mu\nu}\) will be taken to be \((+,-,-,-)\), and the 3-dimensional metric \(\tilde{h}_{\alpha\beta}\) will be negative definite. Under our conventions and labeling, the Einstein constraint equations read

\[
\begin{align*}
\tilde{r} - \tilde{x}^2 + \tilde{\chi}_{\alpha\beta}\tilde{x}^{\alpha\beta} &= 0, \\
\tilde{D}^a\tilde{x}_{\alpha\beta} - \tilde{D}_\beta\tilde{x} &= 0, 
\end{align*}
\]

where \(\tilde{D}\) denotes the connection associated with the 3-metric \(\tilde{h}_{\alpha\beta}\), \(\tilde{r}\) is the corresponding Ricci scalar and we have written \(\tilde{x} = \tilde{x}^\alpha\).

The initial data set \((\tilde{S}, \tilde{h}_{\alpha\beta}, \tilde{\chi}_{\alpha\beta})\) is to be assumed asymptotically flat in the sense that there exists a compact subset of \(\tilde{S}\) such that its complement is the union of disjoint subsets \(\tilde{S}_k, k = 1, 2, \ldots, K\) of \(\tilde{S}\) —the asymptotically flat ends— each of which can be identified diffeomorphically with \(\{\mathbb{R}^3 \mid |y| > r_0\}\) where \(r_0\) is some positive real number, and \(|y| = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}\). We shall require that the coordinates so introduced satisfy

\[
\tilde{h}_{ab} = -\left(1 + \frac{2m_k}{|y|}\right)\delta_{ab} + \mathcal{O}\left(\frac{1}{|y|^2}\right), \quad \text{as } |y| \to \infty, \quad m_k = \text{constant}. \tag{4}
\]

Moreover, we shall require the second fundamental form to decay as

\[
\tilde{x}_{ab} = \mathcal{O}\left(\frac{1}{|y|^2}\right), \quad \text{as } |y| \to \infty. \tag{5}
\]

Besides the above asymptotic flatness requirements, we shall further assume \((\tilde{S}, \tilde{h}_{\alpha\beta})\) to be asymptotically Euclidean and regular in the sense that there exists a 3-dimensional, orientable, smooth, compact Riemannian manifold \((S, h_{\alpha\beta})\) with points \(i_k \in S, k = 1, 2, \ldots, q, q\) some positive integer, a diffeomorphism \(\Phi\) of \(S \setminus \{i_1, \ldots, i_q\}\) onto \(\tilde{S}\) and a function \(\Omega \in C^2(S) \cap C^\infty(S \setminus \{i_1, \ldots, i_q\})\) with the following properties:

\[
\begin{align*}
\Omega &= 0, \quad d\Omega = 0, \quad \text{Hess}(\Omega)_{\alpha\beta} = -2h_{\alpha\beta}, \\
\Omega &> 0 \text{ on } S \setminus \{i_1, \ldots, i_q\}, \\
h_{\alpha\beta} &= \Omega^2\Phi^*\tilde{h}_{\alpha\beta} \text{ on } S \setminus \{i_1, \ldots, i_q\}. \tag{6}
\end{align*}
\]

Note that suitable punctured neighbourhoods of the points \(i_k\) correspond to the asymptotically flat ends of \((\tilde{S}, \tilde{h}_{\alpha\beta})\). Thus, each point \(i_k\) represents a spatial infinity. The discussion in this article will be concerned with the behaviour of the gravitational field in one of these spatial infinities. So, without loss of generality we assume that there is one of them, which we denote by \(i\). The manifold \(S\) will be identified via \(\Phi\) with \(S \setminus \{i\}\).
2 The conformal constraint equations

In the sequel, we shall consider asymptotic expansions near infinity for a wide class of solutions of the constraint equations (3a) and (3b). Instead of working in the physical spacetime manifold, we shall consider a conformally rescaled version thereof

\[ g_{\mu \nu} = \Omega^2 \tilde{g}_{\mu \nu}. \]  

(7)

The first and second fundamental forms induced by the metrics \( g_{\mu \nu} \) and \( \tilde{g}_{\mu \nu} \) on \( \tilde{S} \) are then related via

\[ h_{\alpha \beta} = \Omega^2 \tilde{h}_{\alpha \beta}, \quad \chi_{\alpha \beta} = \Omega(\tilde{\chi}_{\alpha \beta} + \Sigma \tilde{h}_{\alpha \beta}), \]  

(8)

where \( \Sigma \) denotes the derivative of \( \Omega \) along the future directed \( g \)-unit normal of \( \tilde{S} \). As a consequence of the latter two equations the traces \( \chi = h_{\alpha \beta} \chi^{\alpha \beta} \), \( \tilde{\chi} = \tilde{h}_{\alpha \beta} \tilde{\chi}^{\alpha \beta} \) satisfy the relation

\[ \Omega \chi = \tilde{\chi} + 3\Sigma. \]  

(9)

The constraint equations (3a) and (3b) can be written in terms of the conformal fields as —see \[13\]—

\[ 2\Omega D_\alpha D^\alpha \Omega - 3D_\alpha \Omega D^\alpha \Omega + \frac{1}{2} \Omega^2 r - 3\Sigma^2 - \frac{1}{2} \Omega^2 - \frac{1}{2} (\chi^2 - \chi_{\alpha \beta} \chi^{\alpha \beta}) + 2\Omega \Sigma \chi = 0, \]  

(10a)

\[ \Omega^2 D^\alpha (\Omega^{-2} \chi_{\alpha \beta}) - \Omega (D^\beta \chi - 2\Omega^{-1} D_\beta \Sigma) = 0, \]  

(10b)

where \( D_\alpha \) denotes the Levi-Civita connection and \( r \) the Ricci scalar of the metric \( h_{\alpha \beta} \). The latter expressions lead to our first

**Assumption 1 (maximal initial data).** In this work we shall assume that

\[ \Sigma = 0, \quad \chi = 0 \text{ on } \tilde{S}. \]  

(11)

The first condition has to do with the choice of the conformal factor \( \Omega \), which up to now remains unspecified. Together they state that our investigation will be restricted to hypersurfaces \( \tilde{S} \) which are maximal with respect to both metrics \( g_{\mu \nu} \) and \( \tilde{g}_{\mu \nu} \) so that we can make use of the so-called conformal Ansatz to study the solutions of the constraint equations. Under assumption \[11\] the conformal constraint equations on \( \tilde{S} \) reduce to

\[ \left( D_\alpha D^\alpha - \frac{1}{8} r \right) \vartheta = \frac{1}{8} \chi_{\alpha \beta} \chi^{\alpha \beta} \vartheta \text{ with } \vartheta = \Omega^{-1/2}, \]  

(12)

\[ D^\alpha (\Omega^{-2} \chi_{\alpha \beta}) = 0. \]  

(13)

In order to solve in terms of expansions the constraint equations we shall make use of a slight variation of the so-called conformal method consisting of the following steps:

(i) choose a negative definite metric \( h_{\alpha \beta} \) on a 3-dimensional, orientable, smooth, compact manifold \( S \). In \( S \), choose a point \( i \) and set \( S = S \setminus i \).

(ii) Find a symmetric, \( h \)-tracefree tensor field \( \psi_{\alpha \beta} \) on \( \tilde{S} \) satisfying

\[ D^\alpha \psi_{\alpha \beta} = 0. \]  

(14)

A standard way of finding the tensor \( \psi_{\alpha \beta} \) is by means of a York splitting. That is, choose a symmetric, \( h \)-tracefree tensor \( \psi'_{\alpha \beta} \) on \( \tilde{S} \) and set

\[ \psi_{\alpha \beta} = \psi'_{\alpha \beta} + D_\alpha v_\beta + D_\beta v_\alpha - \frac{2}{3} h_{\alpha \beta} D_\gamma v^\gamma, \]  

(15)

where \( v_\alpha \) is some 1-form. It is well known that the latter Ansatz upon substitution on equation \[12\] leads to an elliptic equation for \( v_\alpha \).
(iii) Finally, substitute $\psi_{\alpha\beta} = \vartheta^{-4} \psi_{\alpha\beta}$ into equation (12) and find a positive solution $\vartheta$ of the resulting Licnerowicz equation:

$$
\left( D_\alpha D^\alpha - \frac{1}{8} \right) \vartheta = \frac{1}{8} \psi_{\alpha\beta} \psi^{\alpha\beta} \vartheta^{-7}.
$$

The fields $h_{\alpha\beta}$, $\Omega = \vartheta^2$ and $\chi_{\alpha\beta} = \Omega^2 \psi_{\alpha\beta}$ thus obtained provide a solution to the conformal constraint equations (12) and (13).

The above procedure is to be supplemented with asymptotic (boundary) conditions for the fields $\vartheta$ and $\psi_{\alpha\beta}$ consistent with the conditions (4) and (5). Consider $a > 0$ small enough so that the $h$-metric ball $B_a(i)$ centred on $i$ is a strictly convex normal neighbourhood of $i$, and let $\{x^a\}$ be normal coordinates with origin at $i$ based on an $h$-orthonormal frame $\{e_j\}$ at $i$. Consistent with the conditions (4)-(5) and (6a)-(6c) we shall require that

$$
|x| \vartheta \to 1, \quad \text{as } |x| \to 0.
$$

The tensorial fields $\tilde{\chi}_{\alpha\beta}$, $\chi_{\alpha\beta}$, $\psi_{\alpha\beta}$ are related by

$$
\tilde{\chi}_{\alpha\beta} = \Omega^{-1} \chi_{\alpha\beta} = \Omega \psi_{\alpha\beta}
$$

Thus, the appropriate behaviour for the unphysical fields is given by

$$
\chi_{\alpha\beta} = O(1), \quad \psi_{\alpha\beta} = O\left( \frac{1}{|x|^4} \right) \quad \text{as } |x| \to 0.
$$

In contrast to the analysis carried out in [15], here we shall consider initial data sets —i.e. solutions to the conformal constraint equations—which are not necessarily time symmetric, that is, in general we shall assume that $\chi_{\alpha\beta} \neq 0$. Discussions given in [15, 24, 25, 23] will arise as particular cases of our present analysis. This first article shall be concerned with the analysis of conformally flat initial data sets. Accordingly, we make the following assumption

**Assumption 2 (conformal flatness).** It will be assumed that there is a neighbourhood, $B_a(i) \subset S$ of radius $a$ centred on $i$ and coordinates $\{x^a\}$ for which the (unphysical) conformal metric is of the form

$$
h_{ab} = -\delta_{ab} = \text{diag}(-1,-1,-1).
$$

Results regarding the nonexistence of conformally flat slices imply that the latter assumption eliminates all stationary initial data sets except (time asymmetric) Schwarzschildian ones from our considerations [20, 25, 23]—this observation will play an important role in the interpretation of the results of our investigations. Moreover, based on the results of Dain [7, 8] it seems that any discussion of strictly stationary data —i.e. data whose development is stationary but not static— has to consider conformal metrics which are non-smooth. For example, the conformal metric one obtains for Kerrian initial data form $t = \text{constant}$ slices in Boyer-Lindquist coordinates is just $C^{2,\alpha}$ at infinity\(^4\). This non-smoothness of the conformal metric shall be discussed in extensis in paper II.

### 2.1 Existence of solutions to the conformally flat constraint equations

Under the assumptions (1) and (2), the Einstein constraint equations reduce—in Cartesian coordinates—to

$$
\partial_\alpha \psi^{\alpha\beta} = 0, \quad \text{(21a)}
$$

$$
\Delta \vartheta = \frac{1}{8} \psi_{ab} \psi^{ab} \vartheta^{-7}, \quad \text{(21b)}
$$

\(^4f \in C^{p,\alpha}$ means that the function has $p$-th order derivatives $f$ which are Hölder continuous with exponent $\alpha$. A function $f$ is said to be Hölder continuous with exponent $\alpha$ at a point $x_0$ if there is a constant $C$ such that $|f(x) - f(x_0)| \leq C|x - x_0|^\alpha$, $0 < \alpha < 1$ for $x$ in a neighbourhood of $x_0$.\]
together with the boundary conditions (17) and (19). We want to consider solutions to these equations which can be expanded in powers of $|x|$—note that $f(x^n) = |x|$ is not a smooth function of the coordinates $x^n$ as $\partial_n f = -x^n/|x|$. In particular, we shall exclude from our discussion any solutions whose expansions contain fractional powers of $|x|$ or $\ln |x|$ terms. These considerations lead naturally to the following definition, which we retake from [9]:

**Definition (E∞ spaces).** A function $f \in C^\infty(\tilde{S})$ is said to be in $E^\infty(B_a(i))$ if on $B_a(i)$ one can write $f = f_1 + |x| f_2$ with $f_1, f_2 \in C^\infty(B_a(i))$. 

The type of solutions of the constraint equations that we want to consider are naturally described in terms of these spaces. The existence of the solutions to (14) and (16) are guaranteed by the following two results due to Dain & Friedrich [9]. Note that these results apply to a larger class of data than the one under consideration—here $h_{ab} = \delta_{ab}$, which is trivially smooth. However, these are not general enough to encompass strictly stationary initial data.

**Theorem 1 (Dain & Friedrich, 2001).** Let $h_{ab}$ be a smooth metric on $S$ with non-negative Ricci scalar $r$. Assume that $\psi_{ab}$ is smooth on $\tilde{S}$ and satisfies on $B_a(i)$

$$|x|^8 \psi_{ab} \psi^{ab} \in E^\infty(B_a(i)). \tag{22}$$

Then there exists a unique solution of equation (16) which is positive, satisfies the boundary conditions (17) and (19) and in $B_a(i)$ has the form

$$\vartheta = \hat{\vartheta}/|x|, \quad \hat{\vartheta} \in E^\infty(B_a(i)), \quad \hat{\vartheta}(i) = 1. \tag{23}$$

The condition $|x|^8 \psi_{ab} \psi^{ab} \in E^\infty(B_a(i))$ excludes the possibility of second fundamental forms with linear momentum—the so-called boosted slices. In [9] it has been shown that if one considers a smooth conformal metric and second fundamental forms with linear momentum, then the solution of the Licnerowicz equation will contain logarithmic terms at order $|x|^2$.

Theorem 1 is complemented by the following result stating the conditions for the existence of a second fundamental form of the required type. It reads

**Theorem 2 (Dain & Friedrich, 2001).** Let $h_{ab}$ be a smooth metric on $S$. There exist traceless tensor fields $\psi_{ab} \in C^\infty(S \setminus \{i\})$ satisfying equation (14) with the properties:

(i) in the normal coordinates $\{x^n\}$,

$$\psi_{ab} = A |x|^3 (3n_a n_b - \delta_{ab}) + 3 |x|^3 (n_b \epsilon_{cad} J^d n^c + n_a \epsilon_{dbc} J^n n^d) + O(|x|^{-2}), \tag{24}$$

with $n^a = x^a/|x|$.

(ii) $D^n \psi_{ab} = 0$ on $\tilde{S}$

(iii) $|x|^8 \psi_{ab} \psi^{ab} \in E^\infty(B_a(i))$.

The form of the leading terms of the extrinsic curvature given in the latter theorem is, except for the absence of a term containing linear momentum, that of the Bowen-York Ansatz, which is routinely used in numerical simulations of black hole collisions—see e.g. [13, 14]. Note that the theorem allows for the presence of higher order terms—which, as it shall be shown in the sequel, will encode the multipolar content of the second fundamental form. These terms will play a crucial role in paper II when discussing the structure of stationary data. In relation to this point, we note that the results of [24] strongly suggest that the only stationary data contained in the hypothesis of theorems 1 and 2 are non-time symmetric Schwarzschildian data. We shall return to this point later.
3 The initial slice near $i$

Consider the Cartesian coordinates $\{x^a\}$ discussed in the previous section, and the orthonormal frame $\{e_j\}$ such that $e_j^a = x^a / |x|$. Under the assumptions made in the previous section one has that the solutions of the constraint equations (14) and (16) lead to a conformal factor of the form

$$\Omega = |x|^2 - m|x|^3 + O(|x|)^4. \quad (25)$$

Let $d^i_{jkl} = \Omega^{-1} C^i_{jkl}$ be the rescaled Weyl tensor, where $C^i_{jkl}$ is the Weyl tensor of the metric $g_{\mu\nu}$. The expansion (26) implies for the components of $d^i_{jkl}$ an expansion in $B_\alpha(i)$ of the form:

$$d_{ijkl} = \frac{m(h_{ik} - 3\omega_2^l \delta^i_j)\delta^0_j}{|x|^3} + O\left(\frac{1}{|x|^2}\right), \quad (26)$$

where $h_{ij} \equiv h(e_i, e_j) = h_{ab} e^a_i e^b_j = -\delta_{ij}$. In order to discuss this kind of singular quantities, it is convenient to consider a certain submanifold of the bundle of frames. We present here the crucial points of the construction of this manifold. For full details the reader is remitted to [15], [18] or [16].

