Newtonian limit of the singular $f(R)$ gravity in the Palatini formalism.

A.E. Domínguez and D.E. Barraco
FaM.A.F., Universidad Nacional de Córdoba
Ciudad Universitaria, Córdoba 5000, Argentina

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Abstract

Recently D. Vollick [Phys. Rev. D68, 063510 (2003)] has shown that the inclusion of the $1/R$ curvature terms in the gravitational action and the use of the Palatini formalism offer an alternative explanation for cosmological acceleration. In this work we show not only that this model of Vollick does not have a good Newtonian limit, but also that any $f(R)$ theory with a pole of order $n$ in $R = 0$ and $(d^2 f / d^2 R)(R_0) \neq 0$, where $R_0$ is the scalar curvature of background, does not have a good Newtonian limit.

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1 INTRODUCTION

From recent studies it seems well established that our universe expansion is currently in an accelerating phase. The evidence of cosmic acceleration has arisen not only from the high redshift surveys of type Ia supernovae [1, 2], but also from the anisotropy power spectrum of the cosmic microwave background [3, 4]. One of the most accepted explanations is that the Universe has been dominated by some form of dark energy for a long time. However, none of the existing dark energy models are completely satisfactory.

It is possible to find other explanations for cosmic expansion using field equations other than Einstein’s equations. Recently, some authors have proposed to add a $R^{-1}$ term in the Einstein-Hilbert action to modify general relativity [5, 6, 7]. They obtained the field equations using second-order formalism, varying only the metric field, and thus obtained the so-called fourth-order field equations. Although the models were obtained using corrections of the Einstein-Hilbert Lagrangian of type $R^n$, where $n$ can take a positive or negative value to explain both the inflation at an early time and the expansion at the present time [5, 8], they still suffer from violent instabilities [9]. The Newtonian limit of these fourth-order theories has been studied by Dick [10].
On the other hand, we can consider those theories that are obtained from a Lagrangian density $\mathcal{L}_T(R) = f(R)\sqrt{-g} + \mathcal{L}_M$, which depends on the scalar curvature and a matter Lagrangian that does not depend on the connection, and then we can apply Palatini’s method to obtain the field equations [12, 13]. In Refs. [12, 13], we showed the universality of the Einstein equation using a cosmological constant. More recently, Ferraris, Francaviglia and Volovich published the same result [14, 15]. For these theories we have studied the conserved quantities [16], the spherically symmetric solutions [17], the Newtonian limit [12, 18], and the cosmology described by Friedmann-Robertson-Walker (FRW) metric [19].

In our previous paper [18], it was also shown that it is very difficult to test these models in the (post-)Newtonian approximation. The reason for this is that the departures from Newtonian behavior are both very small and are masked by other effects, because these departures from the Newtonian behavior have to be measured when the body is moving “through” a matter-filled region. In the main applications of these theories, for example, in the Newtonian limit or in the calculation of the observed cosmological parameter, we have considered the analyticity of $f(R)$ at $R = 0$.

Moreover, it is well known [18, 14, 17] that in a vacuum, or in the case of $T = const$, the solutions of these theories are the same as general relativity with a cosmological constant, even when $f(R)$ is not analytical at $R = 0$. On the other hand, solutions corresponding to different cosmological constants are allowed by some of these theories. Therefore, the homogeneous and isotropic vacuum solution for these theories is the de Sitter space-time with different cosmological constants, except when one of the allowed cosmological constants is $\Lambda = 0$, which corresponds to flat space-time.

Recently, Vollick [20] used the corrected Lagrangian of the works of Carroll et al., and Capozziello et al. [5, 7], $f(R) = R - \alpha^2/R$, and the Palatini variational principle—which is a particular case of the above theories but is not analytical at $R = 0$—to prove that the solutions of the field equations approach de Sitter universe at a late time. This result was obtained using the above well known property of the vacuum solutions. Thus, the inclusion of $1/R$ curvature terms in the gravitational Lagrangean provides us with an alternative explanation for the cosmological acceleration. Moreover, Meng and Wang [21, 22] have studied the modified Friedmann equation with the Palatini variational principle and its first-, second-, and third-order approximated equations.

In agreement with this last idea about the maximal symmetric vacuum solution, we have studied the Newtonian limit for $f(R)$ theories as a perturbation of the de Sitter background. The difference from previous works [12, 18] is that there the background was flat space-time, which is only possible in the particular case $f(0) = 0$.