3.1 The manifold $C_\alpha$

In what follows we shall be using a space spinor formalism analogous to a tensorial 3+1 decomposition\(^6\). Consider the (unphysical) spacetime $(M, g_{\mu\nu})$ obtained as the development of the initial data set $(S, h_{\alpha\beta}, \chi_{\alpha\beta})$. Let $SL(S)$ be the set of spin dyads $\delta = \{\delta_A\}_{A=0,1}$ on $S$ which are normalised with respect to the alternating form $\epsilon$ in such a way that

$$\epsilon(\delta_A, \delta_B) = \epsilon_{AB}, \quad \epsilon_{01} = 1. \quad (27)$$

The set $SL(S)$ has a natural bundle structure where $S$ is the base space, and its structure group is given by

$$SL(2, \mathbb{C}) = \{t^A_B \in GL(2, \mathbb{C}) \mid \epsilon_A t^A_B t^C_D = \epsilon_B D\}, \quad (28)$$

acting on $SL(S)$ by $\delta \mapsto \delta \cdot t = \{\delta_A t^A_B\}_{B=0,1}$. Now, let $\tau = \sqrt{2} e_0$, where $e_0$ is the future $g$-unit normal of $S$ and

$$\tau_{A A'} = g(\tau, \delta A \bar{A'}) = \epsilon_A^0 \epsilon_A^0 + \epsilon_A^1 \epsilon_A^1 \quad (29)$$

is its spinorial counterpart — that is, $\tau = \tau^A e_A = \sigma_{AA'}^i \tau^{A'} e_i$ where $\sigma_{AA'}^i$ denote the Infeld-van der Waerden symbols and $\{e_i\}_{i=0,\ldots,3}$ is an orthonormal frame. The spinor $\tau_{A A'}$ enables the introduction of space-spinors — sometimes also called $SU(2)$ spinors, see [1] [12] [22]: it defines a subbundle $SU(S)$ of $SL(S)$ with structure group

$$SU(2, \mathbb{C}) = \{t^A_B \in SL(2, \mathbb{C}) \mid \tau_{AA'} t^A_B \tau_{B B'} = \tau_{BB'}\}, \quad (30)$$

and spatial van der Waerden symbols:

$$\sigma_{i B}^A = \sigma_{i B}^A, \quad \sigma_{AB} = \tau_{(B} A') \sigma_{i A')}, \quad i = 1, 2, 3. \quad (31)$$

The latter satisfy

$$h_{ij} = \sigma_{i AB} \sigma_{j AB}, \quad -\delta_{ij} \sigma_{i AB} \sigma_{j CD} = -\epsilon_{A(C} \epsilon_{D)} B \equiv h_{ABCD}, \quad (32)$$

with $h_{ij} = h(e_i, e_j) = -\delta_{ij}$. The bundle $SU(S)$ can be endowed with a $su(2, \mathbb{C})$-valued connection form $\bar{\omega}^{AB}_{\bar{B}}$ compatible with the metric $h_{\alpha\beta}$ and 1-form $\sigma^{AB}$, the solder form of $SU(S)$\(^6\). The solder form satisfies by construction

$$h = h_{\alpha\beta} dx^\alpha \otimes dx^\beta = h_{ABCD} \sigma^{AB} \otimes \sigma^{CD}, \quad (33)$$

\(^6\)However, it must be said that the use of such a spinorial formalism is not essential. An equivalent discussion can be carried out in terms of frames. See e.g. [12].

\(^7\)To be more precise, these structures are inherited from their analogues on $O_4(S)$, the bundle of positively oriented orthonormal frames via the covering map of $SU(2)$ onto the connected component $SO(3)$ of the rotation group given by $SU(2) \ni t^A_B \Psi = \sigma_{AB}^i c_i A B D \sigma_{i A^*}^D \in SO(3)$, the induced isomorphism of Lie algebras, $\Psi_*$, and suitable contractions with the infeld-van der Waerden symbols.
where $\sigma^{AB} = \sigma^\alpha_a dx^\alpha$ —note that the $\sigma^{AB}$ are not the spatial Infeld-van der Waerden symbols!

Now, given a spinorial dyad $\delta \in SU(\mathcal{S})$ one can define an associated vector frame on $O_+(\mathcal{S})$ via $e_j = e_j(\delta) = \sigma^{AB}_j \delta_A T_B B^\mathcal{S}_{B'}$. It is noted that different dyads $\delta$ and $\delta'$ can give rise to the same frame as long as they are $U(1)$-related\(^7\). We shall restrict our attention to dyads giving rise to frames $\{e_j\}_{j=0,\ldots,3}$ on $\mathcal{B}_a(i)$ such that $e_3$ is tangent to the $h$-geodesics starting at $i$. Let $\bar{H}$ denote the horizontal vector field on $SU(\mathcal{S})$ projecting to the aforementioned radial vectors $e_3$.

The fiber $\pi^{-1}(i) \subset SU(\mathcal{S})$ —the fiber “over” $i$— can be parametrised by choosing a fixed dyad $\delta^*$ and then letting the group $SU(2,\mathbb{C})$ act on it. Let $(-a,a) \ni \rho \mapsto \delta(\rho,t) \in SU(\mathcal{S})$ be the integral curve through the fiber $\pi^{-1}(i)$ such that $e_3$ is tangent to the $h$-geodesics starting at $i$. Let $\bar{H}$ satisfy $\delta(0,t) = \delta(t) \in \pi^{-1}(i)$. With this notation we define the set

$$C_a = \left\{ \delta(\rho,t) \in SU(\mathcal{S}) \mid |\rho| < a, \quad t \in SU(2,\mathbb{C}) \right\}$$

which is a smooth submanifold of $SU(\mathcal{S})$ diffeomorphic to $(-a,a) \times SU(2,\mathbb{C})$. The vector field $\bar{H}$ is such that its integral curves through the fiber $\pi^{-1}(i)$ project onto the geodesics through $i$. From here it follows that the projection map $\pi$ of the bundle $SU(\mathcal{S}) \to \mathcal{S}$ maps $C_a$ into $\mathcal{B}_a(i)$.

Let in the sequel $\mathcal{T}^0 \equiv \pi^{-1}(i) = \{ \rho = 0 \}$ denote the fiber over $i$. It can be seen that $\mathcal{T}^0 \approx SU(2,\mathbb{C})$. On the other hand, for $p \in \mathcal{B}_a(i) \setminus \{ i \}$ it turns out that $\pi^{-1}(p)$ consists of an orbit of $U(1)$ for which $|x(p)|$, and another for which $|x(p)| = |x(p)|$, where $x^A(p)$ denote normal coordinates of the point $p$. Because of the latter, it is convenient, in order to understand better the structure of the manifold $C_a$ to quotient out the effect of $U(1)$ on $C_a$.

The projection map restricted to $C_a \subset SU(\mathcal{S})$ can be factorised as $C_a \xrightarrow{\pi^{-1}} C'_a = \pi^{-1}(\mathbb{C}) \mathcal{B}_a(i) = \pi_2^{-1}(\mathcal{B}_a(i))$. Now, the set $\pi^{-1}(\mathcal{B}_a(i))$ consists of a component $C^+_a$ on which $\rho > 0$ and another, $C^-_a$ on which $\rho < 0$. The projection $\pi_2$ maps both $C^+_a$ and $C^-_a$ into the punctured disk $\mathcal{B}_a(i) \setminus \{ i \}$. This last fact can be used to identify $\mathcal{B}_a(i)$ with $C^+_a$ acquiring in the process a boundary given by $\pi_1(\mathcal{T}^0) = \pi_1^{-1}(i) \approx S^2$. The set $\mathcal{B}_a(i) = (\mathcal{B}_a(i) \setminus \{ i \}) \cup \pi_2^{-1}(i) \approx [0,a) \times S^2$ is a smooth manifold with boundary. Summarising, we have obtained an extension of the initial (physical) manifold $\mathcal{S}$ by blowing up the point the point $i$ into a sphere.

For practical purposes, it is more convenient to make use of the 4-dimensional $U(1)$-bundle

$$\mathcal{C}^+_a = C^+_a \cup \mathcal{T}^0 = \left\{ \delta = \delta(\rho,t) \in C_a \mid |\rho(\delta)| \geq 0 \right\} \approx [0,a) \times SU(2,\mathbb{C}).$$

(35)

The construction of the manifold $C_a$ is such that a number of useful structures are inherited by $C_a$ from $SU(\mathcal{S})$. In particular, the solder and connection forms can be pulled back to smooth 1-forms on $C_a$. These shall be again denoted by $\sigma^{AB}$ and $\omega^A_B$. They satisfy the structure equations:

$$d\sigma^{AB} = -\omega^A_E \wedge \sigma^{EB} - \omega^B_E \wedge \sigma^{AE},$$

(36a)

$$d\omega^A_B = -\omega^A_E \wedge \omega^E_B + \Omega^A_B,$$

(36b)

with

$$\Omega^A_B = \frac{1}{2} r^A_{BCDEF} \sigma^{CD} \wedge \sigma^{EF},$$

(37)

the so-called curvature form determined by the curvature spinor $r_{ABCDEF}$ given by

$$r_{ABCDEF} = \left( \frac{1}{2} s_{ABCE} - \frac{1}{12} r_{ABCE} \right) \epsilon_{DF} + \left( \frac{1}{2} s_{ABDF} - \frac{1}{12} r_{ABDF} \right) \epsilon_{CE},$$

(38)

where $s_{ABCD} = s_{(ABCD)}$ is the tracefree part of the Ricci tensor of $h_{\alpha\beta}$ and $r$ its Ricci scalar. These satisfy the 3-dimensional Bianchi identity

$$D^{AB} s_{ABCD} = \frac{1}{6} D_{CD} r.$$

(39)

\(^7\)Here we shall use the realisation

$$U(1) = \left\{ t \in U(1) \mid t = \left( \begin{array}{cc} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{array} \right), \quad \phi \in \mathbb{R} \right\}.$$
In the case of the data considered in this paper—conformally flat, assumption 2—it follows that \( s_{ABCD} = 0 \) and \( r = 0 \) so that \( \Omega^A_B = 0 \). A detailed analysis of the solutions of the structure equations in the case of non-smooth conformal metrics will be given in paper II.

### 3.2 Lifts to \( \mathcal{C}_a \) and reality conditions

One can use the function \( \rho \) and the matrices \( t = (t^A_B) \in SU(2, \mathbb{C}) \) to coordinatise \( \mathcal{C}_a \). Again, let \( \{ x^\alpha \} \) denote normal coordinates on \( \mathcal{B}_a(i) \) such that \( x^\alpha(i) = 0 \). The projection map \( \pi \) of \( \mathcal{C}_a \) onto \( \mathcal{B}_a(i) \) has the local expression

\[
\pi : (\rho, t) \mapsto x^\alpha(\rho, t) = \sqrt{2} \rho^{\alpha \beta} \sigma^j C^D_D t^D_1. \quad (40)
\]

This last expression can be used to “lift” scalar and spinorial fields on \( \mathcal{B}_a(i) \) to \( \mathcal{C}_a \). In an abuse of notation we shall, generally, denote the lift to \( \mathcal{C}_a \) of a spinorial field \( \mu_{AB \cdots EF} \) on \( \mathcal{B}_a(i) \) again by \( \mu_{AB \cdots EF} \). To which manifold the field belongs will be clear from the context.

The following two maps between spinors will play a role in the sequel: given a “primed” spinorial field \( \mu'_{A'} \), one can associate an unprimed one via \( \mu_A \mapsto \mu_A = \tau_A^{A'} \mu'_{A'} \). The other one is the Hermitian conjugation map \( \mu_A \mapsto \mu^\dagger_A = \tau_A^{A'} \pi'_{A'} \). These maps are extended to higher valence spinors in a direct way. A spinorial field \( \nu_{A_1 A_2 \cdots A_k} \) is said to be spatial if and only if \( \tau^{A_1 A_2} \nu_{A_1 A_2 \cdots A_{j-1} A_{j+1} \cdots A_k} = 0 \) for \( j = 1, \ldots, k \). In which case its “unprimed” version is given by

\[
\mu_{A_1 B_1 \cdots A_k B_k} = \tau_{B_1}^{A'_1} \cdots \tau_{B_k}^{A'_k} \nu_{A_1 A'_1 \cdots A_k A'_k} = \nu_{(A_1, B_1) \cdots (A_k, B_k)}. \quad (41)
\]

Conversely, any spinor field with \( 2k \) unprimed indices corresponds to a spatial spinor. Thus, for a given spinorial field \( \mu_{AA'} \) one obtains an orthogonal splitting of the form

\[
\mu_{AA'} = \tau_{CA'}^{CB'} \mu_{BB'} = \frac{1}{2} \tau_{AA'}^{CB'} \mu_{CC'} - \tau_{A'}^{C} \mu_{CA}, \quad (42)
\]

where \( \mu_{CA} = \tau_{(C}^{B'} \mu_{A)B'} \) corresponds to the spatial part of \( \mu_{AA'} \). Again, this procedure can be extended in a standard way to higher valence spinors. As a final remark we note that, given a spatial spinor field \( \mu_{A_1 B_1 \cdots A_k B_k} = \mu_{(A_1, B_1) \cdots (A_k, B_k)} \) the frame components \( \nu_{j_1 \cdots j_k} = \sigma_{j_1}^{A_1 B_1} \cdots \sigma_{j_k}^{A_k B_k} \mu_{A_1 B_1 \cdots A_k B_k} \) correspond to those of a real tensor field if and only if it satisfies the reality condition

\[
\mu_{A_1 B_1 \cdots A_k B_k} = (-1)^k \mu_{A_1 B_1 \cdots A_k B_k}. \quad (43)
\]

### 3.3 Decompositions in terms of irreducible spinors

A symmetric valence 2 spinor \( \mu_{AB} \) has 3 essential components. In order to make this explicit, we make use of the irreducible spinors

\[
x_{AB} \equiv \sqrt{2} \epsilon^0_A \epsilon^1_B, \quad y_{AB} \equiv \frac{1}{\sqrt{2}} \epsilon^1_A \epsilon^1_B, \quad z_{AB} \equiv \frac{1}{\sqrt{2}} \epsilon^0_A \epsilon^0_B, \quad (44)
\]

to write

\[
\mu_{AB} = \mu_x x_{AB} + \mu_y y_{AB} + \mu_z z_{AB}. \quad (45)
\]

Similarly, a valence 4 spinor \( \nu_{ABCD} = \nu_{(AB)(CD)} \) has 9 essential components and can be written as

\[
\nu_{ABCD} = \nu_0 \epsilon^0_{ABCD} + \nu_1 \epsilon^1_{ABCD} + \nu_2 \epsilon^2_{ABCD} + \nu_3 \epsilon^3_{ABCD} + \nu_4 \epsilon^4_{ABCD} + \nu_5 \epsilon^5_{ABCD} + \nu_6 \epsilon^6_{ABCD} + \nu_7 \epsilon^7_{ABCD} + \nu_8 \epsilon^8_{ABCD} + \nu_9 \epsilon^9_{ABCD}, \quad (46)
\]

where \( \epsilon^i_{ABCD} = \epsilon^i_A \epsilon^j_B \epsilon^k_C \epsilon^l_D \). The notation, which shall be used again in the sequel, \((\cdots)_i\)

means that the indices are to be symmetrised and then \( i \) of them should be set equal to 1. In particular, if \( \nu_{ABCD} \) is associated with a symmetric tensor \( \nu_{\alpha\beta} = \nu_{(\alpha\beta)} \) then \( \nu_x = \nu_y = \nu_z = 0 \). If, furthermore, \( \nu_{\alpha\beta} \) is traceless then \( \nu_h = 0 \).
3.4 Vector fields on \( \mathcal{C}_a \)

Consequently with the discussion of the previous section one has that \( \mathcal{H} = \partial_\rho \). Vector fields relative to the \( SU(2, \mathbb{C}) \)-dependent part of the coordinates are obtained by looking at the basis of the (3-dimensional) Lie algebra \( \mathfrak{su}(2, \mathbb{C}) \) given by

\[
  u_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad u_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad u_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

(47)

In particular, the vector \( u_3 \) is the generator of \( U(1) \). Denote by \( Z_i, i = 1, 2, 3 \) the Killing vectors generated on \( SU(S) \) by \( u_i \) and the action of \( SU(2, \mathbb{C}) \). The vectors \( Z_i \) are tangent to \( T^0 \). On \( T^0 \) we set

\[
  X_+ = -(Z_2 + iZ_1), \quad X_- = -(Z_2 - iZ_1), \quad X = -2iZ_3,
\]

(48)

and extend these vector fields to the rest of \( \mathcal{C}_a \) by demanding them to commute with \( \mathcal{H} = \partial_\rho \).

For latter use we note that

\[
  [X, X_+] = 2X_+, \quad [X, X_-] = -2X_-, \quad [X_+, X_-] = X.
\]

(49)

The vector fields are complex conjugates of each other in the sense that for a given real-valued non-vanishing pairings \( \sigma \in B_a(i) \) we set

\[
  \mathcal{H}, X_\pm \text{ span the tangent space at } p.
\]

More importantly, it can be seen that for \( p \in B_a(i) \setminus \{i\} \) the projections of the fields \( X_\pm \) span the tangent space at \( p \).

We define a frame \( c_{AB} = c_{(AB)} \) dual to the solder forms \( \sigma^{CD} \) and require it not to pick components along the fibres —i.e. along the direction of \( X \). These requirements imply

\[
  \langle \sigma_{AB}, c_{CD} \rangle = \eta_{AB}^{\ CD}, \quad c_{CD} = c_{CD}^{\pm} \partial_\rho + c_{CD}^{\ 0} X_\pm + c_{CD}^{\ -} X_0.
\]

(50)

Let \( \alpha^\pm \) and \( \alpha \) be 1-forms on \( \mathcal{C}_a \) annihilating the vector fields \( \partial_\rho \) and having with \( X_\pm \) the non-vanishing pairings

\[
  \langle \alpha^+, X_+ \rangle = \langle \alpha^-, X_- \rangle = \langle \alpha, X \rangle = 1.
\]

(51)

From the properties of the solder form \( \sigma_{AB} \) one finds that

\[
  c_{AB}^1 = x_{AB}, \quad c_{AB}^{\ +} = \frac{1}{\rho} z_{AB} + \tilde{c}_{AB}^{\ +}, \quad c_{AB}^{\ -} = \frac{1}{\rho} y_{AB} + \tilde{c}_{AB}^{\ -}.
\]

(52)

In particular, for the case under consideration —assumption 2—, one has \( c_{AB}^{\ \pm} = 0 \). Furthermore, one has that

\[
  \sigma^{AB} = \frac{1}{\rho} x^{AB} \rho - 2y^{AB} \alpha^+ - 2z^{AB} \alpha^-,
\]

(53)

so that the projection of \( \mathcal{H} \) onto \( B_a(i) \) renders

\[
  h = -d\rho \otimes d\rho - 2\rho^2 (\alpha^+ \otimes \alpha^- + \alpha^- \otimes \alpha^+) \equiv -d\rho^2 - \rho^2 d\sigma^2,
\]

(54)

so that \( d\sigma^2 = 2(\alpha^+ \otimes \alpha^- + \alpha^- \otimes \alpha^+) \) corresponds to the pull back of the standard metric on \( S^2 \).

3.5 The connection coefficients

The connection coefficients are defined by contracting the connection form \( \tilde{\omega}^A_{\ B} \) with the frame \( c_{AB} \). In general, we write

\[
  \gamma_{CD}^{\ A}_{\ B} \equiv \langle \tilde{\omega}^A_{\ B}, c_{CD} \rangle = \frac{1}{\rho} \tilde{\omega}^A_{\ CDB} + \tilde{\gamma}_{CD}^{\ A}_{\ B},
\]

(55)

where,

\[
  \tilde{\gamma}_{ABCD} = \frac{1}{2} (\epsilon_{AC} x_{BD} + \epsilon_{BD} x_{AC}),
\]

(56)

denotes the singular part of the connection coefficients. The regular part of the connection can be related to the frame coefficients \( c_{AB} \) via commutator equations which under assumption 2 are trivially satisfied as \( \tilde{\gamma}_{ABCD} = 0 \). We defer any further discussion of these matters to paper II.
Let $f$ be a smooth function on $B_a(i)$. We denote again by $f$ its lift. The covariant derivative of $f$ is then given on $C_a$ by

$$D_{AB}f = c_{AB}(f).$$

(57)

Similarly, let $\mu_{AB}$ represent both a smooth spinor field on $B_a(i)$ and its lift to $C_a$. Then the covariant derivative of $\mu_{AB}$ is given by

$$D_{AB}\mu_{CD} = c_{AB}(\mu_{CD}) - \gamma_{AB}^{ECHED} - \gamma_{AB}^{E}D_{ECE}.$$ 

(58)

Analogous formulae hold for higher valence spinors.

### 3.6 Normal expansions

In order to study the behaviour of diverse fields on $C_a$ near $I_0$ in a detailed manner, we shall make use of a certain type of expansions which are obtained by lifting the Taylor expansions (along the radial direction) of the fields and then conveniently symmetrising.

A notion which will be of much help is the following: a smooth function $f$ on $C_a$ is said to have spin weight $s$ if

$$Xf = 2sf$$

(59)

where $2s$ is an integer. If furthermore, $f$ is analytic in $B_a(i)$ then on $C_a$ it admits an expansion of the form

$$f = \sum_{p=|s|}^{\infty} \frac{1}{p!} \sum_{q \in Q_p}^{2q} f_{p;2q,m} T_{2q}^m q^{-s} p^p,$$

(60)

where

$$Q_p = \left\{ q \in \mathbb{Z}^+ \cup \{0\} \mid |s| \leq q \leq p, \quad q \text{ even if } p \text{ is even, } q \text{ odd if } p \text{ is odd} \right\}$$

(61)

and $f_{p;2q,m} \in \mathbb{C}$. The functions $T_{m}^{j,k}$ appearing in the expansion (60) are the functions of $SU(2, \mathbb{C})$ onto $\mathbb{C}$ given by

$$SU(2, \mathbb{C}) \ni t \mapsto T_{m}^{j,k}(t) = \left( \frac{m}{j} \right)^{1/2} \left( \frac{m}{k} \right)^{1/2} e^{\frac{(B_1 \ldots B_m)^j}{(A_1 \ldots A_m)_k}},$$

(62a)

$$T_{0}^{0}(t) = 1, \quad j, k = 0, \ldots, m, \quad m = 1, 2, 3, \ldots,$$

(62b)

where the string of indices with a lower index $k$, say, means that the indices are symmetrised and then $k$ of them are set equal to 1, while the remaining ones are set equal to 0. The function $f$ will be said to have axial symmetry if its expansion (60) is of the form

$$f = \sum_{p=|s|}^{\infty} \frac{1}{p!} \sum_{q \in Q_p} f_{p;2q,q} T_{2q}^q q^{-s} p^p.$$ 

(63)

The following properties of the functions $T_{m}^{j,k}$ are crucial: under complex conjugation they transform as

$$\overline{T_{m}^{j,k}} = (-1)^{j+k} T_{m}^{m-j,k}.$$ 

(64)

Furthermore

$$X_j T_{m}^{j,k} = \sqrt{j} (m-j+1) T_{m}^{j+1,k-1} \quad X_j T_{m}^{j,k} = -\sqrt{(j+1)(m-j)} T_{m}^{j+1,k},$$

$$X T_{m}^{j,k} = (m-2j) T_{m}^{m,k}.$$ 

Also important for our aims is the fact that the set $\left\{ \sqrt{m + 1} T_{m}^{j,k} \right\}$ constitutes an orthonormal basis of the space $L^2(\mu, SU(2, \mathbb{C}))$ where $\mu$ is the Haar measure on $SU(2, \mathbb{C})$. 

12
The functions $T_{m,k}$ are closely related to the spherical harmonics. These are defined as a system of orthogonal functions on $S^2$. They can be extended to $S^3$ as functions with zero spin weight. From this it follows that they can be expanded as $Y_{lm} = \sum c_{j} T_{2k}^{j}$. Indeed, one has

$$Y_{nm} = (-i)^{s+2n-m} \sqrt{\frac{2n+1}{4\pi}} T_{2n}^{n-m}_{n}. \quad (65)$$

Any analytic function on $SU(2,\mathbb{C})$ can be expanded in terms of functions $T_{m,k}$. In particular, the product $T_{2n_1}^{k_1}_{l_1} \times T_{2n_2}^{k_2}_{l_2}$ can be linearised rendering

$$T_{2n_1}^{k_1}_{l_1} \times T_{2n_2}^{k_2}_{l_2} = \sum_{n=0}^{n_1+n_2} (-1)^{n+n_1+n_2} C(n_1,n_1-l_1,n_2,n_2-l_2,n_1+n_2-l_1-l_2)$$

$$\times C(n_1,n_1-l_1,n_2,n_2-l_2,n_1+n_2-l_1-l_2) \times T_{2n}^{n+k_1+k_2-n_1-n_2}_{n+l_1+l_2-n_1-n_2}, \quad (66)$$

where $q_0 = \max\{|n_1-n_2|, n_1+n_2-k_1-k_2, n_1+n_2-l_1-l_2\}$ and $C(l,m_1;l_2,m_2;l,m)$ denote the Clebsch-Gordan coefficients of $SU(2,\mathbb{C})$.

Finally, we note that in occasions we will be in the need of dealing with fields on $\mathcal{B}_a(i)$ which do not project to analytic fields on $\mathcal{B}_a(i)$, but still have a well defined spin weight. Let $g$ be one of such fields with:

$$g = \sum_{p=0}^{n} \frac{1}{p!} g_p g^p, \quad (67)$$

where

$$g_p = \sum_{q=|s|}^{n(p)} \sum_{m=0}^{2q} g_{p;2q,m} T_{2q}^{m}_{q-s}, \quad (68)$$

then we say that the coefficient $g_p$ has expansion type $n(q)$.

We finish by mentioning that the components of a given spinor arising in the decompositions (69) and (70) in terms of elementary irreducible spinors have definite spin weights. More precisely, given the rank 2 spinor $\mu_{AB}$, its components $\mu_{\nu}, \mu_{\nu x}$ and $\mu_{\nu z}$ have spin weight $-1$, $0$ and $+1$ respectively. While for a rank 4 spinor, $\nu_{ABCD}$, the components $\nu_{2}, \nu_{h}$ and $\nu_{g}$ have spin weight $0$; $\nu_{1}, \nu_{s}$ have spin weight $+1$; $\nu_{3}$ and $\nu_{g}$ have spin weight $-1$; while $\nu_{0}$ and $\nu_{q}$ have spin weight $+2$ and $-2$ respectively.

4 Expansions of solutions to the momentum constraint

Using the framework described in the previous section, we proceed now to discuss the solutions to the conformally flat momentum constraint. The essential content of this section is not new, but the presentation is. Other approaches can be found in, for example, 2, 9

We begin with some fairly standard considerations. We shall refer to $\mathbb{R}^3$ endowed in standard (Cartesian) coordinates \( \{ x^a \} \) with the metric $-\delta_{ab}$ as to the Euclidean space $\mathbb{R}^3$. In $\mathbb{R}^3$ it was shown that any smooth solution in $\mathcal{B}_a(i) \setminus \{ i \} \subset \mathbb{R}^3$ of the momentum constraint can be written in the form

$$\psi^{ab} = \psi^{ab}_P + \psi^{ab}_A + \psi^{ab}_Q + \psi^{ab}_X,$$  

where $\psi^{ab}_P = \mathcal{O}(|x|^{-4}), \psi^{ab}_J = \mathcal{O}(|x|^{-3}), \psi^{ab}_A = \mathcal{O}(|x|^{-3}), \psi^{ab}_Q = \mathcal{O}(|x|^{-2})$ correspond, respectively, to the parts of the second fundamental form associated with translational, rotational, expansion, and boost conformal Killing vectors. They all consist of $l = 0, 1$ spherical harmonics. Let $n^a = x^a/|x|$, then if $\psi^{ab}_P \neq 0$ the vector

$$P^{a} = \frac{1}{8\pi} \lim_{\epsilon \to 0} \int_{S_{\epsilon}} |x|^2 \psi_a n^b (2n^c n^a - \delta^{ca}) dS,$$  

$$\epsilon$$Some other notations used in the physics literature are:

$$C(l_1,m_1;l_2,m_2;l,m) = \langle t_1,t_2;m_1,m_2|l,m \rangle = C(l_1,l_2;m_1,m_2,m),$$

$$C(l_1,m_1;l_2,m_2;l,m) = \langle t_1,t_2;m_1,m_2|l,m \rangle = C(l_1,l_2;m_1,m_2,m).$$
corresponds to the ADI linear momentum of the initial data. Similarly, if $\psi^{ab}_j \neq 0$ then

$$J^a = \frac{1}{8\pi} \lim_{\epsilon \to 0} \int_{S_\epsilon} |x| \psi_{bc} n^b c^{cd} n_d dS,$$

(71)

is the ADI angular momentum of the data. In the last two expressions $S_\epsilon$ denotes a sphere of radius $\epsilon$ centred at $i$, $dS_\epsilon$ its volume element, and $n^a$ its unit normal.

The term $\psi^{ab}_j$ in equation (62) is, on the other hand, associated with $l \geq 2$ spherical harmonics and, thus, can be interpreted as providing the higher order momenta content of the second fundamental form. It consists of derivatives of a certain spin-weight 2 potential $\bar{\partial}^2 \lambda$, where $\lambda$ is an arbitrary complex $C^\infty$ function in $B_0(i) \setminus \{i\}$, and $\bar{\partial}$ denotes the standard NP “eth” operator. Important for our purposes is that if $P^a = 0$ and $|x| \lambda \in E^\infty(B_0(i))$ —in which case $\psi^{ab}_j = O(|x|^{-2})$— then $|x|^2 \psi^{ab}_j \psi_{ab} \in E^\infty(B_0(i))$ —see theorem 15 in reference [9]— so that theorem 1 and 2 apply.