In this work we show that the Vollick model [20] does not give the Newtonian limit but also that any $f(R)$ with a pole of order $n$ in $R = 0$ and $(d^2 f/d^2 R)(R_0) \neq 0$, where $R_0$ is the scalar curvature of background, cannot have a Newtonian Limit. This result is equivalent to the condition obtained by Dick [10] for the fourth-order theories. Nevertheless, the theories with singular $f(R)$ and $(d^2 f/d^2 R)(R_0) = 0$ must be studied in the future since they have a correct Newtonian limit and explain the observed cosmological acceleration. In our work we follow the conventions and notation of Synge [11]
2 REVIEW OF THE FIELD EQUATIONS

We review in this section the structure of the theory as presented elsewhere in previous works [12, 18].

Let $\mathcal{M}$ be a manifold with metric $g_{ab}$ and a torsion-free derivative operator $\nabla_a$, both considered as independent variables. Consider a Lagrangian density $L = f(R)\sqrt{-g} + \mathcal{L}_M$, where the matter Lagrangian $\mathcal{L}_M$ does not depend on the connection.

Suppose we have a smooth one-parameter $\lambda$ family of field configurations starting from given fields $g^{ab}$, $\nabla_a$, and $\psi$ (the matter fields), with appropriate boundary conditions. Let $\delta g^{ab}$, $\delta \Gamma^c_{ab}$, $\delta \psi$ be the corresponding variations of those fields, i.e., $\delta g^{ab} = (dg^{ab}/d\lambda)|_{\lambda=0}$, etc. Then, if we vary with respect to the metric, the field equations are

$$f'(R)R_{ab} - \frac{1}{2}f(R)g_{ab} = \alpha_M T_{ab}. \tag{1}$$

where $f'(R) = (df/dR)$, $(\delta S_M/\delta g^{ab}) \equiv -T_{ab}\sqrt{-g}$ and $\alpha_m = -8\pi$. The variation with respect to the connection, recalling that this is fixed at the boundary, gives

$$\nabla_c[\sqrt{-g}g^{ab}f'(R)] = 0. \tag{2}$$

Now, we choose Lagrangian $f(R)$ with $f'(R)$ derivable. Then, the last equation becomes

$$\nabla_c g_{ab} = b_c g_{ab}, \tag{3}$$

where

$$b_c = -[\ln f'(R)]_c. \tag{4}$$

Thus, we have a Weyl conformal geometry with a Weyl field given by Eq. (4).

From Eq. (1) we obtain

$$f'(R)R - 2f(R) = \alpha_M T, \tag{5}$$

which defines $R(T)$, and we suppose that the function $f(R)$ is such that $R(T)$ is derivable with respect to the variable $T$. Therefore, $b_c$ is determined by $T$ and its derivatives except in the case $f(R) = \omega R^2$, for which $Rf' - 2f \equiv 0$, so we must consistently have $T \equiv 0$. It is important to note that $b_c$ has a solution only if $T$ is differentiable in $\mathcal{M}$; this condition on $T$, for the existence of a solution, is not necessary in other theories such as general relativity (GR) or fourth-order theories.

The connection solution to Eq. (3) is

$$\Gamma^a_{bc} = C^a_{bc} - \frac{1}{2}g^a_{bc} + \delta^a_c b_b - g_{bc} b^a, \tag{6}$$

where $C^a_{bc}$ are the Christoffel symbols (metric connection). Then, we have to solve only Eq. (1).

The Riemann tensor can be defined in the usual form, and then, the Ricci tensor and scalar curvature are

$$R_{ab} = R^m_{ab} - \frac{3}{2}D_a b_b + \frac{1}{2}D_b b_a - \frac{1}{2}g_{ab} D \cdot b - \frac{1}{2}b_a b_b + \frac{1}{2}g_{ab} b^2, \tag{7}$$

$$R = R^m - 3D \cdot b + \frac{3}{2}b^2, \tag{8}$$
where \( R_{ab} \), \( R \), and \( D_c \) are the Ricci tensor, scalar curvature, and covariant derivative defined from the metric connection, respectively.