### 4.1 Spinorial version of the momentum constraint

In the space spinor formalism, a symmetric tracefree tensor $\psi_{\alpha\beta}$ is represented by the totally symmetric spinor $\psi_{ABCD} = \psi_{(ABCD)}$ living on $C_\alpha$. The spinorial and tensorial versions are related in a standard way via

$$\psi_{ABCD} = \sigma^i_{AB} \sigma^j_{CD} \psi_{ij} = \sigma_{AB}^i \sigma^j_{CD} \epsilon^i_j \psi_{\alpha\beta}.$$  

(72)

The symmetries of $\psi_{ABCD}$ together with its tracelessness —$\psi^{AB}_{AB} = 0$— imply a decomposition in terms of elementary spinors of the form:

$$\psi_{ABCD} = \chi \delta^A_{ABCD} + \psi_1 \epsilon^1_{ABCD} + \psi_2 \epsilon^2_{ABCD} + \psi_3 \epsilon^3_{ABCD} + \psi_4 \epsilon^4_{ABCD}.$$  

(73)

It can be verified that the components $\psi_0$, $\psi_1$, $\psi_2$, $\psi_3$ and $\psi_4$ are respectively of spin weight $2, 1, 0, -1, -2$. The components of $\psi_{ABCD}$, being those of a spinor associated with a real spatial tensor, must satisfy in virtue of the reality condition (74), the relations

$$\psi_2 = \overline{\psi}_2, \quad \psi_1 = -\overline{\psi}_3, \quad \psi_0 = \overline{\psi}_4.$$  

(74)

The momentum constraint can be rewritten as

$$D^{AB} \psi_{ABCD} = 0.$$  

(75)

### 4.2 The $l = 0, 1$ harmonic solutions

Arguably, the most important solutions to the Euclidean momentum constraint (75) are those having $l = 0, 1$ harmonics. The importance of this class lies in the fact that they are associated to conformal Killing vectors of the Euclidean space. Because of the spin weights of the components of $\psi_{ABCD}$, we can write —cfr. the expansion (72)—

$$\psi_0 = 0,$$

$$\psi_1 = \psi_{1,2,0} T_{2,0} + \psi_{1,1,2,1} T_{2,0} + \psi_{1,2,2,2} T_{2,0},$$

$$\psi_2 = \psi_{2,0,0} T_{2,0} + \psi_{2,2,0} T_{2,0} + \psi_{2,2,1} T_{2,0} + \psi_{2,2,2} T_{2,0},$$

$$\psi_3 = \psi_{3,2,0} T_{2,0} + \psi_{3,3,1} T_{2,0} + \psi_{3,2,2} T_{2,0},$$

$$\psi_4 = 0.$$  

The substitution of the above expressions into equation (75) renders a solution which can be written as —cfr. equation (73)—:

$$\psi_{ABCD} = \psi_{ABCD}^A + \psi_{ABCD}^P + \psi_{ABCD}^Q + \psi_{ABCD}^J.$$  

(76)

where the non-vanishing components of $\psi_{ABCD}^A$ are given by

$$\psi_{2}^A = -\frac{A}{\rho^2} T_{0,0}^{0}.$$  

(77)
Those of $\psi_{ABCD}^p$ are

$$\psi_p^1 = \frac{3}{\rho^4}(P_2 + iP_1)T_{2}^{0} - \frac{3\sqrt{2}}{\rho^4}P_3T_{2}^{1} - \frac{3}{\rho^4}(P_2 - iP_1)T_{2}^{0},$$
$$\psi_p^2 = \frac{9\sqrt{2}}{2\rho^4}(P_2 + iP_1)T_{2}^{0} - \frac{9}{\rho^4}P_3T_{2}^{1} - \frac{9\sqrt{2}}{2\rho^4}(P_2 - iP_1)T_{2}^{0},$$
$$\psi_p^3 = -\frac{3}{\rho^4}(P_2 - iP_1)T_{2}^{2} - \frac{3\sqrt{2}}{\rho^4}P_3T_{2}^{2} + \frac{3}{\rho^4}(P_2 + iP_1)T_{2}^{0}.$$ 

While in the case of $\psi_{ABCD}^q$ one has,

$$\psi_q^1 = \frac{3}{\rho^2}(Q_2 + iQ_1)T_{2}^{0} - \frac{3\sqrt{2}}{\rho^2}Q_3T_{2}^{1} - \frac{3}{\rho^2}(Q_2 - iQ_1)T_{2}^{0},$$
$$\psi_q^2 = \frac{9\sqrt{2}}{2\rho^2}(Q_2 + iQ_1)T_{2}^{0} - \frac{9}{\rho^2}Q_3T_{2}^{1} - \frac{9\sqrt{2}}{2\rho^2}(Q_2 - iQ_1)T_{2}^{0},$$
$$\psi_q^3 = -\frac{3}{\rho^2}(Q_2 - iQ_1)T_{2}^{2} - \frac{3\sqrt{2}}{\rho^2}Q_3T_{2}^{2} + \frac{3}{\rho^2}(Q_2 + iQ_1)T_{2}^{0}.$$ 

And finally for $\psi_{ABCD}^j$,

$$\psi_j^1 = \frac{6}{\rho^3}(-J_1 + iJ_2)T_{2}^{0} + \frac{6\sqrt{2}}{\rho^3}iJ_3T_{2}^{1} - \frac{6}{\rho^3}(J_1 + iJ_2)T_{2}^{0},$$
$$\psi_j^2 = 0$$
$$\psi_j^3 = \frac{6}{\rho^3}(J_1 + iJ_2)T_{2}^{2} - \frac{6\sqrt{2}}{\rho^3}iJ_3T_{2}^{2} - \frac{6}{\rho^3}(-J_1 + iJ_2)T_{2}^{0},$$

In the above expressions $A$, $P_1$, $P_2$, $P_3$, $J_1$, $J_2$, $J_3$, $Q_1$, $Q_2$, $Q_3 \in \mathbb{R}$.

### 4.3 Solutions with higher harmonics

The solutions to the Euclidean momentum constraint with harmonics $l = 0, 1$ constitute essentially what is known as the Bowen-York Ansatz \[3\]. This Ansatz does not exhaust all the possibilities as it excludes solutions with higher harmonics. Some initial data sets of containing these higher harmonics have been recently considered — see \[10\]. The interest of these higher harmonic terms lies in the fact that — as will be discussed in detail in the article II — the second fundamental forms of stationary data, and in particular the Kerr data, generically do contain this kind of terms. An example of conformally flat initial data with higher order multipoles can be found in \[10\]. In that reference, the second fundamental form is essentially that of the $t = constant$ slices of the Kerr spacetime in Boyer-Lindquist coordinates.

Retaking the strategy used for the $l = 0, 1$ harmonics solutions, and because of the linearity of the equations we shall consider the following Ansatz for the solutions of the momentum constraint:

$$\psi_n = \sum_{q \geq 2} \sum_{k=0}^{2q} L_{n;2q;k}(\rho)T_{2q}^{k} \quad \psi_{-2+n},$$

where $n = 0, \ldots, 4$. Substitution of the latter expression into equation (76) yields the following system of ordinary differential equations for the coefficients $L_{1;2q,k}$, $L_{2;2q,k}$ and $L_{3;2q,k}$, $q \geq 2$, $k = 0, \ldots, 2q$:

$$\frac{dL_{1;2q,k}}{dp} - \frac{1}{3\rho}\sqrt{q(q+1)}L_{2;2q,k} + \frac{3}{\rho}L_{1;2q,k} - \frac{2}{\rho}\sqrt{(q+1)(q-1)}L_{0;2q,k} = 0,$$
$$\frac{dL_{2;2q,k}}{dp} - \frac{3}{4\rho}\sqrt{q(q+1)}L_{3;2q,k} - \frac{3}{4\rho}L_{1;2q,k}\sqrt{q(q+1)} + \frac{3}{4\rho}L_{2;2q,k} = 0,$$
$$\frac{dL_{3;2q,k}}{dp} - \frac{1}{3\rho}\sqrt{q(q+1)}L_{2;2q,k} + \frac{3}{\rho}L_{3;2q,k} - \frac{2}{\rho}\sqrt{(q+1)(q-1)}L_{4;2q,k} = 0.$$
Thus, one sees that if one provides the functions \( L_{0,2q,k} = L_{0,2q,k}(\rho) \) and \( L_{4,2q,k} = L_{4,2q,k}(\rho) \), then it is possible to integrate the above equations to obtain \( L_{1,2q,k}, L_{2,2q,k} \) and \( L_{3,2q,k} \). In order to fulfill the reality conditions, one has to require \( L_{0,2q,k} = (-1)^q k T_{4,2q,k} \) — then, the reality conditions of the other components will be automatically satisfied. In particular, one could take a complex function \( \Lambda \in C^\infty(B_n \setminus \{i\}) \) and require

\[
\sum_{q \geq 0} \sum_{k=0}^{2q} L_{0,2q,k}(\rho) = X_+ X_+ \Lambda, \quad \sum_{q \geq 0} \sum_{k=0}^{2q} L_{4,2q,k}(\rho) = X_- X_- \Lambda.
\]

The operator \( X_+ \) corresponds essentially to the \( \tilde{\sigma} \) operator of Newman-Penrose — see [19]. The combination \( |x|^4 \Lambda \) can be identified with the function \( \lambda \) in section 4 of [9]. In order to satisfy the hypothesis of theorem 1 and 2 in section 3.1 — cfr. theorems 15 of [9] — we require that

\[
|x|^4 \Lambda \in E^\infty(B_n(i)), \quad \text{i.e.} \quad \Lambda = \frac{\mu}{|x|^2} + \frac{\nu}{|x|^3}, \quad \mu, \nu \in C^\infty(B_n(i)).
\]

Note that under the above requirements \( X_+ X_+ \mu = O(\rho^2) \) and \( X_+ X_+ \nu = O(\rho^2) \). Thus, one has

\[
\psi_0^\lambda = \frac{1}{\rho^4} \sum_{n=2}^{\infty} \sum_{q=2}^{n} \sum_{k=0}^{2q} L_{0,n-4,2q,k} T_{2q,k} q^- 2 \rho^n, \quad \psi_1^\lambda = \frac{1}{\rho^4} \sum_{n=2}^{\infty} \sum_{q=2}^{n} \sum_{k=0}^{2q} L_{4,n-4,2q,k} T_{2q,k} q^+ 2 \rho^n.
\]

Under these assumptions the solutions to the system (81) - (84) can be readily calculated yielding

\[
\psi_j^\lambda = \frac{1}{\rho^4} \sum_{n=2}^{\infty} \sum_{q=2}^{n} \sum_{k=0}^{2q} L_{j,n-4,2q,k} T_{2q,k} q^- j \rho^n, \quad j = 1, 2, 3,
\]

where

\[
L_{1,n-4,2q,k} = \frac{(L_{4,n-4,2q,k} - L_{0,n-4,2q,k})(q+1)q + 4L_{0,n-4,2q,k}(n+3)^2}{(n+3)(2n+4)(n+2) - (q+2)(q-1)} \sqrt{(q+2)(q-1)}, \quad (86a)
\]

\[
L_{2,n-4,2q,k} = \frac{3(L_{0,n-4,2q,k} + L_{4,n-4,2q,k})}{2(n+3)^2 - q(q+1)} \sqrt{(q+2)(q+1)q(q-1)}, \quad (86b)
\]

\[
L_{3,n-4,2q,k} = \frac{(L_{0,n-4,2q,k} - L_{4,n-4,2q,k})(q+1)q + 4L_{4,n-4,2q,k}(n+3)^2}{(n+3)(2n+4)(n+2) - (q+2)(q-1)} \sqrt{(q+2)(q-1)}, \quad (86c)
\]

and \( n = 2, 3, 4, \ldots \). We conclude by summarising,

**Assumption 3.** The spinor \( \psi_{ABCD} \) associated with the second fundamental form will be assumed to be on \( C_n \) of the form \( \psi_{ABCD} = \psi_{ABCD}^A + \psi_{ABCD}^B + \psi_{ABCD}^Q + \psi_{ABCD}^\lambda \) where \( \psi_{ABCD}^A \) is given by equation (79), \( \psi_{ABCD}^Q \) by equations (78a)-(78c), \( \psi_{ABCD}^\lambda \) by equations (79a)-(79c) and \( \psi_{ABCD}^\lambda \) by equations (82).  

5 **Expansions of solutions to the Hamiltonian constraint**

The Licnerowicz equation (10) is a scalar equation which can be very easily translated into the space spinor language rendering:

\[
D_{AB} D_{AB} \vartheta = \frac{1}{8} \psi_{ABCD} \psi_{ABCD} \vartheta^{-7}.
\]

Throughout this section we assume that \( \psi_{ABCD} \) satisfies the assumption 8 and adopt the local parametrisation

\[
\vartheta = \frac{1}{|x|} + W,
\]

with \( W(i) = m/2 \), where \( m \) is the ADM mass of the initial data set \( (\tilde{S}, \tilde{\nu}, \tilde{\omega}, \tilde{\chi}, \tilde{\alpha}) \). The term \( 1/|x| \) corresponds to the Green function of the flat Laplacian \( \Delta = D_{AB} D_{AB} \). In the case of
non-conformally flat 3-geometries the term $1/|x|$ has to be replaced by an expression of the form $U/|x|$, where the function $U$ can be calculated recursively by means of the so-called Hadamard parametrix construction —this again, will be discussed at length in part II. Thus, the term $1/|x|$ contains information about the local geometry near infinity. On the other hand, the function $W$ contains information which is global in nature —the mass in particular.

Under the assumptions of theorem 1 the following expansion for the function $W$ follows:

$$W = m/2 + \sum_{p=q=0}^{\infty} \sum_{k=0}^{2q} \frac{1}{p!} w_{p;2q,k} T_{2q,k} k \rho^p,$$  

(89)

with $w_{p;2q,k} \in \mathbb{C}$, and satisfying the conditions $w_{p;2q,k} = (-1)^q w_{p;2q-2q-k}$ so as to guarantee that $W$ is real. Substitution of the above into equation (87) allows us to calculate an expansion consistent with the Hamiltonian constraint, but which not necessarily satisfies it. This is already quite different with what happened in the time symmetric case where any polynomial of degree $N$ of the form

$$W = m/2 + \sum_{p=2}^{N} \sum_{k=0}^{2q} \frac{1}{p!} w_{p;2p,k} T_{2p,k} k \rho^p,$$  

(90)

was an actual solution of the (time symmetric, conformally flat) Hamiltonian constraint

$$D^{AB} D_{AB} \vartheta = 0.$$  

(91)

On the other hand, in the case upon consideration in this article, the coefficients $w_{p;2q,k}$ depend, in general, in a nontrivial way on the parameters determining the second fundamental form. As an example, consider a conformally flat slice in the Schwarzschild spacetime which is not time symmetric. In this case $\psi_{ABCD} = \psi_{ABCD}^A$. It follows then that the Hamiltonian constraint implies

$$\Delta W = \frac{A^2}{48 |x|^6 (\vartheta + W)^r}, \quad \vartheta = \frac{1}{|x|} + \frac{m}{2}.$$  

(92)

A solution to the latter equation can formally be written in terms of the Green function of $\Delta$ as:

$$W = \frac{A^2}{192 \pi} \int_{\mathbb{R}^3} \frac{1}{|x'|^6 |x - x'| (\vartheta + W)^r} d^3 x'.$$  

(93)

In this specific case, it is possible to find the explicit form of the expansion —e.g. by iterations— up to a given order. It reads

$$W = -\frac{1}{576} A^2 \rho^2 + \frac{7}{1920} A^2 m \rho^4 - \frac{7}{1440} A^2 m^2 \rho^5 + \frac{1}{192} A^2 m^3 \rho^6 - \frac{1}{22184} A^2 (1080 m^4 - A^2) \rho^7 + O(\rho^8).$$  

(94)

For more general second fundamental forms, these similar expansions have been calculated up to order $O(\rho^8)$ using the scripts written in the computer algebra system Maple V. Because of their size, these will not be presented here.

### 5.1 Some simplifying considerations

The conformal factor $\vartheta$ and the symmetric tensor $\psi_{ab}$ obtained by virtue of theorems 1 and 2 contain some terms which are pure gauge. The freedom remaining in our setting consists essentially in elements of the conformal group. In particular we have at our disposition a rotation and a translation. We shall make due use of them. Firstly, a rotation can be used so that the Cartesian coordinates $\{x^a\}$ are such that the ADM-angular momentum vector $J^q$ is “aligned with the positive z-axis” —that is $J_1 = J_2 = 0$.

With regard to the translation freedom, this can be used to eliminate certain dipolar terms from the expansions of the conformal factor $\vartheta$. Consider the inversion $x^a = y^a / |y|$ mapping $\mathcal{B}_{\pm}(i)$ to the asymptotic end $\mathbb{R}^3 \setminus \mathcal{B}_{1/a}(0)$, and consider the conformal factor $\phi = |x| \vartheta$. This last function
satisfies, under the hypothesis of theorems 1 and 2, an equation of the form \( \Delta \phi = O(|y|^{-5}) \). The assumptions of theorems 1 and 2—using standard arguments of potential theory—lead to

\[
\phi = 1 + \frac{m}{2|y|} + \frac{d_a y^a}{|y|^3} + O\left(\frac{1}{|y|^3}\right),
\]

where \( d_a \) is a constant (dipolar) vector. Now, consider the translation \( z^a = y^a + c^a \), where \( c^a \) is a constant vector. One has that,

\[
\frac{1}{|y|} = \frac{1}{|z|} \left(1 - \frac{c_a z^a}{|z|^2} + O\left(\frac{1}{|z|^2}\right)\right), \quad \frac{1}{|y|^3} = \frac{1}{|z|^3} \left(1 + O\left(\frac{1}{|z|}\right)\right).
\]

Where it follows that

\[
\phi = 1 + \frac{m}{2|z|} + \frac{1}{|z|^3}(d_a z^a - mc_a z^a) + O\left(\frac{1}{|z|^3}\right)
\]

so that the dipolar term in the expansions can be removed if one sets \( c_a = d_a/m \) —“centre of mass”. The final step now consists in a further inversion \( z^a = \tilde{x}^a/|x|^2 \). Thus, without loss of generality one can assume that

\[
W = m/2 + q_{ab}x^a x^b + O(|x|^3),
\]

where \( q_{ab} \) is a “quadrupolar term”. Summarising,

**Assumption 4.** The Cartesian (normal) coordinates \( \{x^a\} \) on \( B_a(i) \) are chosen so that the only non-vanishing component of the ADM angular momentum is \( J_3 \) —and we shall write \( J = J_3 \)—and the expansion of \( \vartheta \) contains no dipolar terms, that is: \( W = m/2 + O(|x|^2) \).

### 6 The conformal propagation equations

In the sequel, we shall make use of the dynamical formulation of the Conformal Einstein field equations developed in [14, 15]. This formulation of the field equations allows the implementation of a regular initial value problem near spacelike infinity. Besides the Levi-Civita connection \( \nabla \) of the spacetime \( (\mathcal{M}, \mathring{g}_{\mu
u}) \) arising as the development of the initial data \( (\mathring{S}, \mathring{h}_{\mu
u}, \mathring{\chi}_{\mu
\nu}) \), we shall consider two other connections: the Levi-Civita connection \( \nabla \) of a metric \( g_{\mu
u} = \Theta^2 \mathring{g}_{\mu
u} \) in the same conformal class of \( \mathring{g}_{\mu
u} \); and a conformal connection (Weyl connection)\(^{10} \nabla \), which is not necessarily metric. Now, given the connections \( \nabla, \mathring{\nabla} \) and \( \nabla \), there are 1-forms \( b_\mu, f_\mu \) such that

\[
\mathring{\nabla} - \nabla = S(b), \quad \nabla - \mathring{\nabla} = S(f),
\]

where for a given 1-form \( d_\mu \) the tensor field \( S(d) \) is given by

\[
S(d)_\mu^\nu = \delta_\mu^\nu d_\mu + \delta_\nu^\mu d_\mu - g_{\mu\rho} g^{\nu\lambda} d_\lambda.
\]

The Conformal Einstein equations are essentially equations for a frame, the connection coefficients of \( \mathring{\nabla} \), the Ricci tensor of \( \mathring{\nabla} \), and the rescaled Weyl tensor of \( g_{\mu
u} \).

The Conformal Einstein equations written in terms of Weyl connections allow the implementation of a certain gauge based on the properties of certain curves called conformal geodesics—conformal Gauss coordinates. The conformal geodesic equations for a vacuum spacetime are given by

\[
\begin{align*}
\mathring{\nabla}_\nu \mathring{x}^\mu &= -2b_\mu \mathring{x}^\nu \mathring{x}_\mu + \mathring{g}_{\mu\lambda} \mathring{x}^\nu \mathring{x}^\lambda \mathring{g}^{\mu\sigma} b_\sigma, \\
\mathring{x}_\nu \mathring{\nabla}_\mu b_\mu &= b_\nu \mathring{x}^\nu b_\mu - \frac{1}{2} \mathring{g}^{\mu\lambda} b_\mu b_\lambda \mathring{g}_{\sigma\rho} \mathring{x}^\sigma, \\
\mathring{x}_\nu \mathring{x}_\mu b_\mu &= b_\nu e^\nu_\mu - b_\nu \mathring{x}^\nu \mathring{x}_\mu + \mathring{g}_{\mu\lambda} \mathring{x}^\nu \mathring{x}_\mu \mathring{g}^{\mu\sigma} b_\sigma.
\end{align*}
\]

\(^{9}\)It is noted that without the theorems 1 and 2, \( \Delta \phi = O(|y|^{-5}) \) would actually imply that the order of the remainder in the expansion of \( \phi \) is \( O(|x|^{-3} \ln |x|) \)—see for example lemma A in the appendix of [21].

\(^{10}\)A Weyl connection is a torsionfree connection for which parallel transport preserves the causal nature.
where \( x^\mu = \tau^\mu(\tau), b_\mu = b_\mu(\tau), \) and \( \{ e^\mu_j = e^\mu_j(\tau) \}_{j=0,...,3} \) are, respectively, a curve, a 1-form and a frame. For \( q \in \tilde{\mathcal{S}} \), these equations will be supplemented by the initial conditions at \( \tau = 0 \):

\[
\begin{align*}
\tau(0) &= q, \quad \dot{x}(0) = e_0 \text{ future directed and orthogonal to } \tilde{\mathcal{S}}, \\
\kappa^{-1} \Omega \tilde{g}(e_j, e_k) &= \eta_{jk}, \quad \kappa > 0, \\
b_\mu \dot{x}^\mu &= 0, \quad b_\mu = \Omega^{-1} \tilde{\nabla}_\mu \Omega.
\end{align*}
\]

The solutions to the conformal geodesic equations with these initial conditions satisfy —see e.g. [14, 17]—

\[
\dot{x}(\tau) = e_0(\tau), \quad \Theta^2 \tilde{g}(e_j, e_k) = \eta_{jk},
\]

with a conformal factor \( \Theta \) given by

\[
\Theta(\tau) = \kappa^{-1} \Omega \left( 1 - \tau^2 \frac{\kappa^2}{\omega^2} \right), \quad \omega = \frac{2\Omega}{\sqrt{\det \Omega}}, \quad (104)
\]

The function \( \kappa \) contains the remaining conformal freedom in this construction. The 1-form \( b_\mu \) is such that

\[
d_j(\tau) = \Theta b_\mu x^\mu_j = \left( -2 \tau \frac{\kappa \Omega}{\omega^2}, \kappa^{-1} e_k(\Omega) \right), \quad (105)
\]

where in this last equation \( j = 0, \ldots, 3 \) and \( k = 1, 2, 3 \).

### 6.1 The manifold \( \mathcal{M}_{a,\kappa} \)

We introduce some useful terminology. Let \( CSL(2, \mathbb{C}) = \mathbb{R}^+ \times SL(2, \mathbb{C}) \). Further, let \( CSL(\tilde{\mathcal{M}}) \) denote the fibre bundle with fibres \( CSL(2, \mathbb{C}) \). Note that \( SL(2, \mathbb{C}) \) can be regarded in a natural way as a subgroup of \( CSL(2, \mathbb{C}) \), thus \( CSL(\tilde{\mathcal{M}}) \) may be identified with the set of spin frames \( \{ \kappa \delta_A \}_{A=0,1} \), where \( \{ \delta_A \}_{A=0,1} \in SL(\tilde{\mathcal{M}}), \kappa \in \mathbb{R}^+ \). Now, \( \mathcal{C}_a \subset SU(S) \subset SL(S) \), from which it follows that \( \mathcal{C}_a \) is also a submanifold of \( CSL(\tilde{\mathcal{M}}) \). On the bundle \( CSL(\tilde{\mathcal{M}}) \) we shall be considering its solder form \( \sigma^{AB} \), its \( g \)-connection form \( \omega^A, \) and the connection form associated to the \( \tilde{\nabla} \). We write

\[
\Gamma_{CC'}^A = \langle \omega^A, c_{CC'} \rangle, \quad \tilde{\Gamma}_{CC'}^A = \langle \omega^A, c_{CC'} \rangle,
\]

where the the connection coefficients \( \Gamma_{CC'}^A \) satisfy

\[
\Gamma _{CC'} = \tilde{\Gamma}_{CC'} = \Gamma_{CC'} + \epsilon_{CA} f_{BC'}, \quad (107)
\]

Because of the divergent nature of the rescaled Weyl tensor \( d_{ijkl} = O(|x|^{-3}) \), and accordingly of its spinorial counterpart \( \phi_{ABCD} = O(\rho^{-3}) \)—cfr. equation (26)—, we shall be in the need of considering rescalings, \( \delta \mapsto \kappa^{1/2} \delta \), of the spinor dyads over \( \mathcal{B}_a(i) \), in order to obtain quantities which are regular at \( \rho = 0 \). A choice of the form \( \kappa = \rho \kappa' \), where \( \kappa' \) is a smooth function such that \( \kappa'(i) = 1 \), would do the trick. The restriction of this rescaling defines a diffeomorphism of \( \mathcal{C}_a \) to \( \mathcal{C}_a, \kappa = \kappa^{1/2} \mathcal{C}_a \). This diffeomorphism can be used to carry to \( \mathcal{C}_a, \kappa \) the coordinates \( \rho, t^A, \) and the vector fields \( \partial_\rho, X_+, X_-, \) defined on \( \mathcal{C}_a \).

The conformal factor \( \Theta \) given by equation (104) can be used to define the following sets:

\[
\mathcal{M}_{a,\kappa} = \left\{ (\tau, q) \mid q \in \mathcal{C}_{a,\kappa}, -\frac{\omega(q)}{\kappa(q)} \leq \tau \leq \frac{\omega(q)}{\kappa(q)} \right\}, \quad (108a)
\]

\[
\mathcal{I} = \left\{ (\tau, q) \in \mathcal{M}_a \mid \rho(q) = 0, \ |\tau| < 1 \right\}, \quad (108b)
\]

\[
\mathcal{I}^\pm = \left\{ (\tau, q) \in \mathcal{M}_a \mid \rho(q) = 0, \ \tau = \pm 1 \right\}, \quad (108c)
\]

\[
\mathcal{J}^\pm = \left\{ (\tau, q) \in \mathcal{M}_a^+ \mid q \in \mathcal{C}_{a,\kappa}^+, \ \tau = \pm \frac{\omega(q)}{\kappa(q)} \right\}, \quad (108d)
\]
Let $\sigma^{AA'}$ denote the solder form of $\mathcal{M}_{a,n}^+$. We use it to define the frame fields $c_{AA'}$ via the conditions

$$
\langle \sigma^{AA'}, c_{BB'} \rangle = \epsilon_B^A \epsilon_{B'}^{A'},
$$

$$
c_{AA'} = c_{AA'}^0 \partial_\tau + c_{AA'}^1 \partial_\rho + c_{AA'}^+ X_+ + c_{AA'}^- X_-, \tag{109b}
$$

These fields can be split in the form

$$
c_{AA'} = \frac{1}{2} \tau_{AA'} \tau_{CC'} c_{CC'} - \tau^B_{\cdot A} c_{AB}, \tag{110}
$$

with

$$
\tau^{AA'} c_{AA'} = \sqrt{2} \partial_\tau, \tag{111a}
$$

$$
c_{AB} = \tau_{(A}^{B'} c_{B')B} = c_{AB}^0 \partial_\tau + c_{AB}^1 \partial_\rho + c_{AB}^+ X_+ + c_{AB}^- X_-, \tag{111b}
$$

and $c_{AB}^0 = 0$ at $\tau = 0$. Again, the functions $\rho, t^A_{\cdot B}, T^j_m$ are extended off $\mathcal{C}_{a,n}$ by requiring that they remain constant, for fixed $q$ along the curves $\tau \mapsto (\tau, q) \in \mathcal{M}_{a,n}^+$.

The spatial part of the 1-form $d_j$ given in equation (105) has a spinorial counterpart $d_{AB} = \tau_{(A}^{B'} d_{B')B}$. The latter can be calculated in terms of the functions $U$ and $W$. Namely,

$$
d_{AB} = 2\rho \left( \frac{U x_{AB} - \rho D_{AB}U - \rho^2 D_{AB}W}{(U + \rho W)^3} \right). \tag{112}
$$

6.2 The propagation equations

Using the conformal geodesic gauge and the 2-spinor decomposition, it can be shown that the extended conformal field equations given in [14, 15] under the conformal Gauss coordinates described in the preceeding sections imply propagation equations for:

(i) The components of the frame $c_{AB}^\mu, \mu = 0, 1, \pm$.

(ii) The space-spinor connection coefficients of $\nabla$, $\Gamma_{ABCD} = \tau_{(A}^{B'} \Gamma_{AB'CD}$. In [14] it was shown that in the gauge that we are using

$$
\tau^{AA'} \Gamma_{AA'BC} = 0, \quad \tau^{AA'} f_{AA'} = 0. \tag{113}
$$

The latter imply the decomposition

$$
\Gamma_{ABCD} = \frac{1}{\sqrt{2}} \left( \xi_{ABCD} - \chi_{(AB)CD} \right) - \frac{1}{2} \epsilon_{AB} \xi_{CD}, \tag{114}
$$

where the spinor $\chi_{ABCD}$ agrees on $S$ with the second fundamental form of the initial hypersurface.

(iii) The spinor $\Theta_{AA'B'B'}$ associated with the tensor

$$
A_{jk} = \frac{1}{2} \tilde{R}(jk) - \frac{1}{12} \eta^{il} \tilde{R}_{il} \eta_{jk} + \frac{1}{4} \tilde{R}_{[jk]}, \tag{115}
$$

where $\tilde{R}_{jk}$ is the Ricci tensor of the Weyl connection $\tilde{\nabla}$. In the above expression all indices range $0, \ldots, 3$. The spinor $\Theta_{AA'B'B'}$ can be decomposed as

$$
\Theta_{AA'B'B'} = \Phi_{AA'B'B'} + \Lambda \epsilon_{AB} \epsilon_{A'B'} + \Phi_{AB} \epsilon_{A'B'} + \overline{\Phi}_{A'B'} \epsilon_{AB}. \tag{116}
$$

Now, in our gauge

$$
\Theta^{B'B'} \Theta_{AA'B'B'} = 0. \tag{117}
$$

The propagation equations do not contain $\Theta_{AA'B'B'}$, but its space spinor counterpart $\Theta_{ABCD} = \tau_C^{A'} \tau_D^{B'} \Theta_{AA'B'B'}$. Moreover, $\Theta_{ABC} \epsilon_C = 0$, so we can write

$$
\Theta_{ABCD} = \Theta_{(AB)CD} - \frac{1}{2} \epsilon_{AB} \Theta_{G}^G \Theta_{CD}, \tag{118}
$$

with $\Theta_{(AB)CD} = \Theta_{(AB)(CD)}$. 
(iv) The components of the Weyl spinor $\phi_{ABCD} = \phi(ABCD)$, where

$$
d_{ijkl} = \sigma^A_i \sigma^B_j \sigma^C_k \sigma^D_l \left( \phi_{ABCD} \epsilon_{A'B'C'D'} + \phi_{A'B'C'D'} \epsilon_{ABCD} \right),
$$

(119)

The propagation equations group naturally in two sets: the propagation equations for what will be known as the $v$-quantities, $v = (\phi^0_{\mu AB}, \xi_{ABCD}, \eta_{ABCD}, \Theta_{(AB)CD}, \Theta_{G (CD) E})$, $\mu = 0, 1, \pm$

\[
\begin{align*}
\partial_\tau \phi^0_{AB} &= -\chi_{(AB)}^{\ EF} c^E_{\ EF} - f_{AB}, \\
\partial_\tau \xi_{ABCD} &= -\chi_{(AB)}^{\ EF} \xi_{EFCD} + \frac{1}{\sqrt{2}} (\epsilon_A^C \chi_{(BD)EF} + \epsilon_B^D \chi_{(AC)EF}) f^{EF} \\
&\quad - \sqrt{2} \chi_{(AB)(C)} f_{DE} - \frac{1}{2} (\epsilon_C^A \Theta^F_{DE} + \epsilon_D^B \Theta^F_{AC}) - i \Theta_{ABCD}, \\
\partial_\tau f_{AB} &= -\chi_{(AB)}^{\ EF} f f^{EF} + \frac{1}{\sqrt{2}} \Theta^{C F}_{\ AB},
\end{align*}
\]

(120a, 120b, 120c, 120d, 120e, 120f, 120g)

where

\[
\eta_{ABCD} = \frac{1}{2} (\phi_{ABCD} + \phi^+_{ABCD}), \quad \mu_{ABCD} = -\frac{i}{2} (\phi_{ABCD} - \phi^+_{ABCD}),
\]

(121)
denote, respectively, the electric and magnetic parts of $\phi_{ABCD}$. The quantities $\Theta$, $\partial_\tau \Theta$ correspond to the conformal factor given by equation (111) awhile $d_{AB}$ is given by equation (112). These quantities are available a priori from a knowledge of the solutions to the constraint equations. Thus, the equations (120a-120g) are essentially ordinary differential equations for the components of the vector $v$.