Because the matter action must be invariant under diffeomorphisms and the matter fields satisfy the matter field equations, \( T_{ab} \) is conserved [18]

\[
D^a T_{ab} = 0.
\] (9)

Therefore, a test particle will follow the geodesics of the metric connection. Using Eqs.(4) and (5) we have

\[
b_c = -\frac{f'' \alpha_M \nabla_c T}{f'(Rf'' - f')}.
\] (10)

Except for the case of GR, where \( f'' \equiv 0 \), the Weyl field is nonzero wherever the trace of the energy-momentum tensor varies with respect to the coordinates. If \( T \) is constant, then \( R \) is also constant, \( b_c = 0 \) and (5) takes the form

\[
G_{ab} - \frac{1}{2} \Lambda g_{ab} = KT_{ab},
\] (11)

where \( \Lambda \) and \( K \) are two functions of \( R \). All those cases with constant trace of the energy-momentum tensor are equivalent to GR for a given cosmological constant. This is the so-called universality of the Einstein equations for matter for which \( T \) is constant [12, 13]. In the case \( T = 0 \), the scalar \( R \) is any of the roots, \( R_i \), of the equation \( f'(R)R - 2f(R) = 0 \). For each root the solutions of the field equations are the solutions of GR with cosmological constant \( \Lambda = -R_i/4 \). Therefore, the maximal symmetric vacuum solution of these theories is the de Sitter space-time.

3 THE NEWTONIAN LIMIT

The above theories, which explain the cosmological acceleration, must be checked in the Newtonian limit, i.e., in the slow field and slow velocity approximation. In this approximation, these theories must be in agreement with the present data.

The Newtonian limit, in our case, must be taken as a perturbation of the homogeneous isotropic vacuum background. As we have seen above, this background space-time is the de Sitter space-time, except when \( \Lambda = 0 \), in which case we must perturbate around the flat space-time. The case \( \Lambda = 0 \) has just been studied [12, 18] and the general solution of the problem with \( \Lambda \neq 0 \) must agree with the previous result when \( \Lambda = 0 \).

The background metric and the background Ricci tensor are

\[
^0 g_{ab} = -dt^2 + e^{2t \sqrt{\Lambda/3}}(dx^2 + dy^2 + dz^2),
\] (12)

\[
^0 R_{ab} = -\Lambda ^0 g_{ab}.
\] (13)

We assume that the metric can be written in the form

\[
^0 g_{ab} = ^0 g_{ab} + h_{ab}
\] (14)
where \( h_{ab} \) is the perturbation of the metric. The first order of the field equations (1) are

\[
f'(R) R_{ab} + f'(R) \frac{1}{2} f(R) g_{ab} - \frac{1}{2} f(R) h_{ab} = \alpha M T_{ab} \tag{15}
\]

where the 0 superscript means zero-order quantities, i.e., background quantities, and the superscript 1 means the first-order quantities. We have to calculate each term of the above equation.

According to (6) the connection \( \Gamma_{ab}^c \) can be split into two parts,

\[
\Gamma_{ab}^c = \Gamma_{ab}^{0c} + \Gamma_{ab}^{1c}, \tag{16}
\]

where \( \Gamma_{ab}^{0c} \) is the connection corresponding to the background metric and \( \Gamma_{ab}^{1c} \) has two terms, one depending on the metric perturbation \( h_{ab} \) and the other depending on \( b_a \) which has an order higher than zero.

In order to be able to write the linear field equations we must calculate the linear Ricci tensor by using (16). We obtain

\[
R_{ab} = R_{ab} + \frac{1}{2} R_{ab} \tag{17}
\]

where the \( R_{ab} \) part is

\[
\frac{1}{2} R_{ab} = - \nabla_c \Gamma_{ab}^{0c} + \nabla_{(a} \Gamma_{b)c}^{0c}. \tag{18}
\]

By \( \nabla \) we mean the covariant derivative associated with the background metric.

As we have just said, the first order of connection can be written as follows:

\[
\Gamma_{ab}^{1c} = C_{ab}^{1c} + A_{ab}^{1c}, \tag{19}
\]

where the first term is

\[
C_{ab}^{1c} = g^{cd} \nabla_{(a} h_{b)d} - \frac{1}{2} g^{cd} \nabla_d h_{ba} \tag{20}
\]

and the second term is

\[
A_{ab}^{1c} = - \frac{1}{2} (\delta_{a}^{c} b_{b} + \delta_{b}^{c} b_{a} - \frac{1}{2} g_{ab} \delta^{c}). \tag{21}
\]

By substituting (19) in (18) we obtain

\[
\frac{1}{2} R_{ab} = \nabla_{(a} C_{b)c}^{1c} - \nabla_{c} C_{ab}^{1c} + \nabla_{(a} A_{b)c}^{1c} - \nabla_{c} A_{ab}^{1c}. \tag{22}
\]