The second set of equations is, arguably, the most important part of the propagation equations and corresponds to the evolution equations for the spinor $\phi_{ABCD}$ derived from the Bianchi identities \textit{Bianchi propagation equations}:

\[
\begin{align*}
(\sqrt{2} - 2 c_{01}^0) \partial_\tau \phi_0 + 2 c_{00}^0 \partial_\tau \phi_1 - 2 c_{01}^0 \partial_\tau \phi_0 + 2 c_{00}^0 \partial_\tau \phi_1 \\
&= (2 \Gamma_{0011} - 8 \Gamma_{1010}) \phi_0 + (4 \Gamma_{0001} + 8 \Gamma_{1000}) \phi_1 - 6 \Gamma_{0000} \phi_2,
\end{align*}
\]

(122a)

\[
\sqrt{2} \partial_\tau \phi_1 - c_{11}^0 \partial_\tau \phi_1 + c_{00}^0 \partial_\tau \phi_2 - c_{01}^0 \partial_\tau \phi_1 + c_{00}^0 \partial_\tau \phi_2 \\
= - (4 \Gamma_{1110} + f_{11}) \phi_0 + (2 \Gamma_{0011} + 4 \Gamma_{1000} - 2 f_{01}) \phi_1 + 3 f_{00} \phi_2 - 2 \Gamma_{0000} \phi_3,
\]

(122b)

\[
\sqrt{2} \partial_\tau \phi_2 - c_{11}^0 \partial_\tau \phi_1 + c_{00}^0 \partial_\tau \phi_2 - c_{11}^0 \partial_\tau \phi_1 + c_{00}^0 \partial_\tau \phi_2 \\
= - \Gamma_{1111} \phi_0 - 2 (\Gamma_{1101} + f_{11}) \phi_1 + 3 (\Gamma_{0011} + \Gamma_{1100}) \phi_2 - 2 (\Gamma_{0001} - f_{00}) \phi_4 - \Gamma_{0010} \phi_4,
\]

(122c)

\[
\sqrt{2} \partial_\tau \phi_3 - c_{11}^0 \partial_\tau \phi_1 + c_{00}^0 \partial_\tau \phi_2 - c_{11}^0 \partial_\tau \phi_1 + c_{00}^0 \partial_\tau \phi_2 \\
= - 2 \Gamma_{1111} \phi_0 - 3 f_{11} \phi_2 + (2 \Gamma_{1100} + 4 \Gamma_{0011} + 2 f_{01}) \phi_3 - (4 \Gamma_{0001} - f_{00}) \phi_4,
\]

(122d)

\[
(\sqrt{2} + 2 c_{01}^0) \partial_\tau \phi_4 - 2 c_{01}^0 \partial_\tau \phi_3 + 2 c_{00}^0 \partial_\tau \phi_4 - 2 c_{01}^0 \partial_\tau \phi_3 \\
= - 6 \Gamma_{1111} \phi_2 + (4 \Gamma_{1110} + 8 \Gamma_{0111}) \phi_3 + (2 \Gamma_{1100} - 8 \Gamma_{0010}) \phi_4,
\]

(122e)

where

\[
\phi_j = \phi(ABCD)_j, \quad j = 0, \ldots, 4.
\]

(123)
The subindex $j$ in $(ABCD)_j$ indicates that after symmetrisation, $j$ indices are to be set equal to 1. To the equations (122a-122e) we add yet another set of three equations, also implied by the
Bianchi identities which we refer to as the Bianchi constraint equations

\[
\begin{align*}
\mathcal{E}^\alpha_1 \partial_\alpha \phi_0 - 2\mathcal{E}^\alpha_0 \partial_\alpha \phi_1 + \mathcal{E}^\alpha \partial_\alpha \phi_2 - 2\mathcal{E}^\alpha_0 \partial_\alpha \phi_1 + \mathcal{E}^\alpha_0 \partial_\alpha \phi_2 \\
= -(2\Gamma_{(01)11} - 4\Gamma_{1110})\phi_0 - (2\Gamma_{0011} - 4\Gamma_{01(01)} - 4\Gamma_{1100})\phi_1 + 6\Gamma_{00100}\phi_2 - 2\Gamma_{00000}\phi_3, \quad (124a)
\end{align*}
\]

\[
\begin{align*}
\mathcal{E}^\alpha_1 \partial_\alpha \phi_1 - 2\mathcal{E}^\alpha_0 \partial_\alpha \phi_2 + \mathcal{E}^\alpha \partial_\alpha \phi_1 - 2\mathcal{E}^\alpha_0 \partial_\alpha \phi_3 + \mathcal{E}^\alpha_0 \partial_\alpha \phi_4 \\
= \Gamma_{1111} \phi_0 - (4\Gamma_{(01)11} - 2\Gamma_{1101})\phi_1 + 3(\Gamma_{0011} - \Gamma_{1100})\phi_2 - (2\Gamma_{0001} - 4\Gamma_{(01)00})\phi_3 \\
- \Gamma_{00000} \phi_4, \quad (124b)
\end{align*}
\]

\[
\begin{align*}
\mathcal{E}^\alpha_1 \partial_\alpha \phi_2 - 2\mathcal{E}^\alpha_0 \partial_\alpha \phi_1 + \mathcal{E}^\alpha \partial_\alpha \phi_3 - 2\mathcal{E}^\alpha_0 \partial_\alpha \phi_4 + \mathcal{E}^\alpha_0 \partial_\alpha \phi_4 \\
= 2\Gamma_{1111} \phi_1 - 6\Gamma_{(01)11} \phi_2 + (4\Gamma_{0011} + 4\Gamma_{01(01)} - 2\Gamma_{1100}) \phi_3 - (4\Gamma_{0001} - 2\Gamma_{(01)00}) \phi_4. \quad (124c)
\end{align*}
\]

6.3 The initial data for the conformal propagation equations

Assume for the moment that one has a solution \((\Omega, \chi_{\alpha\beta})\) of the constraint equations \((105a)\) and \((105b)\). How do we calculate the initial data for the propagation equations \((120a)-(120g)\) and \((122a)-(122c)\)? In order to do this, one has to make use of the so-called conformal constraint field equations — see e.g. [14].

Let \(\Theta\) be the conformal factor induced by the F-gauge. We consider the further conformal rescaling \(\Theta \rightarrow \kappa^{-1}\Theta\) where \(\kappa\) is a function on \(S\) such that \(\kappa = \kappa' \rho\) with \(\kappa'(i) = 1\). This conformal rescaling induces a rescaling in the spinor frame of the form \(\delta \mapsto \kappa^{1/2}\delta\). Using the procedure described in [14] one arrives to

\[
\begin{align*}
\Theta_{ABCD} &= -\frac{\kappa^2}{\Omega} D_{(AB} D_{CD)}\Omega + \frac{\kappa^2}{12} \left( r + \chi_{EFGH} \chi^{EFGH} \right) h_{ABCD} + \frac{\kappa^2}{\sqrt{2} \Omega} \varepsilon_{ABCD} \varepsilon^{EF} D_{EF}\Omega, \quad (125a)
\end{align*}
\]

\[
\phi_{ABCD} = \frac{\kappa^3}{\Omega} D_{(AB} D_{CD)}\Omega + \frac{\kappa^3}{12} s_{ABCD} + \frac{\kappa^3}{\Omega} \chi_{EF(AB} \chi_{CD)}^{EF} + \sqrt{2} \frac{\kappa^3}{\Omega} \mathcal{D}_{(A} \chi_{BCD)E}, \quad (125b)
\]

\[
c^0_0 = 0, \quad c^1_{AB} = \kappa x_{AB}, \quad (125c)
\]

\[
c^+_{AB} = \kappa \left( \frac{1}{\rho} z_{AB} + \hat{e}^+_{AB} \right), \quad c^-_{AB} = \kappa \left( \frac{1}{\rho} y_{AB} + \hat{e}^-_{AB} \right), \quad (125d)
\]

\[
\xi_{ABCD} = \sqrt{2} \left( \kappa \xi_{ABCD} - \frac{1}{2} (\xi_{AC} \kappa_{BD} + \xi_{BD} \kappa_{AC}) \right), \quad (125e)
\]

\[
\chi_{(AB)CD} = \kappa \Omega^2 \psi_{ABCD}, \quad (125f)
\]

\[
f_{AB} = \kappa_{AB}, \quad (125g)
\]

with

\[
\kappa_{AB} = \left( x_{AB} \partial_\rho + \left( \frac{1}{\rho} z_{AB} + \hat{e}^+_{AB} \right) X_+ + \left( \frac{1}{\rho} y_{AB} + \hat{e}^-_{AB} \right) X_- \right) \kappa. \quad (126)
\]

In the above expressions it has been assumed that the initial hypersurface is maximal but not necessarily conformally flat.

7 Transport equations at \(I\)

The system of equations \((120a)-(120g), (122a)-(122e), (124a)-(124c)\), allow us to introduce a special kind of (asymptotic) expansions in the region of spacetime near null and spatial infinity. Writing as before \(v = (c^+_{AB}, \xi_{ABCD}, f_{AB}, \chi_{(AB)CD}, \Theta_{(AB)CD}, \Theta_G, \mu) = (\phi_0, \phi_1, \phi_2, \phi_3, \phi_4)\), then the equations \((120a)-(120g)\) can be concisely written as

\[
\partial_r v = K v + Q(v, v) + L \phi, \quad (127)
\]

where \(K, Q\) denote respectively linear and quadratic functions with constant coefficients, and \(L\) denotes a linear function with coefficients depending on the coordinates through the functions \(\Theta, \partial_r \Theta, d_{AB}\) and such that \(L|_I = 0\). For the Bianchi propagation equations one can write

\[
\sqrt{2} E \partial_r \phi + A^{AB} c^{\mu}_{AB} \partial_\mu \phi = B(\Gamma_{ABCD}) \phi, \quad (128)
\]
where $E$ denotes the $(5 \times 5)$ unit matrix, $A^{AB}c^{\mu}_{AB}$ are $(5 \times 5)$ matrices depending on the coordinates, and $B(\Gamma_{ABCD})$ is a linear $(5 \times 5)$ matrix valued function with constant entries of the connection coefficients $\Gamma_{ABCD}$. On similar lines, the Bianchi constraint equations can be written as

$$F^{AB}c^{\mu}_{AB}\partial_{\mu}\phi = H(\Gamma_{ABCD}),$$

(129)

where now $F^{AB}c^{\mu}_{AB}$ denote $(3 \times 5)$ matrices, and $H(\Gamma_{ABCD})$ is a $(3 \times 5)$ matrix valued function of the connection with constant entries.

The system (127), (128) can be formally evaluated on $\mathcal{I}$ rendering an interior system of equations which, due to $L|_{\mathcal{I}} = 0$ is decoupled, and can be easily solved. The solutions $(v^{(0)}, \phi^{(0)})$ to this system are universal — i.e. independent of the initial data — and can be regarded as the leading terms in Taylor-like expansions of the form

$$v \sim \sum_{p=0}^{1} \frac{1}{p!} v^{(p)} \rho^p, \quad \phi \sim \sum_{p=0}^{1} \frac{1}{p!} \phi^{(p)} \rho^p. \quad (130)$$

The higher order coefficients in these expansions can be determined via a recursive procedure. To do so, we differentiate the systems (127), (128) and (129) $p$ times and then evaluate at $\mathcal{I}$. The resulting equations are of the form

$$\partial_{\tau}v^{(p)} = K v^{(p)} + Q(v^{(0)}, v^{(p)}) + Q(v^{(p)}, v^{(0)})$$

$$+ \sum_{j=1}^{p-1} \left( \frac{p}{j} \right) \left( Q(v^{(j)}, v^{(j-p)}) + L^{(j)} \phi^{(p-j)} \right) + L(p) \phi^{(0)}. \quad (131)$$

From the transport Bianchi propagation equations (122a) - (122c) one gets

$$\left( \sqrt{2} E + A^{AB}(c^{\mu}_{AB})^{(0)} \right) \partial_{\tau} \phi^{(p)} + A^{AB}(c^{C}_{AB})^{(0)} \partial_{C} \phi^{(p)} = B(\Gamma^{(0)}_{ABCD}) \phi^{(p)}$$

$$+ \sum_{j=1}^{p} \left( \frac{p}{j} \right) \left( B(\Gamma^{(j)}_{ABCD}) \phi^{(p-j)} + A^{AB}(c^{\mu}_{AB})^{(j)} \partial_{\mu} \phi^{(p-j)} \right), \quad (132)$$

where $C = \pm$. Similarly, from the Bianchi constraint equations (124a) - (124c) one obtains

$$F^{AB}(c^{\mu}_{AB})^{(0)} \partial_{\tau} \phi^{(p)} + F^{AB}(c^{C}_{AB})^{(0)} \partial_{C} \phi^{(p)} = H(\Gamma^{(0)}_{ABCD}) \phi^{(p)}$$

$$+ \sum_{j=1}^{p} \left( \frac{p}{j} \right) \left( H(\Gamma^{(j)}_{ABCD}) \phi^{(p-j)} \right) - F^{AB}(c^{\mu}_{AB})^{(j)} \partial_{\mu} \phi^{(p-j}) \right). \quad (133)$$

We will refer to these equations as to the transport equations of order $p$. An interesting feature of the transport equations is that their principal part is universal — i.e. independent of the order $p$. For $p \geq 1$ they are linear differential equations for the unknowns of order $p$. Furthermore, note that the subsystem (131) consists only of ordinary differential equations. Now, the expressions (125a) - (125g) for the initial data of the conformal propagation equations can be used to determine the expansion types of the quantities $v^{(p)}$ and $\phi^{(p)}$ at $\mathcal{I}$. For initial data satisfying assumptions 1-4, it is found that they have expansion type $p - 1$ and $p$ respectively. The transport equations (131) and (132) can be used in turn to show that the expansion type is preserved during the evolution.

The transport equations (131), (132) and (133) are decoupled in the following sense: a knowledge of $v^{(j)}$, $\phi^{(j)}$, $j = 0, \ldots, p - 1$ together with the initial data $v^{(p)}|_{\mathcal{I}}$ allows us to solve the subsystem (131) to obtain the quantities $v^{(p)}$. With $v^{(k)}$, $\phi^{(l)}$, $k = 0, \ldots, p$ and $l = 0, \ldots, p - 1$ and the initial data $\phi^{(p)}$ at hand one could, in principle, solve the equations (132) to get $\phi^{(p)}$. A major complication in the aforesaid procedure is that the coefficient accompanying the $\tau$-derivative in equation (132) is such that

$$\left( \sqrt{2} E + A^{AB}(c^{\mu}_{AB})^{(0)} \right) = \sqrt{2} \text{diag}(1 - \tau, 1, 1, 1, 1 + \tau), \quad (134)$$

where $\text{diag}(1 - \tau, 1, 1, 1, 1 + \tau)$
that is, it looses rank at the sets $I^{\pm}$ — in other words, the system degenerates. Intuitively, one would expect this degeneracy of the system to leave some sort of imprint on the behaviour of its solutions on the critical sets, $I^{\pm}$. From a partial differential equations’ point of view, the degeneracy arises from the transversal intersection of a total characteristic of the Conformal Einstein field equations, the cylinder $I$, with standard characteristics — null infinity, $\mathcal{I}^+$.

In order to obtain a detailed analysis of the behaviour of the solutions to the propagation Bianchi transport equations at $I^{\pm}$ one has to make use of the transport equations implied by the Bianchi constraint equations (133).

The equations (132) & (133) can be, consistently with the spin weight of its diverse terms, decomposed in terms of the functions $T_{i,k}^j$. In particular, one has

$$\phi^{(p)}_j = \sum_{q=1}^{2q} \sum_{k=0}^p a_{j,p;2q,k} T_{2q}^k_{q-2+j},$$

with $a_{j,p;2q,k} = a_{j,p;2q,k}(\tau)$. The latter strategy “reduces” the problem to the analysis of a system of ordinary differential equations in $\tau$. Note however, that the left hand sides of the equations (132) and (133) involve products of the components of the vectors $v^{(k)}$ and $\phi^{(l)}$ which need to be linearised, that is, one needs to write them as a linear combination of $T_{i,k}^j$’s according to formula (135). Now, proceeding sector by sector, one can use the transport constraint equations to further reduce the problem to the analysis of a system of two ordinary differential equations for the coefficients $a_{0,p;2q,k}$ and $a_{4,p;2q,k}$ of $\phi^{(p)}_0$ and $\phi^{(p)}_4$. The remaining coefficients can be calculated algebraically from these two. The equations can be written in the form

$$a_{0,p;2q,k}'' + (1 + 2(p - 1)\tau) a_{0,p;2q,k}' + (q + p)(q - p + 1) a_{0,p;2q,k} = b_{0,p,q}(\tau),$$

$$a_{4,p;2q,k}'' + (4 + 2(p - 1)\tau) a_{4,p;2q,k}' + (q + p)(q - p + 1) a_{4,p;2q,k} = b_{4,p,q}(\tau),$$

where we have written $a_0$ and $a_4$ instead of $a_{0,p;2p,k}$ and $a_{4,p;2q,k}$, and where the terms $b_{0,p,q}$ and $b_{4,p,q}$ can be derived from the right hand sides of (132) and (133). Their solution can be written as

$$a_{0,p;2q,k}(\tau) = X_{p,q}(\tau) \left[ X_{p,q}^{-1}(0) \left(a_{0,p;2q,k}(0)\right) + \int_0^{\tau} X_{p,q}^{-1}(\tau') B_{p,q}(\tau')d\tau' \right],$$

where $X_{p,q}(\tau)$ denotes the fundamental matrix of the system (135) and the vector $B_{p,q}(\tau)$ is to be derived from $b_{0,p,q}$ and $b_{4,p,q}$. The matrix $X_{p,q}(\tau)$ has been given explicitly in terms of Jacobi polynomials. More importantly,

$$\det X_{p,q}(\tau) = f(\tau)(1 - \tau^2)^{p-2}$$

with $f(\tau)$ a second order polynomial such that $f(\pm 1) \neq 0$. This means that the integrand in equation (137) will have poles at $\tau = \pm 1$ unless $B_{p,q}(\tau)$ is very special. The main hurdle to be overcome in this procedure is that the explicit form of $B_{p,q}(\tau)$ becomes more and more complicated as the order $p$ of the expansions increases. Thus, in order to gain some understanding of its structure, one has to make use of computer algebra methods and calculate explicitly the solutions up to a given order.

### 7.1 A regularity condition

A detailed analysis of the expansion types of the diverse quantities involved in equations (132) and (133) reveals that $B_{p,q}(\tau) = 0$. Moreover,

$$a_{0,p;2p,k}(\tau) = (1 - \tau)^{p+2}(1 + \tau)^{p-2} \left(a_{0,p;2p,k}(0) + \frac{(p+1)(p+2)}{4p} [a_{0,p;2p,k}(0) - a_{4,p;2p,k}(0)] I_+ \right),$$

$$a_{4,p;2p,k}(\tau) = (1 + \tau)^{p+2}(1 - \tau)^{p-2} \left(a_{4,p;2p,k}(0) + \frac{(p+1)(p+2)}{4p} [a_{4,p;2p,k}(0) - a_{0,p;2p,k}(0)] I_- \right),$$

where

$$I_\pm = \int_0^{\tau} \frac{d\tau'}{(1 - \tau')^{p-1}(1 + \tau')^{p+3}} = A_0 \ln(1 - \tau) + \frac{A_{p+2}}{(1 - \tau)^{p+2}} + \cdots + \frac{A_{1}}{1 - \tau} + B_0 \ln(1 + \tau) + \frac{B_{p+2}}{(1 + \tau)^{p+2}} + \cdots + \frac{B_1}{1 + \tau} + C,$$
where the $A$'s, $B$'s and $C$ are some constants. Hence, the coefficients $a_{0,p;2p,k}(\tau)$ and $a_{4,p;2p,k}(\tau)$ will not be smooth at $\tau = \pm 1$ unless

$$a_{0,p;2p,k}(0) = a_{4,p;2p,k}(0).$$

(141)

We shall refer to the latter as to a **regularity condition**. Although the original derivation of this result in [15] was given with time symmetric initial data in mind, the statement holds true even if the data is not time symmetric\(^{11}\).

Given the aforesaid regularity condition, the question arises whether it is possible to rewrite it in terms of the freely specifiable data available when solving the constraint equations using the conformal method. This is the content of the following theorem.

**Theorem 3.** For conformally flat data such that $|x|^8 \psi_{\alpha\beta} \psi^{\alpha\beta} \in \mathcal{E}^\infty(B_{\alpha}(i))$, and independently of the choice of the function $\kappa$ appearing in the conformal factor [106], the following conditions are equivalent:

(i) $\phi_{0,p;2p,k}(0) = \phi_{4,p;2p,k}(0), p = 2, 3, \ldots, k = 0, \ldots, 2p$;

(ii) $L_{0,p-4;2p,k} + L_{4,p-4;2p,k} = 0, p = 2, 3, \ldots, k = 0, \ldots, 2p$.

**Proof.** The result follows from looking at formula (125b) and counting arguments regarding the expansion types of the quantities involved.

**Remark.** Note that the regularity condition (141) is trivially satisfied for time symmetric, conformally flat data.

The analysis described in the present article will be concerned with initial data rendering solutions to the transport equations which are as smooth as possible. Accordingly, we make the further

**Assumption 5.** The second fundamental form satisfies the condition (ii) in theorem 3.

## 8 Solutions to the transport equations and obstructions to the smoothness of null infinity

From the discussion in the previous section it follows that the origin of the non-smoothness of the solutions of the transport equations lies in Bianchi subsystem [132]. In this section we shall describe in a qualitative fashion the structure of the solutions to this subsystem for initial data satisfying the assumptions 1-5. In order to ease our calculations we choose

$$\kappa = \rho,$$

(142)

any other choice with $\kappa = \rho \kappa'$, where $\kappa'$ is smooth and such that $\kappa'(i) = 1$ would not alter the essence of the results.

All the results here presented have been obtained by means of scripts written for the computer algebra system **Maple V**. In what follows, we shall be systematically using the decomposition

$$\phi_{j}^{(p)} = \sum_{q=[2-j]}^{p} \sum_{k=0}^{2q} a_{j,p;2q,k} T_{2q}^{k} q_{2+q,j}.$$  

(143)

of the components of the Weyl spinor $\phi_{A B C D}$. We shall use $Q(\tau)$ to denote a generic polynomial in $\tau$, while $P_k(\tau)$ will denote a generic polynomial of degree $k$ in $\tau$ such that $P_k(\pm 1) \neq 0$. It will be understood that $Q(\tau)$ and $P_k(\tau)$ appearing in different equations are in principle unrelated.

\(^{11}\)In [15] Friedrich, in a *tour de force*, has shown that the condition $\phi_{0,p;2p,k}(0) = \phi_{4,p;2p,k}(0)$ for $p = 2, 3, \ldots, k = 0, \ldots, 2p$ is equivalent to the vanishing of the Bach tensor and its symmetrised, tracefree derivatives to all orders at $i$—cfr. also with footnote 1 on the present article.
8.1 Lower order solutions

The first observation that has to be made is that if the assumptions 1-5 are satisfied then the lower order solutions are fully regular. Indeed,

**Theorem 4.** Under assumption 1-5, the solutions of the transport equations for $p = 0, 1, 2, 3, 4$ have polynomial dependence in $\tau$. Thus, they extend smoothly to the sets $I^\pm$.

It should be pointed out that assumption 5 plays a crucial role here. The original analysis in [15] suggests that if the regularity condition involving the Cotton-York tensor does not hold, then logarithmic divergences in the solutions to the transport equations can appear at order $p = 2$.

8.2 Solutions at order $p = 5$

As it happens in the time symmetric case, the first logarithmic divergences in the solutions to the transport equations appear at order $p = 5$. More precisely, the solutions to the $p = 5$ v-transport equations are polynomial in $\tau$. On the other hand, the solutions of the $p = 5$ Bianchi transport equations are such that

$$a_{j;5;4,k}(\tau) = \Upsilon_{5;4,k}(1 - \tau)^{7-j}P_j(\tau) \ln(1 - \tau) + (1 + \tau)^{3+j}P_{4-j}(\tau) \ln(1 + \tau) + O(\tau),$$

with

$$\Upsilon_{5;4,0} = 18m^2w_{2;4,0} - \frac{3099}{199}\sqrt{6m^2}L_{0,-2,4,0}$$

$$+ \frac{1023}{199}\sqrt{6m^2}\left(L_{0,-1;4,0} - L_{4,-1;4,0}\right) + \frac{224}{199}\sqrt{6}\left(L_{0,0;4,0} - L_{4,0;4,0}\right)$$

$$\Upsilon_{5;4,1} = 18m^2w_{2;4,1} - \frac{3099}{199}\sqrt{6m^2}L_{0,-2,4,1}$$

$$+ \frac{1023}{199}\sqrt{6m^2}\left(L_{0,-1;4,1} - L_{4,-1;4,1}\right) + \frac{224}{199}\sqrt{6}\left(L_{0,0;4,1} - L_{4,0;4,1}\right)$$

$$\Upsilon_{5;4,2} = 18m^2w_{2;4,2}m^2 + \frac{37602}{199}mJ^2 - \frac{3099}{199}\sqrt{6m^2}L_{0,-2,4,2}$$

$$+ \frac{1023}{199}\sqrt{6m^2}\left(L_{0,-1;4,2} - L_{4,-1;4,2}\right) + \frac{224}{199}\sqrt{6}\left(L_{0,0;4,2} - L_{4,0;4,2}\right)$$

$$\Upsilon_{5;4,3} = 18m^2w_{2;4,3} - \frac{3099}{199}\sqrt{6m^2}L_{0,-2,4,3}$$

$$+ \frac{1023}{199}\sqrt{6m^2}\left(L_{0,-1;4,3} - L_{4,-1;4,3}\right) + \frac{224}{199}\sqrt{6}\left(L_{0,0;4,3} - L_{4,0;4,3}\right)$$

$$\Upsilon_{5;4,4} = 18m^2w_{2;4,4} - \frac{3099}{199}\sqrt{6m^2}L_{0,-2,4,4}$$

$$+ \frac{1023}{199}\sqrt{6m^2}\left(L_{0,-1;4,4} - L_{4,-1;4,4}\right) + \frac{224}{199}\sqrt{6}\left(L_{0,0;4,4} - L_{4,0;4,4}\right).$$

The coefficients for the remaining sectors: $a_{j;5,k}(\tau), a_{j;5,2,k}(\tau), a_{j;5,6,k}(\tau), a_{j;5,8,k}(\tau)$ and $a_{j;5,10,k}(\tau)$, $j = 0, \ldots, 4$, are polynomial in $\tau$. We refer to the coefficients $\Upsilon_{5;4,k}$ as to the quadrupolar obstructions at order 5. Note that although the initial data set is time asymmetric, the obstructions obtained at this order are time symmetric.

The quadrupolar sector at order $p = 5$ is the only one rendering logarithmic divergences. More precisely

**Lemma 1.** For initial data satisfying the assumptions 1-5, the coefficients $a_{j;5,2q,k}$, $j = 0, \ldots, 4$, $q = 0, \ldots, 5$, $q \neq 2$, $k = 0, \ldots, 2q$ are polynomials in $\tau$.

8.3 Expansions at order $p = 6$

Some experimentation reveals that if one uses the solutions (144) to the $p = 5$ Bianchi transport equations then $\ln(1 \pm \tau)$ terms will be present in the vector $\psi^{(6)}$. Moreover, $\phi^{(6)}$ would also
contain, besides the \( \ln(1 \pm \tau) \) terms, expressions involving \( \ln^2(1 + \tau) \), \( \ln(1 + \tau)\ln(1 - \tau) \) and \( \ln^2(1 - \tau) \). The discussion of these non-regular solutions falls beyond the scope of the present investigation. Hence, in the following calculation we assume that

\[
\Upsilon_{5,4,k} = 0, \quad k = 0, \ldots, 4. \tag{150}
\]

The latter condition could be used to solve for, say, \( w_{2,4,k} \). Now, at first glance it is not fully clear whether it is possible to have some initial data for which the obstructions vanish. As already mentioned in section 5, the coefficients \( w_p2,0,k \) in the expansions of the function \( W \) are of global nature and may depend in a non-trivial way on the freely specifiable data. It may be the case that in order to have initial data for which the obstruction vanishes further restrictions on the free data are necessary. Indeed, explicit calculations render the following

**Lemma 2.** Consider initial data satisfying the assumptions 1-5. Let \( \Upsilon_{5,4,k} = 0, k = 0, \ldots, 4 \). Then the \( p = 6 \) Bianchi constraint transport equations are satisfied if and only if

\[
\frac{1023}{199} \sqrt{6} \left[ m \left( L_{0,-1;4,k} - L_{4,-1;4,k} \right) + \frac{224}{199} \sqrt{6} \left( L_{0,0;4,k} - L_{4,0;4,k} \right) - \frac{3099}{199} \sqrt{6} m^2 L_{0,-2;4,k} \right] = 0, \tag{151}
\]

for \( j = 0, \ldots, 4 \). The latter implies that

\[
mw_{2;4,2} = -\frac{2089}{199} j^2, \tag{152}
\]

while the other \( w_{2;4,k} \)'s vanish.

If \( \Upsilon_{5,4,k} = 0 \) and the previous theorem is satisfied, then the solution to the Bianchi transport equations are of the form:

\[
a_{j,6,4,k}(\tau) = \Upsilon^+_{6,4,k}(1 - \tau)^{q - j} \mathcal{P}_j(\tau) \ln(1 - \tau) + \Upsilon^-_{6,4,k}(1 + \tau)^{q + j} \mathcal{P}_{4-j}(\tau) \ln(1 + \tau) + \mathcal{Q}(\tau), \quad (153)
\]

with \( k = 0, \ldots, 6 \).

These obstructions do not have a counterpart when considering expansions of time symmetric initial data. We call the coefficients \( \Upsilon^\pm_{6,4,k} \) the *quadrupolar obstructions at order \( p = 6 \).* Furthermore, the obstructions are symmetric with regard to the two disconnected parts of null infinity. That is,

\[
\Upsilon^+_6(4,4, k) \neq 0 \neq \Upsilon^-_{6,4,k} = 0, \tag{154}
\]

generically.