In turn, straightforward calculations lead us to

\[
\nabla_{(b} C_{a)c}^{1c} = \frac{1}{2} \nabla_{(a} \nabla_{b)} h, \tag{23}
\]

\[
\nabla_{c} C_{ab}^{1c} = \nabla_{(a} C_{b)c}^{1c} + \nabla_{(a} h_{b)c} + R_{(b)a}^{e} c h_{ec} - \nabla_{(a} h_{b)c} - \frac{1}{2} \nabla h_{ab}. \tag{24}
\]
Hence, by replacing Eqs. (23) and (24) in (22) we obtain

\[
\begin{align*}
\frac{1}{2} R_{ab} &= \frac{1}{2} \nabla_a \nabla_b h - \nabla_a \nabla^c h_{bc} - R_{(ba)}^c h_{ec} + R_{(a) b}^c h_{ec} \\
&\quad + \frac{1}{2} \nabla h_{ab} - 2 \nabla_{(ab)} + \frac{1}{2} (\nabla_a b + \nabla_b a - g_{ab} \nabla^c b^c).
\end{align*}
\]

(25)

As is well known, there is a gauge freedom in any geometrical theory of gravitation corresponding to the group of diffeomorphisms of space-time. In practice, these diffeomorphisms may be viewed as coordinate freedom which may be used to impose coordinate conditions. For instance, we may choose coordinates \(x^a\) so that, in the linear approximation, the perturbation \(h_{ab}\) and the vector field \(b^a\) satisfy the gauge condition

\[
\nabla c h_{bc} - \frac{1}{2} \nabla b h + \frac{1}{2} b = 0.
\]

(26)

In this gauge the linearized Ricci tensor simplifies to become

\[
\begin{align*}
\frac{1}{2} R_{ab} &= \frac{1}{2} \nabla h_{ab} - R_{(ba)}^c h_{ec} + R_{(a) b}^c h_{ec} - \frac{1}{2} g_{ab} \nabla b^c.
\end{align*}
\]

(27)

Finally, to rewrite the linear field equations (15) we have to take into account the factors \(\frac{1}{f'} (R)\) and \(\frac{0}{f} (R)\) which depend on \(T\). From (17) we can split the curvature scalar into a zero order part and a first order part:

\[
R = R_0 + \frac{1}{T},
\]

(28)

where \(R_0 = -4\Lambda\). Therefore, using Eq. (5) for first terms we can obtain \(R\) as a function of \(T\):

\[
R = \frac{\alpha_M}{f''(R_0)R_0 - f'(R_0)} T
\]

(29)

From this result we can easily obtain \(\frac{0}{f} (R)\) and \(\frac{1}{f'} (R)\).

Now we are ready to rewrite the first order field equations, in the gauge (26), using Eqs. (10), (15), (27), (29), and (13):

\[
\begin{align*}
\frac{1}{2} \nabla h_{ab} + \frac{4\Lambda}{3} h_{ab} - \frac{1}{2} h_{gb} g_{ab} - \frac{1}{2} g_{ab} \nabla T &= \frac{\alpha_M}{2} \frac{f''}{f'} (f'' + 4\Lambda) h_{ab} \\
- \frac{1}{2} \frac{f}{f'} h_{ab} &= \frac{\alpha_M}{2} \frac{f''}{f'} T_{ab} - \frac{1}{2} \frac{f''}{f'} (1 + \frac{2 f''}{f'} \Lambda) \left(1 + \frac{2 f''}{f'} \Lambda\right)
\end{align*}
\]

(30)

It is important to note that when \(\Lambda = 0\) we recover the result obtained in Refs. [12, 18].

In the Newtonian limit, the equation of motion of a test particle is given by (9) with \(u^a \simeq \delta_0^a\), and the proper time of the particle may be approximated by the coordinate time, \(t\). Since the sources are "slowly varying", we expect the space-time geometry to change slowly as well, so that the time derivative can be assumed to be negligible. Thus, we find

\[
\Gamma^{\mu}_{00} = -\frac{1}{2} e^{2t} \sqrt{-h_{00}^2} h_{00,\mu}, \quad \mu = 1, 2, 3.
\]