In order to analyse with more detail the structure of the obstructions, we shall further assume that the initial data is *axially symmetric*. In this case, only the obstructions with \( k = 2 \) are in principle non-vanishing. They are given by

\[
\Upsilon^+_{6,4,2} = \frac{7722}{7} i Q_3 - \frac{2198208}{6965} j^2 - \frac{20817}{14} A_j^2 + \frac{62961}{4816} \sqrt{6} A \left[ L_{4,-1;4,2} - L_{0,-1;4,2} \right] - \sqrt{6} \left( \frac{7559711}{126420} L_{4,-1;4,2} + \frac{863019}{126420} L_{0,-1;4,2} \right)
\]

\[
+ \frac{58}{43} \sqrt{6} A \left[ L_{4,0;4,2} - L_{0,0;4,2} \right] + \sqrt{6} \left( \frac{3282401}{27090} L_{4,0;4,2} + \frac{3648769}{27090} L_{0,0;4,2} \right)
\]

\[
+ \frac{144}{13} \sqrt{6} \left[ L_{4,1;4,2} + L_{0,1;4,2} \right]. \tag{155a}
\]

\[
\Upsilon^-_{6,4,2} = \frac{7722}{7} i Q_3 + \frac{2198208}{6965} j^2 - \frac{20817}{14} A_j^2 + \frac{62961}{4816} \sqrt{6} A \left[ L_{4,-1;4,2} - L_{0,-1;4,2} \right] - \sqrt{6} \left( \frac{7559711}{126420} L_{0,-1;4,2} + \frac{863019}{126420} L_{4,-1;4,2} \right)
\]

\[
+ \frac{58}{43} \sqrt{6} A \left[ L_{4,0;4,2} - L_{0,0;4,2} \right] + \sqrt{6} \left( \frac{3282401}{27090} L_{0,0;4,2} + \frac{3648769}{27090} L_{4,0;4,2} \right)
\]

\[
+ \frac{144}{13} \sqrt{6} \left[ L_{4,1;4,2} + L_{0,1;4,2} \right]. \tag{155b}
\]
The asymmetry in the obstructions can be observed in much simpler situations. Namely, consider initial data sets for which
\[ Q_3 = L_{0,-1;4,2} = L_{4,-1;4,2} = L_{0,0;4,2} = L_{4,0;4,2} = L_{0,1;4,2} = L_{4,1;4,2} = 0. \] (156)
In that case the obstructions read
\[ \Upsilon_{6;4,2}^+ = -\frac{2198208}{6965} J^2 - \frac{20817}{14} AJ^2, \quad \Upsilon_{6;4,2}^- = \frac{2198208}{6965} J^2 - \frac{20817}{14} AJ^2, \] (157)
which clearly cannot be satisfied simultaneously.

Now, assume one has an initial data such that \( \Upsilon_{6;4,2}^+ = \Upsilon_{6;4,2}^- = 0 \). One can use these two conditions to solve for \( L_{0,-2;4,2}, L_{4,-1;4,2} \). The latter implies that
\[ L_{0,-2;4,2} = \frac{14}{23} (L_{0,0;4,4} - L_{4,0;4,4}) - \sqrt{6} \frac{19096}{4377} J^2. \] (158)
The regularity condition (ii) of theorem 3 together with the reality condition imply that the coefficient \( L_{0,-2;4,2} \) is a pure imaginary number. Similarly, because of the reality conditions, the first term on the right hand side is also pure imaginary, while the second is clearly a real number. This implies that
\[ J = 0, \quad w_{2;4,2} = 0. \] (159)
This result holds true also for initial data without axial symmetry. More precisely, one has the following

**Theorem 5.** Initial data sets satisfying assumptions 1-5, if \( \Upsilon_{5;4,k} = 0, k = 0, \ldots, 4 \) with the consistency condition of lemma 2 fulfilled, and moreover, with \( \Upsilon_{6;4,k}, k = 0, \ldots, 4 \) are such that
\[ J = 0, \quad w_{2;4,k} = 0, \quad \text{with } k = 0, \ldots, 4. \] (160)

Following a pattern already observed in the case of time symmetric initial data, the sector \( a_{j,6,8,k}(\tau) = 0 \) will also render obstructions—which, in this case, are of octupolar nature and time symmetric. More precisely
\[ a_{j,6,6,k}(\tau) = \Upsilon_{6;6,k} \left( (1 - \tau)^{8-j} P_{j+1}(\tau) \ln(1 - \tau) + (1 + \tau)^{4+j} P_{5-j}(\tau) \ln(1 + \tau) \right) + Q(\tau). \] (161)
In particular, one has for example that
\[ \Upsilon_{6;6,3} = 24 w_{3,6,3} - \frac{565753248}{82585} i J^3 \]
\[ + \frac{36399}{830} \sqrt{6} i J \left( L_{4,-1,1,2} - L_{0,-1;4,2} \right) + \frac{1904}{415} \sqrt{6} i J \left( L_{4,0;4,2} - L_{0,0;4,2} \right) \]
\[ - \frac{2727}{415} \sqrt{30} L_{0,-1;6,3} + \frac{408}{83} \sqrt{30} \left( L_{0,0;6,3} - L_{4,0;6,3} \right) + \frac{576}{415} \sqrt{30} \left( L_{0,1;6,3} - L_{4,1;6,3} \right). \] (162)
The other obstructions have a similar structure. These can be used to solve for the octupolar quantities \( w_{6,6,k}, k = 0, \ldots, 6 \). In a similar way as to what happened when setting \( \Upsilon_{5;4,k} = 0 \), it is to be expected that consistency conditions will arise in the expansions at order \( p = 7 \). In complete affinity with lemma 2 one has the following

**Lemma 3.** For initial data complying with assumptions 1-5 and such that \( \Upsilon_{5;4,k} = 0, k = 0, \ldots, 4 \), if the consistency conditions of lemma 2 are satisfied, then the coefficients \( a_{j,6,2q,k}(\tau) \) with \( j = 0, \ldots, 4, q = 0, \ldots, 6, q \neq 2, 3, k = 0, \ldots, 2q \) are polynomial in \( \tau \).
8.4 Expansions at order $p = 7$ and beyond

The analogue of lemma 2 for the expansions at order $p = 7$ is the following

**Lemma 4.** Consider initial data conforming with assumptions 1-5. If, moreover, $\Upsilon_{5,4,k_2} = 0$, and $\Upsilon_{6,6,k_3} = 0$ for $k_i = 0, \ldots, 2i$ and the consistency condition in lemma 5 is observed, then the $p = 7$ Bianchi constraint equations are satisfied if and only if

$$Q_1 = Q_2 = Q_3 = 0;$$

$$L_{0,0,4,k} = L_{4,0,4,k}, \text{ with } k = 0, \ldots, 4;$$

$$85 \left( L_{0,6,6,k} - L_{4,0,6,k} \right) + 24 \left( L_{0,1,6,k} - L_{4,1,6,k} \right) - 303 L_{0,-1;6,k} = 0, \text{ with } k = 0, \ldots, 6.$$

(163a, 163b, 163c)

**Remark.** The conditions given in the previous lemma imply that

$$L_{0,-2;4,k_2} = 0, \quad w_{3,6,k_3} = 0,$$

(164)

with $k_i = 0, \ldots, 2i$.

If the consistency requirements given in the previous lemma are satisfied then it turns out that the sectors $a_{j,7,0,k_0} (\tau), a_{j,7,2,k_1} (\tau), a_{j,7,10,k_3} (\tau), a_{j,7,12,k_5} (\tau)$ and $a_{j,7,14,k_7} (\tau)$ have polynomial dependence in $\tau$, while the sectors $a_{j,7,4,k_2} (\tau), a_{j,7,6,k_3} (\tau)$ and $a_{j,7,8,k_4} (\tau)$ exhibit a distinctive structure. Namely,

$$a_{j,7,4,k} (\tau) = \Upsilon_{7,4,k}^+ (1 - \tau)^{9-j} \mathcal{P}_j (\tau) \ln (1 - \tau) + \Upsilon_{7,4,k}^- (1 + \tau)^{5+j} \mathcal{P}_{4-j}(\tau) \ln (1 + \tau) + \mathcal{Q}(\tau),$$

(165)

with $k = 0, \ldots, 4$ and the obstructions $\Upsilon_{7,4,k}^\pm$ time asymmetric in the sense described in the previous subsection. They depend on $A, L_{0,0,4,k}, L_{4,0,4,k}, L_{0,1,4,k}, L_{4,1,4,k}, L_{0,2,4,k}, L_{4,2,4,k}$.

The conditions $\Upsilon_{7,4,k}^+ = \Upsilon_{7,4,k}^- = 0$ can be used to solve for, say, $L_{0,0,4,k}, L_{4,0,4,k}$.

Similarly, one has

$$a_{j,7,6,k} (\tau) = \Upsilon_{7,6,k}^+ (1 - \tau)^{9-j} \mathcal{P}_{j+1}(\tau) \ln (1 - \tau) + \Upsilon_{7,6,k}^- (1 + \tau)^{5+j} \mathcal{P}_{5-j}(\tau) \ln (1 + \tau) + \mathcal{Q}(\tau),$$

(166)

with $k = 0, \ldots, 6$ and $\Upsilon_{7,6,k}^\pm$ also time asymmetric and depending on $A, L_{0,0,6,k}, L_{4,0,6,k}, L_{0,1,6,k}, L_{4,1,6,k}, L_{0,2,6,k}$ and $L_{4,2,6,k}$.

Finally, we have

$$a_{j,7,8,k} (\tau) = \Upsilon_{7,8,k}^+ \left( (1 - \tau)^{9-j} \mathcal{P}_{j+2}(\tau) \ln (1 - \tau) + (1 + \tau)^{5+j} \mathcal{P}_{6-j}(\tau) \ln (1 + \tau) \right) + \mathcal{Q}(\tau),$$

(167)

for $k = 0, \ldots, 8$ which renders a time symmetric obstruction depending on $w_{3,6,k}, A, L_{0,0,8,k}, L_{4,0,8,k}, L_{0,1,8,k}, L_{4,1,8,k}, L_{0,2,8,k}$ and $L_{4,2,8,k}$. The condition $\Upsilon_{7,8,k}$ can be used to solve for $w_{3,6,3}$.

As it is to be expected, requiring that the obstructions obtained at order $p = 7$ all vanish gives rise to consistency conditions on the expansions at order $p = 8$. In particular, one has the following

**Lemma 5.** Assume that the hypothesis and conditions of lemma 4 are satisfied. If one has initial data such that $\Upsilon_{7,4,k_2}^\pm = 0$, $\Upsilon_{7,6,k_3}^\pm = 0$, and $\Upsilon_{7,8,k_4} = 0$, with $k_i = 0, \ldots, 2i$ then a necessary condition for the existence of solutions to the $p = 8$ constraint equations is that

$$L_{0,-1,6,k} = L_{4,-1,6,k} = 0, \text{ with } k = 0, \ldots, 6.$$

(168)

From the results here presented it is possible to hint a general pattern in the structure of the solutions to the transport equations at an arbitrary order. Nevertheless, it must be said that a rigorous proof of such a pattern, given the intricacy and length of the equations, would constitute a remarkable feat which cannot be attained with the current tools and understanding of the problem.
8.5 Asymptotic Schwarzschildian data

The vanishing of the obstructions and the fulfillment of the associated compatibility conditions greatly restrict the type of conformally flat initial data under consideration. We summarise the results of our calculations in the following

Theorem 6. An initial data set \((\mathcal{S}, h_{\alpha\beta}, \chi_{\alpha\beta})\) for which

\[
\begin{align*}
\Upsilon_{5,4,k_2} &= \Upsilon_{6,4,k_2}^\pm = \Upsilon_{7,4,k_2}^\pm = 0, \\
\Upsilon_{6,6,k_3} &= \Upsilon_{7,6,k_3}^\pm = 0, \\
\Upsilon_{7,8,k_3} &= 0,
\end{align*}
\]

with \(k_i = 0, \ldots, 2i\), and for which the consistency conditions in lemmas 2, 4 and 5 are satisfied is such that

\[
\begin{align*}
W &= \frac{m}{2} + \mathcal{O}(\rho^4), \\
\psi_{ABCD} &= -\frac{A}{\rho^3} \epsilon_{ABCD}^2 + \mathcal{O}(1).
\end{align*}
\]

Remark. Initial data sets of the sort discussed in the previous theorem can rightfully be called Schwarzschildian up to order \(p = 3\). Their essential octupolar (2\(^3\) = 8) content vanishes. More generally, we shall say that an initial data set is Schwarzschildian to order \(p\) near infinity if

\[
\begin{align*}
W &= \frac{m}{2} + \mathcal{O}(\rho^{p+1}), \\
\psi_{ABCD} &= -\frac{A}{\rho^3} \epsilon_{ABCD}^2 + \mathcal{O}(\rho^{p-3}).
\end{align*}
\]

That the solutions of the transport equations for the time asymmetric data discussed in section 5 extend smoothly to the critical sets \(I^\pm\) can be shown through an induction argument. Because of the manifest spherical symmetric of the set up, the only non-zero field quantities are those with spin-weight 0—that is, the only non-trivial sector is that given by \(T_{0,0,0}^0\). It is not difficult to show that the fundamental matrix of the system of \(v\)-transport equations (131) has entries that are polynomial in \(\tau\)—this result is also true even without assuming spherical symmetry. This implies that the solutions of the \(v\)-transport equations will be polynomial if the lower order terms are polynomial. More crucially is that under the current assumptions, the only non-trivial Bianchi propagation equation is

\[
\partial_\tau \phi_2 = -\frac{1}{2} \chi_2 \phi_2 - 3 \chi_h \phi_2,
\]

where \(\chi_2\) and \(\chi_h\) are, respectively, the \(\epsilon_{ABCD}^2\) and \(h_{ABCD}\) components of \(\chi_{ABCD}\)—see section 3. From here it also trivially follows that if the lower order terms in, say, the \(p\)-th order Bianchi transport equation are polynomial, then also \(\phi_2^{(p)}\) will be polynomial in \(\tau\) and, thus, will extend through the critical sets. This simple argumentation is possible only because of the spherical symmetry. This is to be contrasted with the case of, say, the solutions to the transport equations for Kerrian initial data, for which—as it will be discussed in paper II—all the components of the Weyl spinor \(\phi_{ABCD}\) are non-zero.

To prove that the solutions of the transport equation for the time asymmetric Schwarzschildian data render series of the form (130) which are solutions to the propagation equations is a bit more sophisticated. It can be done using techniques similar to the ones used by Friedrich in section 6 of [18]. In any case, th smoothness follows from the analysis of [18].

9 Conclusions

Theorem 6 and the structure observed in the expansions up to order \(p = 7\) readily suggest the following generalisation of the conjecture given in [24]. Namely,
Conjecture. The time development of an asymptotically Euclidean, initial data set which is conformally flat in a neighbourhood $B_a(i)$ of infinity and for which $|x|^k \psi_{\alpha\beta} \psi^{\alpha\beta} \in E^\infty(B_a(i))$ admits a conformal extension to both future and past null infinity of class $C^k$, with $k$ a non-negative integer, if and only if the initial data is Schwarzschild to order $p_\star$, in $B_a(i)$, where $p_\star = p_\star(k)$ is a non-negative integer. If the development admits an extension of class $C^\infty$, then the initial data has to be exactly Schwarzschild on $B_a(i)$.

Remarks. Firstly, it is important to notice —as it follows from our discussion in section 8— that the Schwarzschild data is not necessarily time symmetric. The condition $|x|^k \psi_{\alpha\beta} \psi^{\alpha\beta} \in E^\infty(B_a(i))$ is used here for convenience. Any condition which would ensure that the conformal factor arising from the Lienerowicz equation is of the form $\vartheta = \vartheta/|x|$ with $\vartheta \in E^\infty$ —a conformal factor that is not necessarily smooth, but whose expansions in $B_a(i)$ do not contain logarithmic terms— would serve as well. One could think of generalisations of the conjecture where one requires that $\vartheta \in E^m(B_a(i))$ for some given non-negative integer $m$. If the condition is sharp, it is to be expected that it would generate its own type of obstructions to the smoothness of null infinity. Finally, we notice that —as seen from the results of our calculations— future and past null infinity could have different degrees of regularity, say $C^{k_+}$ and $C^{k_-}$. This extra structure is lost in the conjecture as clearly $k = \min\{k_+, k_-\}$.

A first step toward proving this conjecture would be to show that the solutions to the $p$th order transport equations extend smoothly to the critical sets $I^\pm$ if and only if the initial data is Schwarzschild to a certain order $p_\star = p_\star(p)$. The calculations shown in this article suggest that at least $p_\star = p - 4$. This is to be contrasted with the results in [24] which suggest that for conformally flat, time symmetric data the corresponding value ought to be $p_\star = p - 3$. This result which should provide sharp conditions could then be used in turn to prove the necessary existence results required for the conjecture.

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