(31)
As a consequence, we have to study the Newtonian limit of $h_{00}$ using Eq. (30). Taking into account the expressions for $\Box h_{00}$ and $\Box T$, and the gauge condition (26) in order to substitute $h_{12,2}$, we obtain the field equation for $h_{00}$

$$e^{-2t\sqrt{\Lambda/3}} \nabla^2 \left( \frac{h_{00}}{2} - A \rho \right) - B \left[ \frac{h_{00}}{2} - A \rho \right] = C \rho,$$

where

$$A = \frac{\alpha M_0 f''}{2 f' (4 f'' \Lambda + f')},$$

$$C = \frac{(8 \Lambda f'' f' + (f')^2)}{(f')^2 (4 \Lambda f'' + f')}.$$

Finally, the candidate to be the Newtonian potential is

$$\frac{h_{00}}{2}(\vec{x}) = e^{2t\sqrt{\Lambda/3}} C \int \frac{\rho(\vec{x}') e^{-(-|\vec{x} - \vec{x}'| e^t \sqrt{\Lambda/3} \sqrt{2\Lambda}})}{d^3 \vec{x}'} d^3 \vec{x}' + A \rho(\vec{x}).$$

According to the observed cosmological acceleration, we have to choose $\Lambda \simeq 10^{-53} \text{ m}^{-2}$. Therefore, the last result is consistent with our previous assumption about the time derivative. Moreover, the first term in (35) behaves as the Newtonian potential for any current experiment or observation.

In order to ignore the second term, we have to assume $|A/C| \ll 1$. After a little algebra, this inequality can be rewritten as

$$|f''(R_0)/f'(R_0)| \ll \frac{1}{1 + 2R_0}.$$  

(36)

When $f(R)$ is analytical in $R = 0$ we can fulfill this condition without problems since $R_0 \simeq 0$; but we are interested in the singular case, i.e. when $f(R)$ is singular in $R = 0$.

If $f(R)$ has a pole of order $n$ in $R = 0$, the leading order singularity of $f''(R)$ and $f'(R)$ are $R^{-n-2}$ and $R^{-n-1}$, respectively. Therefore, the inequality (36) cannot be fulfilled except for the particular case

$$f''(R_0) = 0.$$  

(37)

The inconsistency of the Newtonian limit cannot be attributed to the gauge selection since the former is already present in the perturbation equation (15), which was derived previous to choosing the gauge. In reality, the problem is originated by $b_a$ whose first order of perturbation around $R_0$ has the factor $f''(R_0)/f'(R_0)$. In the singular case this factor becomes greater as $R_0$ gets smaller.

It is interesting to notice that the condition $f''(R_0) = 0$ is the same as Dick’s condition [10] for the case of the singular fourth order theories considered by him. This similarity is not surprising on account of the fact, which we will prove elsewhere, that if $f(R) = R + \alpha g(R)$, then for the first-order of $\alpha$ the field equations of the fourth order theories and the field equations of the Palatini theories are the same.
It is not difficult to show that if the Lagrangian density $R - \alpha/3R$ does not satisfies the condition (37), then the perturbation around the maximally symmetric vacuum solution does not give a Newtonian limit. Nevertheless, as Dick has shown, there exist Lagrangian densities that satisfy this condition, for example, $f(R) = R + R^2/9\mu^2 - 3\mu^4/R$.

4 CONCLUSIONS

We have studied the first order of perturbation from the maximally symmetric vacuum solutions, de Sitter space-time, in a family of theories which are obtained by using the Palatini formalism on a general Lagrangian $f(R)$.

We have proved that the first order of perturbation does not give a good Newtonian limit when $f(R)$ has a pole of order ”$n$” in $R = 0$, and $f''(R_0) \neq 0$. However, except for a small correction proportional to $\rho$, the perturbation in the analytical case has a good Newtonian limit. In this analytical case we have also recovered the result of Refs. [12, 18] when $\Lambda = 0$.

While we were working on this paper Meng and Wang [23] claimed to have shown that the Newtonian limit is obtained not only for analytical Lagrangian densities but also for the singular case. However, we disagree with their results. We suppose that the main problem is that they were not consistent with the first-order of perturbation, since they introduced terms $\nabla_b \nabla_a f'(R)$ instead of using the first order forms of these terms, namely, $f''(R_0)/f'(R_0)[R_0f''(R_0) - f'(R_0)]\nabla_b \nabla_a T$. Another point of disagreement, but not essential to show that the Newtonian limit has problems, is about the gauge. They chose the traceless gauge but this gauge does not exist when there are sources.

Finally, the Lagrangian of Carroll et al., and Capozziello et al. [5, 7] does not satisfy the condition $f''(R_0) = 0$, and thus we have proved that the theory of Vollick [20] is not in accordance with the experimental data. Nevertheless, it is worthwhile to study the $f(R)$ theories in the Palatini formulation with an $f(R)$ singular when they fulfill condition (37), because they can give a good explanation of the observed cosmological acceleration.

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